Interpolation of monotone functions in lattices

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Abstract

We show that for every lattice L_0 and for every cardinal κ there is a lattice $L \supseteq L_0$ on which every monotone function can be interpolated by a polynomial on any set of size κ.

Introduction

Let $(L, \wedge, \vee)$ be a lattice. The **polynomials** over L are the well-formed expressions involving constants from L and/or formal variables $\mathtt{x}_1$, $\mathtt{x}_2$, ... together with parentheses and the formal symbols $\wedge$, $\vee$. The set of n-ary **polynomial functions** is the smallest set of functions from L^n to L which contains all the constant functions as well as the projection functions and is closed under (pointwise) meet and join. Every polynomial in which at most the variables $\mathtt{x}_1, \ldots, \mathtt{x}_n$ occur defines in an obvious way an n-ary polynomial function. Clearly every polynomial function f is monotone.

We call a lattice L **n-order polynomially complete** if

$(*)$ every monotone n-ary function is a polynomial function

and we say that L is order polynomially complete if L is n-order polynomially complete for every n.

The finite order polynomially complete lattices were investigated and classified by Schweigert [7], Wille [8], Kindermann [6] and others. The question whether infinite order polynomially complete lattices exist is still open. In [5], Kaiser and Sauer showed

(a) Every order polynomially complete lattice is bounded (i.e., has a largest and a smallest element)

(b) Every infinite lattice contains an infinite chain or an infinite antichain

(c) If L contains an antichain of size κ or a chain of order type $\kappa+1$ or $(\kappa+1)^*$, then L admits at least 2^κ many monotone unary functions,

and concluded that an infinite 1-order polynomially complete lattice (if one exists at all) cannot be countable, but must have size at least continuum. Note that (b) and (c) above are true not only for lattices but for arbitrary partial orders, so the theorem is only partially of an "algebraic" nature. The search for a algebraic proof of the conjecture "there is no order polynomially complete infinite lattice" naturally leads to the following problem:

> Find a "structural" property of polynomial functions and show that in every infinite lattice there is a monotone function not satisfying this property.

In this paper we show that there are no "local" properties of polynomial functions which are not shared by all monotone functions. Formally, we prove:

Theorem A 0.1. *Let L_0 be a lattice, n a natural number, $f_0 : L_0^n \to L_0$ a monotone partial function. Then there is a lattice L which contains L_0 as a sublattice and a polynomial function f on L which extends f_0.*

Thus, any monotone function on a lattice can be realized as the behaviour of a polynomial on a suitable extension lattice. Is it then possible to represent ALL monotone functions by polynomials? We can give a "local" answer to this question in the next theorem. Noticing that the lattice L in theorem A can be chosen to be of the same cardinality as L_0, we can iterate theorem A to obtain:

Theorem B 0.2. *Let λ and κ be infinite cardinals, and let L_0 be a lattice of size $\leq \lambda^\kappa$. Then there is a lattice L of size λ^κ, $L_0 \subseteq L$ such that for every n, for every partial monotone function $f : L^n \to L$ with domain of size $\leq\kappa$ there is a polynomial function f' on L which extends f, i.e., L is "κ-locally order polynomially complete."*

In particular, for any lattice L_0 we can (choosing $\kappa = \lambda = |L_0|$) get a lattice $L \supseteq L_0$ of size $2^{|L_0|}$ such that every monotone function $f : L_0 \to L_0$ is induced by a polynomial over L. (In the terminology of [4] we could phrase this as follows: Every lattice is "extension order polynomially complete.")

For example, letting $\lambda = \kappa = \aleph_0$, we get a lattice of size continuum on which every monotone function can be interpolated by a polynomial on every countable set.

Notation 0.3. Lattices are denoted by L, L', L_1, etc. When we consider several lattices, we use the self-explanatory notation $\wedge_1$, $\leq_2$, etc. for the operations/relation in L_1, L_2, etc. We read lattice operations from left to right: $a \vee b \wedge c = (a \vee b) \wedge c$, $a \wedge b \vee c = (a \wedge b) \vee c$. For $a \in L$ we let $(a] := \{b \in L : b \leq a\}$.

