

# Analytic Combinatorics: Complex-analytic Methods and Applications

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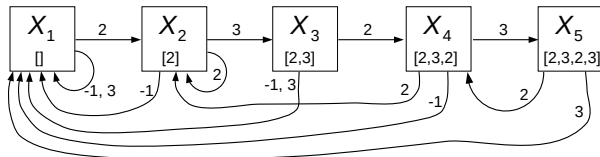
## The automaton and its adjacency matrix

Lattice path is a word  $w$ ,  $p = [a_1, \dots, a_\ell]$ .

Automaton  $T = (V, E)$  has  $\ell$  states  $X_1, \dots, X_\ell$ , where  $X_1 = \varepsilon$  and  $X_i = [a_1, \dots, a_{i-1}]$  for  $2 \leq i \leq \ell$ .

Transitions (edges):  $i, j \in \{1, 2, \dots, \ell\}$ ,  $\lambda \in S$ . Then

$$(X_i, X_j, \lambda) \in E \iff j = \max\{k \mid X_k \text{ is suffix of } X_i.\lambda\}$$



$$p = [2, 3, 2, 3, -1]$$

└ The automaton and its adjacency matrix

Adjacency matrix of  $T$ :

$$A_{i,j} = \sum_{\lambda: (X_i, X_j, \lambda) \in E} u^\lambda$$

Properties:

- $A_{i,j} = 0$  for  $j > i + 1$
- $A_{i,i+1} = u^{a_i}$
- $A_{i,j} = u^{a_{j-1}}$  or  $A_{i,j} = 0$  for  $2 \leq j \leq i$

■

$$\sum_{j=1}^{\ell} A_{i,j} = \begin{cases} P(u) & \text{for } i < \ell \\ P(u) - u^{a_\ell} & \text{for } i = \ell \end{cases}$$

└ The automaton and its adjacency matrix

Choose  $0 \leq d \leq \ell - 2$  and assume

$$[a_1, a_2, \dots, a_{\ell-d-1}] = [a_{d+2}, a_{d+3}, \dots, a_{\ell}]$$

and we want to go from  $X_{j+d}$  to  $X_j$ . Sequence read:

$$\begin{array}{ccccccccccc}
 & & & & & X_j & & & & X_{j+d} \\
 [a_1 & a_2 & \dots & a_{j-2} & a_{j-1} & | & a_j & \dots & a_{j+d-1} & | \\
 [a_{d+2} & a_{d+3} & \dots & a_{j+d-1} & \textcolor{red}{a_{j+d}} & & & & & & 
 \end{array}$$

Must read  $a_{j+d}$ , but then we go to state  $X_{j+d+1}$ .

Thus  $A_{j+d,j} = 0$  for  $j \geq 2$ .

└ The automaton and its adjacency matrix

Assume that we transit from  $X_i = X_{d+k+1}$  to  $X_j = X_{k+1}$ .  
Sequence read:

$$\begin{array}{cccccccccc}
 & & & & & & X_j & & & X_i \\
 [a_1 & a_2 & \dots & \dots & a_{j-2} & a_{j-1} & | & \dots & a_{i-1} & | \\
 [a_1 & \dots & a_{d+2} & a_{d+3} & \dots & \dots & \dots & \dots & a_{d+k} & a_{d+k+1} \\
 [\dots & \dots & a_1 & a_2 & \dots & \dots & \dots & \dots & a_{j-2} & a_{j-1}
 \end{array}$$

Must read  $a_{j-1} = a_k$  and we must have  $a_k \neq a_{d+k+1}$  and

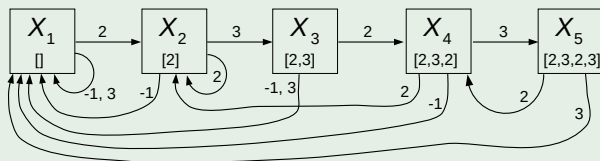
$$[a_1, a_2, \dots, a_{k-1}] = [a_{d+2}, a_{d+3}, \dots, a_{d+k}]$$