$L_1 \subseteq L_2$ or $L_2 \supseteq L_1$ always means that L_1 is a sublattice of L_2.

We use α, β, γ to range over ordinals, κ, λ, μ for cardinals. The set of natural numbers is denoted by ω. $|A|$ is the cardinality of the set A. λ^κ (cardinal exponentiation) is the cardinality of $\{f : f : \kappa \to \lambda\}$.

f, g, h are always unary functions, $\mathrm{dom}(f)$ is the domain of f and $\mathrm{ran}(f)$ is the range of f. $f : L_1 \to L_2$ means $\mathrm{dom}(f) = L_1$, $\mathrm{ran}(f) \subseteq L_2$, unless we explicitly call f "partial", in which case we only have $\mathrm{dom}(f) \subseteq L_1$.

$f \restriction A$ is the restriction of f to A. "f extends g" or "$g \subseteq f$" means $g = f \restriction \mathrm{dom}(g)$. $f[A] = \mathrm{ran}(f \restriction A)$.

We write ☺$_x$ to denote the end of the proof of theorem (lemma, etc.) x, and we write 😐$_y$ if (most of) a proof is left to the reader. ☹ marks the end of a wrong proof.

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1 Continuous join-homomorphisms

First we remark that it is enough to prove theorem A for the case $n = 1$. This follows easily from fact 1.2 below:

Definition and Fact 1.1. Let L be a lattice, $n > 0$. Define $\delta_n : L \to L^n$ by $\delta_n(x) = \langle x, \dots, x\rangle$. Then δ_n is an embedding of lattices.

Fact 1.2. Let L be a bounded lattice, $n > 0$. Then there is a polynomial π_n over L^n such that the induced function from $(L^n)^n \to L^n$ (which we again call π_n) satisfies

$$\pi_n(\delta_n(a_1), \dots, \delta_n(a_n)) = \langle a_1, \dots, a_n\rangle$$

for all $a_1, \dots, a_n \in L$.

Proof: Let $a = \inf L$, $b = \sup L$. Let $e_1 := \langle b, a, \dots, a\rangle, \dots, e_n := \langle a, \dots, a, b\rangle$. Let π_n be the polynomial $(\mathtt{x}_1 \wedge b_1) \vee \cdots \vee (\mathtt{x}_n \wedge b_n)$. ☺ 1.2

Conclusion 1.3. It is enough to prove theorem A for the case $n = 1$.

Proof: Without loss of generality assume that L_0 is bounded. Let f_0 be an n-ary function on L_0. So $\delta_n \circ f_0$ is a unary function on L_0^n. Find a lattice $L \supseteq L_0^n$ and a polynomial function $f' : L \to L$ extending $\delta_n \circ f_0$. Now let $f : L^n \to L$ be defined by $f(x_1, \dots, x_n) = f'(\pi_n(x_1, \dots, x_n))$. It is clear that (identifying L_0 with $\delta_n[L_0] \subseteq L_0^n \subseteq L$) f is a polynomial function extending f_0. ☺ 1.3

So from now on we will only on concentrate on unary functions.

Definition 1.4. A lattice L is **complete**, if every subset A of L has a greatest lower bound, called $\inf A$. In a complete lattice, also every set has a least upper bound, $\sup A$. In particular, L must be bounded: $0 = \inf L = \sup \emptyset$, $1 = \inf \emptyset = \sup L$.

Fact 1.5. For every lattice L there is a complete lattice $\bar{L}$ such that $L \subseteq \bar{L}$.

Proof: Well known (see, e.g., [3]). Let $\bar{L}$ be the set of ideals of L, i.e., the set of downward closed upward directed subsets of L (including the empty set), and embed L via $x \mapsto \{y : y \leq x\}$. ☹ 1.5

Fact 1.6. (a) If $f : L_1 \to L_2$ is a partial monotone function, and if L_2 is a complete lattice, then f can be extended to a total monotone function.
(b) If f is a partial monotone function from L to L then there is a lattice $\bar{L}$ of the same cardinality as L and a total monotone function $\bar{f} : \bar{L} \to \bar{L}$ extending f.

Proof: (a): Let $\bar{f}(x) = \sup\{f(y) : y \le x, y \in \mathrm{dom}(f)\}$.
(b): Let $L' \supseteq L$ be a complete lattice, and let $f' : L' \to L'$ be a total function extending f. Now let $\bar{L} \subseteq L'$ be the structure generated from L by the binary operations $\wedge$ and $\vee$ and the unary function f'. ☺ 1.6

To explain the motivation for the following definition, consider a complete lattice L and the polynomial $p(\mathtt{x}) = \mathtt{x} \vee a$ for some fixed $a \in L$. Clearly $p(x \vee y) = p(x) \vee p(y)$ and moreover $p(\sup A) = (\sup A) \vee a = \sup(A \cup \{a\}) = \sup\{x \vee a : x \in A\} = \sup p[A]$ for any nonempty $A \subseteq L$.

Definition 1.7. $f : L_1 \to L_2$ is called **join-homomorphism** if for all $x, y \in L_1$ we have $f(x) \vee f(y) = f(x \vee y)$.
We call $f : L_1 \to L_2$ a **continuous join-homomorphism** if

(a) f is monotone

and there is a function $f^* : L_2 \to L_1$ such that

(b) $\forall y \in L_2\ f(f^*(y)) \le y$

(c) $\forall x \in L_1\, \forall y \in L_2\ f(x) \le b \Rightarrow a \le f^*(y)$.

We define "meet-homomorphism" and "continuous meet-homomorphism" dually.

Remark 1.8.

1. Every join homomorphism is monotone.
2. Every continuous join-homomorphism is a join-homomorphism.
3. If f is a continuous join-homomorphism, then f^* is unique, namely, $f^*(y)$ must be the greatest element of

$$\{x \in L_1 :\ f(x) \le y\}$$

4. (a)&(b)&(c) above can be replaced by (a)&(bc), where

   (bc) $\forall a \in L_1 \, \forall b \in L_2 \; f(a) \leq b \Leftrightarrow a \leq f^*(b)$.

5. If L_1 is a complete lattice, then $f : L_1 \to L_2$ is a continuous join-homomorphism iff $f(\sup A) = \sup f[A]$ for all $A \subseteq L_1$.

Proof: (1), (3) and (4) are clear.

(2): Since f is monotone, $f(x \vee y) \geq f(x) \vee f(y)$. Conversely, let $z := f(x) \vee f(y)$. Then $f(x) \leq z$, so $x \leq f^*(z)$, and similarly $y \leq f^*(z)$. Hence $x \vee y \leq f^*(z)$, so $f(x \vee y) \leq ff^*(z) \leq z = f(x) \vee f(y)$ and we are done.

(5): If $f(\sup A) = \sup f[A]$, then f is clearly a join homomorphism, hence monotone. Let $f^*(y) := \sup\{x : f(x) \leq y\}$. It is clear that $ff^*(y) \leq y$, and that $f(x) \leq y$ implies $x \leq f^*(y)$.

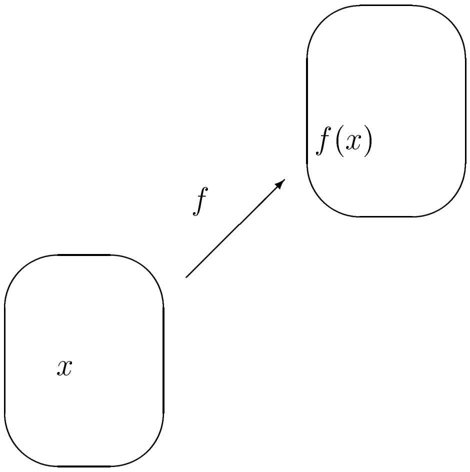
Conversely, assume that f is a continuous join-homomorphism. Let $A \subseteq L_1$, and let $a = \sup A$. The monotonicity of f shows that $f(a)$ is an upper bound for $f[A]$. Now let b be any upper bound for $f[A]$. We have $\forall x \in A \; f(x) \leq b$, hence $\forall x \in A \; x \leq f^*(b)$, so $a = \sup A \leq f^*(b)$, hence $f(a) \leq f(f^*(b)) \leq b$. So $f(a)$ is indeed the least upper bound of $f[A]$.