and there must not be a smaller  $d$  with that property. Then

$$A_{d+k+1, k+1} = u^{a_k}$$

└ The automaton and its adjacency matrix

## Example



$$p = [2, 3, 2, 3, -1]$$

$$A_{i,i+1} = u^{a_i}$$

$$\begin{pmatrix} u^{-1} + u^3 & u^2 & 0 & 0 & 0 \\ u^{-1} & u^2 & u^3 & 0 & 0 \\ u^{-1} + u^3 & 0 & 0 & u^2 & 0 \\ u^{-1} & u^2 & 0 & 0 & u^3 \\ u^3 & 0 & 0 & u^2 & 0 \end{pmatrix} \quad \begin{array}{l} d = 0 \\ d = 1 \\ d = 2 \\ d = 3 \end{array}$$

$$\begin{array}{l} a_1 = 2 \quad 3 \quad 2 \quad 3 \\ a_2 = 3 \quad 2 \quad 3 \quad -1 \\ a_1 = 2 \quad 3 \quad 2 \\ a_3 = 2 \quad 3 \quad -1 \\ a_1 = 2 \quad 3 \\ a_4 = 3 \quad -1 \\ a_1 = 2 \\ a_5 = -1 \end{array}$$

$$A_{2,2} = u^{a_1} = u^2$$

$$A_{5,4} = u^{a_3} = u^2$$

$$A_{4,2} = u^{a_1} = u^2$$

$$A_{5,2} = 0$$

$A_{5,2} = 0$ , because  $A_{5,4} = u^2$  (smaller  $d$  in that case!)

# Structure of the kernel

We have

$$(I - tA)^{-1} = \frac{\text{adj}(I - tA)}{\det(I - tA)}$$

## Proposition (Structure of the kernel)

*The structure of the kernel is given by the following equations:*

$$(1, 0, \dots, 0) \text{adj}(I - tA) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = R(t, u),$$

and

$$\det(I - tA) = K(t, u) = (1 - tP(u))R(t, u) + t^{|p|} u^{\text{alt}(p)}.$$

# Structure of the kernel

Recall  $A_{i,j} = \sum_{\lambda: (X_i, X_j, \lambda) \in E} u^\lambda$

Let  $W_i(t, u)$  be the g.f. of walks terminating in state  $X_i$ . Then

$$W_i(t, u) = \delta_{i,1} + t \sum_{j=1}^{\ell} W_j(t, u) A_{j,i}$$

and thus  $(W_1, \dots, W_\ell) = (1, 0, \dots, 0) + t(W_1, \dots, W_\ell)A$ .

This implies  $(W_1, \dots, W_\ell)(I - tA) = (1, 0, \dots, 0)$  and

$$\begin{aligned} W(t, u) &= \sum_{i=1}^{\ell} W_i(t, u) = (W_1, \dots, W_\ell) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \\ &= \frac{(1, 0, \dots, 0) \operatorname{adj}(I - tA) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}}{\det(I - tA)} \end{aligned}$$

# Structure of the kernel

On the other hand: Let  $W^{(p)}(t, u)$  be the g.f. of walks ending with  $p$  and no further occurrence of  $p$ . Then

$$W(t, u) + W^{(p)}(t, u) = 1 + tP(u)W(t, u)$$

and

$$W(t, u)t^\ell u^{\text{alt}(p)} = W^{(p)}(t, u)R(t, u).$$

Together these equations imply  $W(t, u) = R(t, u)/K(t, u)$  and thus

$$\frac{(1, 0, \dots, 0) \operatorname{adj}(I - tA) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}}{\det(I - tA)} = \frac{R(t, u)}{K(t, u)}$$

As the denominators are polynomials in  $t$  of degree  $\ell$  and constant term 1, they coincide.

# The autocorrelation vector

## Proposition

The *autocorrelation vector* satisfies

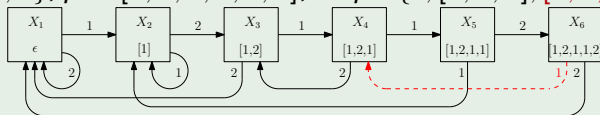
$$\mathbf{v} := \text{adj}(I - tA) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = R(t, u) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} - \begin{pmatrix} s_1(t, u) \\ \vdots \\ s_\ell(t, u) \end{pmatrix}$$

where  $s_j(t, u)$  is the generating function of the finite set  $S_j$  of all walks  $w.a_\ell.\bar{q}$  such that  $|w.a_\ell.\bar{q}| < \ell$  and

- $w$  is a walk starting in state  $X_j$  and ending in  $X_\ell$
- $a_\ell$  is the last step of  $p$
- $\bar{q}$  is the complement of a presuffix  $q$  of  $p$ .

# Example ( $S_j: w.a_\ell.\bar{q}, X_j \rightarrow X_\ell$ )

$$S \supseteq \{1, 2\}, p = [1, 2, 1, 1, 2, 1]; \quad \bar{q} \in \{\varepsilon, [1, 2, 1], [2, 1, 1, 2, 1]\}$$



$$\mathbf{v} = (1 + t^3 u^4 + t^5 u^7) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ t^5 u^7 \\ t^4 u^5 \\ t^3 u^4 + t^5 u^7 \\ t^2 u^3 + t^5 u^7 \\ tu + t^4 u^5 \end{pmatrix}$$

$S_1: X_1 \rightsquigarrow X_6$  and  $a_\ell \implies |w| \geq 6$ ,  $S_1 = \emptyset$

$S_2: w: X_2 \rightsquigarrow X_6 \implies S_2 = \{[2, 1, 1, 2].1.\varepsilon\}$

$S_3 = \{[1, 1, 2].1.\varepsilon\}$ ,  $S_5 = \{2.1.\varepsilon, 2.1.[1, 2, 1]\}$ ,

└ The autocorrelation vector – proof

$$(I - tA)\mathbf{v} = K(t, u) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad A \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = P(u) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} - u^{a_\ell} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} s_1(t, u) \\ \vdots \\ s_\ell(t, u) \end{pmatrix} = R(t, u) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} - \mathbf{v}$$

This implies  $((I - tA)\mathbf{s} = -K(t, u) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + R(t, u) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} - tR(t, u)A \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix})$

$$\begin{pmatrix} s_1(t, u) \\ \vdots \\ s_\ell(t, u) \end{pmatrix} = \underbrace{(I - tA)^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{\begin{pmatrix} V_1(t, u) \\ \vdots \\ V_\ell(t, u) \end{pmatrix}} \underbrace{tu^{a_\ell} R(t, u)}_{\text{add } a_\ell \cdot \bar{q}} - \underbrace{(I - tA)^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}}_{\begin{pmatrix} W(t, u) \\ W^{(2)}(t, u) \\ \vdots \\ W^{(\ell)}(t, u) \end{pmatrix}} \underbrace{t^\ell u^{\text{alt}(p)}}_{\text{add } p}$$

where  $V_j(t, u)$  is the g.f. of walks  $X_j \rightsquigarrow X_\ell$  and  $W^{(j)}(t, u)$  the g.f. of walks that start in  $X_j$ .

Single  $p$  and two overlapping  $p$  at the end counted by both expressions, too short walks only by first term.

## Generating function for bridges

We have  $B(t) = [u^0]W(t, u)$  and thus

$$\begin{aligned} B(t) &= \frac{1}{2\pi i} \oint_{|u|=\varepsilon} \frac{W(t, u)}{u} du = \sum_{i=1}^e \operatorname{Res}_{u=u_i(t)} \frac{W(t, u)}{u} \\ &= \sum_{i=1}^e \operatorname{Res}_{u=u_i(t)} \frac{R(t, u)}{uK(t, u)} = \sum_{i=1}^e \left. \frac{R(t, u)}{\frac{d}{du}(uK(t, u))} \right|_{u=u_i(t)} \end{aligned}$$

As  $K(t, u_i(t)) = 0$ , we get  $K_u(t, u_i(t)) = -K_t(t, u_i(t))/u'_i(t)$  and hence

$$\operatorname{Res}_{u=u_i(t)} = -\frac{u'_i(t)R(t, u)}{u_i(t)K_t(t, u_i(t))}$$

which yields the result.

# Generating function for meanders

## Generating function for meanders

Start from

$$(M_1, \dots, M_\ell) = (1, 0, \dots, 0) + t(M_1, \dots, M_\ell)A - t\{u^{<0}\}(M_1, \dots, M_\ell)A$$

and rewrite as

$$\begin{aligned}(M_1, \dots, M_\ell)(I - tA) &= (1, 0, \dots, 0) - t\{u^{<0}\}(M_1, \dots, M_\ell)A \\ &= (F_1, \dots, F_\ell)\end{aligned}$$

where  $F_\alpha = F_\alpha(t, u) \in \mathbb{R}[[t]][\frac{1}{u}]$  of degree at most  $c$ .