☺ 1.8

The next lemma shows that the property isolated in definition 1.7 more or less characterizes the possible behaviours of the polynomial function $x \mapsto x \vee a$, at least for complete lattices.

Definition 1.9. Let L_1 and L_2 be disjoint lattices, $f : L_1 \to L_2$. We define $L_1 +_f L_2 = (L_1 \cup L_2, \leq)$ as follows:

$x \leq y$ iff: either $x, y \in L_1$, $x \leq_1 y$
or $x, y \in L_2$, $x \leq_2 y$
or $x \in L_1$, $y \in L_2$, $f(x) \leq_2 y$



Lemma 1.10. *Assume that $f : L_1 \to L_2$ is a continuous join-homomorphism. Let $L := L_1 +_f L_2$. Then*

(a) *L is a lattice.*

(b) *$L_1, L_2 \subseteq L$.*

(c) *If L_2 has a smallest element a_2, then for all $x \in L_1$ we have $f(x) = x \vee a_2$.*

Proof: Transitivity of $\leq$ follows from the monotonicity of f: Let $x \leq y \leq z$. The only nontrivial case is when $x \leq_1 y$ and $f(y) \leq_2 z$. But then also $f(x) \leq_2 f(y) \leq_2 z$.

Now let $f^* : L_2 \to L_1$ be such that $f^*(y)$ is always the greatest element of $\{x \in L_1 : f(x) \leq y\}$. We claim that for $x \in L_1$, $y \in L_2$ we have $\inf\{x, y\} = x \wedge_1 f^*(y)$. For a proof, note that z is a lower bound for $\{x, y\}$ iff $z \leq_1 x$ and $f(z) \leq_2 y$, which is equivalent to $z \leq_1 x$ and $z \leq_1 f^*(y)$, i.e., $z \leq x \wedge_1 f^*(y)$.

We leave it to the reader to check the other cases of the following table:

$$\inf\{x, y\} = \begin{cases} x \wedge_1 y & \text{if } x, y \in L_1 \\ x \wedge_2 y & \text{if } x, y \in L_2 \\ x \wedge_1 f^*(y) & \text{if } x \in L_1,\ y \in L_2 \end{cases}$$

$$\sup\{x, y\} = \begin{cases} x \vee_1 y & \text{if } x, y \in L_1 \\ x \vee_2 y & \text{if } x, y \in L_2 \\ f(x) \vee_2 y & \text{if } x \in L_1,\ y \in L_2 \end{cases}$$

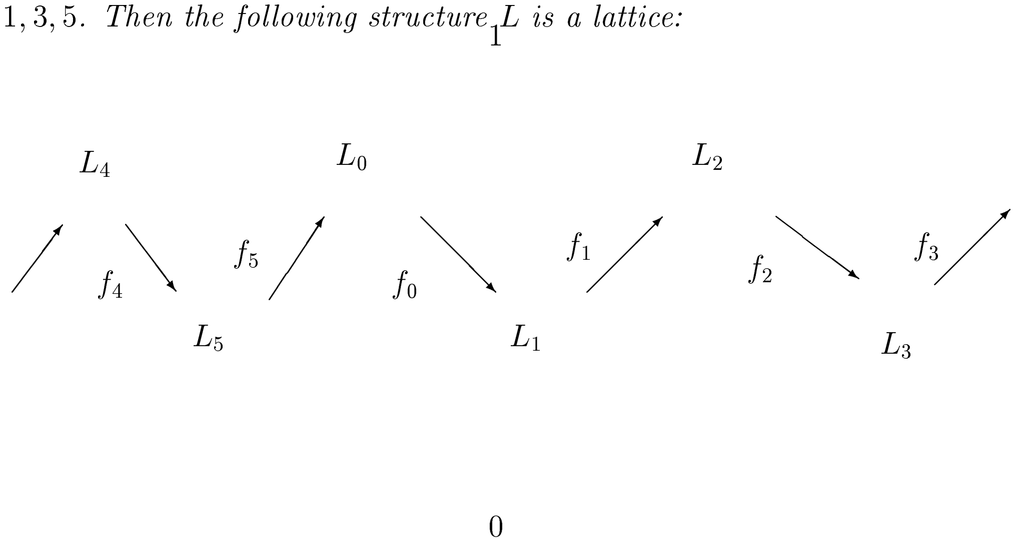
(b) and (c) are now clear. ☺ 1.10

Conversely, it is easy to check the following: If $f : L_1 \to L_2$, and L_1 has a largest element b_1, then we also have:

> If $L_1 +_f L_2$ is a lattice, then f is a continuous join-homomorphism, witnessed by $f^*(y) = y \wedge b_1$.

The construction in lemma 1.10 deals with the situation where the function which we want to represent by a polynomial has disjoint domain and range. Below we give a slight extension of the previous lemma that deals with the case $f : L \to L$. (The following lemma is also a generalization of a construction considered in [2].)

Lemma 1.11. *Let L_0, L_1, ..., L_5 be disjoint lattices, and let $f_i : L_i \to L_{i+1}$, $i = 0, \ldots, 5$ be monotone functions. Assume that each f_i is a continuous meet-homomorphism for $i = 0, 2, 4$ and a continuous join-homomorphism for $i = 1, 3, 5$. Then the following structure L is a lattice:*



The universe of L is the disjoint union $L_0 \cup L_1 \cup \cdots \cup L_5 \cup \{0, 1\}$. The partial order on L is defined as follows:

$x \leq y$ *iff* $x = 0$ *or* $y = 1$
or for some $i \in \{0, \ldots, 5\}$, $x, y \in L_i$ *and* $x \leq_i y$
or for some $i \in \{1, 3, 5\}$, $x \in L_i$, $y \in L_{i+1}$ *and* $f_i(x) \leq_{i+1} y$
or for some $i \in \{0, 2, 4\}$, $x \in L_{i+1}$, $y \in L_i$ *and* $x \leq_{i+1} f_i(y)$.

Moreover, if $a_i = \inf L_i$, $b_i = \sup L = i$, *then for all* $x \in L_0$ *we have*

$$f_5\, f_4\, f_3\, f_2\, f_1\, f_0(x) = x \wedge b_1 \vee a_2 \wedge b_3 \vee a_4 \wedge b_5 \vee a_0.$$

(It is understood that addition of indices is performed modulo 6, i.e. $L_6 = L_0$, $L_7 = L_1$, *etc.)*

Proof: We leave it to the reader to check that join and meet can be computed accoding to the following rules:

- $\inf\{x, 0\} = 0$, $\sup\{x, 0\} = \inf\{x, 1\} = x$, $\sup\{x, 1\} = 1$.
- If $x, y \in L_i$, then $\inf\{x, y\} = x \wedge_i y$, $\sup\{x, y\} = x \vee_i y$.
- If $x \in L_i$, $y \in L_{i+1}$, i even, then $\inf\{x, y\} = f_i(x) \wedge_{i+1} y$, $\sup\{x, y\} = x \vee_i f_i^*(y)$.
- If $x \in L_i$, $y \in L_{i+1}$, i odd, then $\inf\{x, y\} = x \wedge_i f_i^*(y)$, $\sup\{x, y\} = f_i(x) \vee_{i+1} y$.
- If $x \in L_i$, $y \in L_{i+2}$, i odd, then $\inf\{x, y\} = 0$, $\sup\{x, y\} = f_i(x) \vee_{i+1} f_{i+1}^*(y)$.
- If $x \in L_i$, $y \in L_{i+2}$, i even, then $\inf\{x, y\} = f_i(x) \wedge_{i+1} f_{i+1}^*(y)$, $\sup\{x, y\} = 1$.
- If $x \in L_i$, $y \in L_{i+3}$, then $\inf\{x, y\} = 0$, $\sup\{x, y\} = 1$.