Multiplying by  $(I - tA)^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \frac{1}{\det(I - tA)} \mathbf{v}$  from right gives

$$M(t, u) = \frac{(F_1, \dots, F_\ell) \mathbf{v}}{K(t, u)}$$

Plug  $u_i(t)$  into

$$(M_1, \dots, M_\ell)(I - tA) = (F_1, \dots, F_\ell).$$

The matrix  $(I - tA(u_i(t)))$  is then singular and

$$\mathbf{v}|_{u=u_i(t)} = \text{adj}(I - tA(u_i(t))) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\iff (I - tA(u_i(t)))\mathbf{v}|_{u=u_i(t)} = \det(I - tA(u_i(t))) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 0$$

and so  $\mathbf{v}|_{u=u_i(t)}$  is an eigenvector of  $I - tA(u_i(t))$  belonging to the eigenvalue 0.

Recall the shape of  $A$ : We get

$$I - tA = \begin{pmatrix} 1 - tP(u) + tu^{a_1} & -tu^{a_1} & 0 & 0 & \dots & 0 \\ * & 1 - tu^\alpha & -tu^{a_2} & 0 & \dots & 0 \\ \vdots & \vdots & & & & 0 \\ & & & & & -tu^{a_\ell} \\ * & \dots & & \dots & * & \end{pmatrix}$$

and thus  $I - tA(u_i(t))$  has rank  $\ell - 1$ , meaning that  $\mathbf{v}$  is unique up to scaling.

The equation  $(M_1, \dots, M_\ell)(I - tA) = (F_1, \dots, F_\ell)$  becomes then

$$[(F_1, \dots, F_\ell)\mathbf{v}]|_{u=u_i(t)} = 0.$$

Note that the  $F_\alpha$  and  $\mathbf{v}$  are Laurent polynomials in  $u$ , and so is

$$\Phi(t, u) := u^e(F_1(t, u), \dots, F_\ell(t, u))\mathbf{v}(t, u).$$

By the red equation before we get

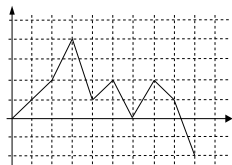
$\Phi(t, u) = u^e M(t, u) K(t, u) \in \mathbb{R}[[t]][u]$  and from before  $\Phi(t, u_i(t)) = 0$ , a *polynomial* equation. Thus

$$\Phi(t, u) = G(t, u) \prod_{i=1}^e (u - u_i(t))$$

and

$$M(t, u) = \frac{G(t, u)}{u^e K(t, u)} \prod_{i=1}^e (u - u_i(t))$$

## Generating function for meanders – special cases

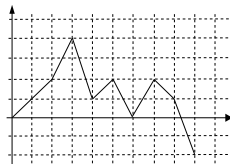


Quasi-meanders:

Note:

- $R(t, u) = 1 + \sum_i t^{\alpha_i} u^{\beta_i}$  then  $\beta_i \geq 0$  for all  $i$ .
- If  $p$  is not a meander, then  $R(t, u) = 1$ .
- $\text{alt}(p) \geq -c$ . If  $\text{alt}(p) = -c$ , then  $R(t, u) = 1$  and thus  $[u^{-c}](1 - tP(u) + t^\ell u^{-c}) \neq 0$ . Thus  $c = e$ .

└ Generating function for meanders – special cases



Quasi-meanders:

Turn back to

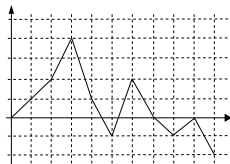
$$\begin{aligned}(M_1, \dots, M_\ell)(I - tA) &= (1, 0, \dots, 0) - t\{u^{<0}\}(M_1, \dots, M_\ell)A \\ &= (F_1, \dots, F_\ell)\end{aligned}$$

Assume  $w$  ends in state  $X_i$ ,  $i \geq 2$ : Then  $w = m.a_{i-1}$ ,  $m$  meander, and so  $\text{alt}(w) \geq 0$ . This implies  $F_2 = F_3 \cdots = F_\ell = 0$ . Thus

$$\Phi(t, u) = u^c(F_1, 0, \dots, 0)\mathbf{v} = u^c F_1(t, u)R(t, u)$$

$$\text{Then } [u^0]F_1 = 1 \quad \implies \quad \Phi(t, u) = R(t, u) \prod_{i=1}^c (u - u_i).$$

# Generating function for meanders – special cases



Reversed meanders:

Note:

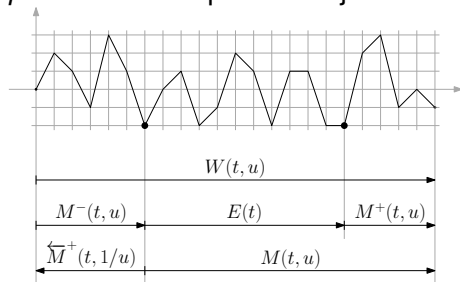
- If  $R(t, u) = 1 + \sum_i t^{\alpha_i} u^{\beta_i}$  then  $\beta_i < 0$  for all  $i$ .
- $\text{alt}(p) < 0$ .
- These two observations imply  $f = d$ .

# Generating function for meanders – special cases

Generalization of the Wiener-Hopf decomposition:

$$w = m^- . e . m^+ = m^- . m$$

$p$  cannot overlap with the junction of any two factors.

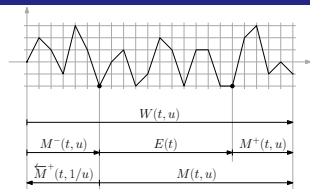


$$M(t, u) = E(t)M^+(t, u),$$

$$W(t, u) = M^-(t, u)M(t, u) = M^-(t, u)E(t)M^+(t, u),$$

$$\implies M(t, u) = \frac{W(t, u)}{M^-(t, u)} = \frac{R(t, u)}{K(t, u)M^-(t, u)}$$

# Generating function for meanders – special cases



$$M(t, u) = E(t)M^+(t, u),$$

$$W(t, u) = M^-(t, u)M(t, u) = M^-(t, u)E(t)M^+(t, u),$$

$$M(t, u) = \frac{W(t, u)}{M^-(t, u)} = \frac{R(t, u)}{K(t, u)M^-(t, u)}.$$

Find  $M^-(t, u)$  by time reversal:

$$-S = \{-s : s \in S\}, \quad \overleftarrow{p} = [-a_\ell, \dots, -a_2, -a_1] : M^-(t, u) = \overleftarrow{M}^+(t, 1/u)$$

and similarly  $\overleftarrow{M}(t, u) = \overleftarrow{E}(t)\overleftarrow{M}^+(t, u)$  and so  
 $M^-(t, u) = \overleftarrow{M}(t, 1/u)/\overleftarrow{E}(t).$

# Generating function for meanders – special cases

As  $\overleftarrow{p}$  is a meander, we get

$$\overleftarrow{M}(t, 1/u) = \frac{u^d \overleftarrow{R}(t, 1/u)}{\overleftarrow{K}(t, 1/u)} \prod_{j=1}^d \left( \frac{1}{u} - \frac{1}{v_j(t)} \right) = \frac{u^d R(t, u)}{K(t, u)} \prod_{j=1}^d \left( \frac{1}{u} - \frac{1}{v_j(t)} \right)$$

and

$$\overleftarrow{E}(t) = \frac{(-1)^{d+1}}{t} \prod_{j=1}^d \frac{1}{v_j(t)}.$$

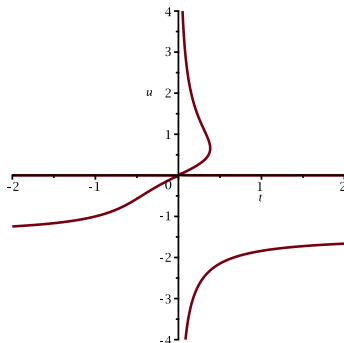
Combining everything and

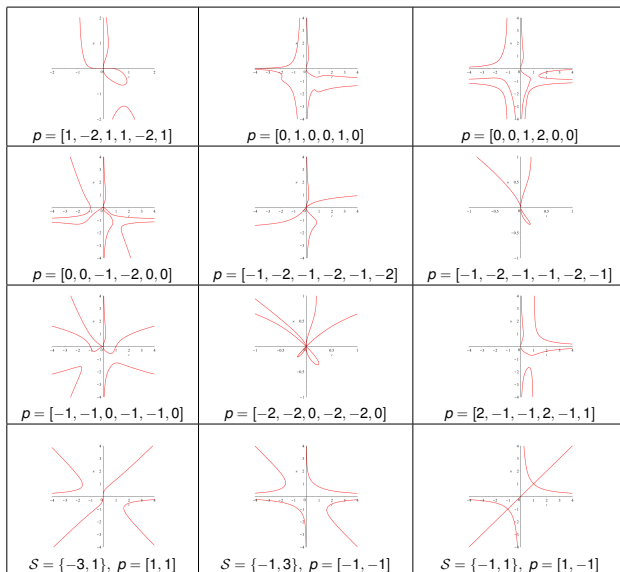
$u^e K(t, u) = -t \prod_{i=1}^e (u - u_i(t)) \prod_{j=1}^d (u - v_j(t))$  gives

$$M(t, u) = \frac{1}{u^e K(t, u)} \prod_{i=1}^e (u - u_i(t)).$$

# Asymptotics for $p$ -avoiding walks

Branches in the classical case ([Banderier, Flajolet \(2002\)](#)):