☹ 1.11

2 Factoring monotone functions

By what we have seen so far, theorem A is true if we change the condition "f monotone" to "f a continuous join-homomorphism". The dual is of course also true. Our next aim is to show that every monotone function can be written as a composition of a continuous join-homomorphism and a continuous meet-homomorphism.

Definition 2.1. For any lattice L let $\hat{L}$ be the set of all downward closed subsets of L:

$$\hat{L} = \{A \subseteq L : \forall a \in A \, \forall b \leq a \; b \in A\}.$$

Fact 2.2. For any lattice L the partial order $(\hat{L}, \subseteq)$ is a complete lattice, with operations

$$\begin{aligned} A \vee B &= A \cup B \quad \text{(more generally, } \sup \mathcal{A} = \bigcup \mathcal{A} \text{ for any } \mathcal{A} \subseteq \hat{L}) \\ A \wedge B &= \{z : \exists x \in A \, \exists y \in B \; z \leq x \wedge y\} \end{aligned}$$

Proof: Straightforward. ☺ 2.2

Lemma 2.3. *Assume that L is a complete lattice. Then the map $g : L \to \hat{L}$, defined by $g(x) = (x] := \{y \in L : y \leq x\}$, is a continuous meet-homomorphism.*

Proof: We will use the dual of remark 1.8(5). $x \leq y$ implies $(x] \subseteq (y]$, hence g is monotone.
Let $A \subseteq L$, $a = \inf A$. We have to show $g(a) = \inf g[A]$. It is clear that $g(a)$ is a lower bound for $g[A]$, since g is monotone. So let $B \in \hat{L}$ be any lower bound for $g[A]$. Then $B \subseteq (x]$ for all $x \in A$, hence each $b \in B$ is a lower bound for A. So each $b \in B$ is $\leq a$, so $B \subseteq (a] = g(a)$. ☺ 2.3

Factor Lemma 2.4. *Assume that $f : L_0 \to L_2$ is monotone, and L_2 is complete. Then there is a function $h : \hat{L}_0 \to L_2$ satisfying*

- *h is a continuous join-homomorphism*
- *$h \circ g = f$ (where g is the continuous meet-homomorphism from the previous lemma)*

$$\begin{array}{ccc} L_0 & \xrightarrow{f} & L_2 \\ g \searrow & & \nearrow h \\ & \hat{L}_0 & \end{array}$$

Proof: Let $h(A) = \sup f[A]$. Clearly h is monotone. Also, since f is monotone we get

$$h(g(x)) = \sup f[\{y : y \leq x\}] = f(x)$$

For any $\mathcal{A} \subseteq \hat{L}$ we have $\sup \mathcal{A} = \bigcup \mathcal{A}$ and

$$\sup h[\mathcal{A}] = \sup\{\sup\{f(x) : x \in A\} : A \in \mathcal{A}\} = \sup\{f(x) : x \in \bigcup \mathcal{A}\} = h(\sup \mathcal{A}).$$

☺ 2.4

Now we can put the pieces together:

Proof of theorem A: By conclusion 1.3 we only have to consider the case of a unary function. Let L_0 be a lattice, $f : L_0 \to L_0$ a partial monotone function. By facts 1.5 and 1.6 we may assume that L_0 is complete, and that f is a total function. Let $L_1 := \hat{L}_0$ as in definition 2.1, and let $f_0 : L_0 \to L_1$ be the continuous meet-homomorphism defined in lemma 2.3, $f_0(x) = (x]$. Let L_2 be an isomorphic copy of L_0, $i : L_0 \to L_2$ an isomorphism. The map $i \circ f : L_0 \to L_2$ is monotone, so we can find a continuous join-homomorphism $f_1 : L_1 \to L_2$ such that $f_1 \circ f_0 = i \circ f$.

$$\begin{array}{ccc} L_0 & \xrightarrow{i \circ f} & L_2 \\ f_0 \searrow & & \nearrow f_1 \\ & L_1 & \end{array}$$

Let L_3, L_4, L_5 be disjoint isomorphic copies of L_2 with isomorphisms

$$\begin{array}{ccccccc} L_2 & & & & L_4 & & & & L_0 \\ & f_2 \searrow & & f_3 \nearrow & & f_4 \searrow & & f_5 \nearrow & \\ & & L_3 & & & & L_5 & & \end{array}$$

such that $f_5 \circ f_4 \circ f_3 \circ f_2 \circ i : L_0 \to L_0$ is the identity map on L_0. Then $f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1 \circ f_0 = f : L_0 \to L_0$.