Asymptotics for  $p$ -avoiding walks


## Definition

A walk model is **generic** if the following five properties hold:

- Property 1.  $B(t)$ ,  $M(t)$  and  $E(t)$  are algebraic, not rational.
- Property 2. They have a unique dominant singularity, which is algebraic, not a pole.
- Property 3.  $G(t, u)$  is a polynomial in  $t$ .
- Property 4. Let  $\rho > 0$  be the smallest number s.t. a large branch meets a small branch at  $t = \rho$ . No large negative branch meets a small negative branch at  $t = \rho$ .
- Property 5. The smallest positive root of  $K(t, 1)$  is simple.

## Lemma

*For any generic model, the dominant singularity of  $B(t)$  and  $E(t)$  is  $\rho$  and it is a square root singularity.*

## Theorem

*Let  $\rho_K$  be the smallest positive root of  $K(t, 1)$ . The asymptotic number of walks of length  $n$  avoiding a pattern  $p$  is*

$$W_n \sim -\frac{R(\rho_K, 1)\rho_K^{-n}}{\rho_K K_t(\rho_K, 1)}.$$

## Theorem

*The asymptotic number of excursions of length  $n$  avoiding a pattern  $p$  is*

$$E_n \sim (-1)^{e-1} \frac{Y(\rho)G(\rho, 0)}{D(\rho)} \sqrt{\frac{K_t(\rho, \tau)}{2\pi\rho K_{uu}(\rho, \tau)}} \cdot n^{-3/2} \rho^{-n},$$

where  $\tau := u_1(\rho)$ ,  $Y(t) := u_2(t) \cdots u_e(t)$ , and  $D(t) := [u^0]u^e K(t, u)$ .

*The asymptotic number of bridges of length  $n$  avoiding a pattern  $p$  is*

$$B_n \sim -\frac{R(\rho, \tau)}{\tau K_t(\rho, \tau)} \sqrt{\frac{K_t(\rho, 1)}{2\pi\rho K_{uu}(\rho, 1)}} \cdot n^{-1/2} \rho^{-n}.$$

## Theorem

Let  $\rho, \rho_K, \tau$  as before.

- *Zero drift:*  $\tau = 1$  and  $\rho_K = \rho$ .
- *Negative drift:*  $\tau > 1$ ,  $u_1(\rho_K) = 1$ , all large roots  $v$  satisfy  $v(t) \neq 1$  for  $\rho_K < t < \rho$ .
- *Positive drift:* Either
  - $\tau < 1$  or
  - $\tau = 1$  but  $\rho_K < \rho$  or
  - $\tau > 1$  but some large root  $v$  satisfies  $v(\rho_K) = 1$ .

Then,  $M_n = [t^n]M(t)$  with

$$M(t) = \frac{(1 - u_1(t))Y(t)G(t, 1)}{K(t, 1)} \quad \text{with } Y(t) := \prod_{i=2}^c (1 - u_i(t))$$

satisfies

$$M_n \sim G(\rho, 1)Y(\rho) \sqrt{\frac{2}{\pi \rho K_t(\rho, 1) K_{uu}(\rho, 1)}} \cdot n^{-1/2} \rho^{-n} \quad (\text{"zero drift"}),$$

$$M_n \sim -\frac{G(\rho, 1)Y(\rho)}{K(\rho, 1)} \sqrt{\frac{\rho K_t(\rho, \tau)}{2\pi K_{uu}(\rho, \tau)}} \cdot n^{-3/2} \rho^{-n} \quad (\text{"negative drift"}),$$

$$M_n \sim \frac{(1 - u_1(\rho_K))Y(\rho_K)G(\rho_K, 1)}{\rho_K K_t(\rho_K, 1)} \cdot \rho_K^{-n} \quad (\text{"positive drift"}).$$