Letting $L := L_0 \cup \cdots \cup L_5 \cup \{0, 1\}$ as in lemma 1.11, we get a lattice L. Letting $a_i := \inf(L_i)$, $b_i := \sup(L_i)$ we get

$$\forall x \in L_0 \ f(x) = x \wedge b_1 \vee a_2 \wedge b_3 \vee a_4 \wedge b_5 \vee a_0$$

☺ 0.1

Remark 2.5. We may assume that the lattice L is of the same cardinality as L_0. (Assuming, of course, that L_0 is infinite. If L_0 is finite the L will also be finite of size $\leq 2^{|L_0|} + 5|L_0| + 2$.)

Proof: Let L' be the sublattice of L generated by the elements of the original (possibly non-complete) L_0 plus the coefficients of the polynomial p.

☺ 2.5

3 Proof of theorem B

Let $\mu := \lambda^\kappa$. Let $\Gamma : \mu \times \mu \times \omega \to \mu$ be a bijection with the property

$$(**) \forall \beta, \gamma < \mu \ \forall n : \ \Gamma(\beta, \gamma, n) \geq \gamma$$

Start with an arbitrary infinite lattice L_0 of cardinality $\leq \mu$. We will construct a continuous increasing sequence $(L_\alpha : \alpha \leq \mu)$ of lattices and a sequence $(p_\alpha : \alpha < \mu)$ of polynomials, where p_α has coefficients in $L_{\alpha+1}$, and $|L_\alpha| \leq \max(|\alpha|, |L_0|)$.

First we deal with successor steps: L_α has size $\leq \mu$ so for any n the set of partial functions from L_α^n to L_α whose domain has size $\leq \kappa$ has cardinality at most $\mu^\kappa = \mu$. Let $(f^\beta_{\alpha,n} : \beta < \mu)$ be an enumeration of these functions.
We can find $\beta_\alpha, \gamma_\alpha \in \mu$, $n_\alpha \in \omega$ such that $\Gamma(\beta_\alpha, \gamma_\alpha, n_\alpha) = \alpha$. As $\gamma_\alpha \leq \alpha$, we already know what $f^{\beta_\alpha}_{\gamma_\alpha, n_\alpha}$ is, namely, a partial function from $L_{\gamma_\alpha}^{n_\alpha}$ to L_{γ_α}, hence also a partial function from $L_\alpha^{n_\alpha}$ to L_α.

By theorem A we can find a lattice $L_{\alpha+1} \supseteq L_\alpha$ of again the same size and a polynomial p_α with coefficients in $L_{\alpha+1}$ such that the polynomial function induced by p_α (which we again call p_α) extends f_α and hence also $f^{\beta_\alpha}_{\gamma_\alpha,n_\alpha}$.

This concludes the construction of $L_{\alpha+1}$ and p_α.

If α is a limit ordinal, we let $L_\alpha := \bigcup_{\beta<\alpha} L_\beta$.

Finally, let $L := L_\mu = \bigcup_{\alpha<\mu} L_\alpha$. L is of size μ. Let f be an arbitrary partial monotone function from L^n to L with domain of cardinality $\leq \kappa$. We have to find a polynomial which extends f. Since $\mu^\kappa = \mu$ we have $\kappa < cf(\mu)$, so there is a γ such that $\mathrm{dom}(f) \cup \mathrm{ran}(f) \subseteq L^n_\gamma$. Let $f = f^\beta_{\gamma,n}$. Letting $\alpha := \Gamma(\beta, \gamma, n) \geq \gamma$ we get a polynomial p_α with coefficients in $L_{\alpha+1}$ such that the induced polynomial function (in $L_{\alpha+1}$ as well as in L) extends f.

☺ 0.2

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