MOST ALGEBRAS HAVE THE INTERPOLATION PROPERTY

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ABSTRACT. We show that for "almost all" algebras on a given set A, every function on A can be interpolated by a polynomial on any set of cardinality κ (for reasonable κ).

1. Introduction

Let $\mathfrak{A} = (A, f_1, \ldots, f_n)$ be a universal algebra. Let $F(A)$ be the set of all functions from some A^t to A ($t \in \{0, 1, 2, 3 \ldots\}$), $P(\mathfrak{A})$ the set of polynomial functions.

If A is a finite set, then it is possible to have $F(A) = P(\mathfrak{A})$. In fact, "most" algebras will have this property, see [6]. For infinite A this is impossible by cardinality considerations: $|F(A)| = 2^{|A|} > |A| = |P(\mathfrak{A})|$.

We can still ask if we can "interpolate" all functions by polynomial functions on sufficiently large sets: Let $L_k P(\mathfrak{A})$ be the set of "k-local" polynomial functions, [that is, functions which can be interpolated by a polynomial on any set containing less than $1 + k$ (i.e., $\leq k$) many points], then we have

$$F(A) = L_1 P(\mathfrak{A}) \supseteq L_2 P(\mathfrak{A}) \supseteq \cdots \supseteq L_{\aleph_0} P(\mathfrak{A})$$

where $L_{\aleph_0} P(\mathfrak{A})$ is the set of all "local" polynomial functions, i.e., functions that can be interpolated by a polynomial on any finite set of points.

This sequence can also be continued into the transfinite by considering $L_\kappa P(A)$ for infinite cardinals κ. We define $L_\kappa P(\mathfrak{A})$ to be the set of all functions from some A^t into A which can be interpolated by a polynomial on any set of $< \kappa$ many points. (So $L_{\kappa^+} P(\mathfrak{A})$ is the set of functions that can be interpolated by a polynomial on any set of $\leq \kappa$ many points. See also 6.1.3)

Trivially, $\kappa < \kappa'$ implies $L_\kappa P(\mathfrak{A}) \supseteq L_{\kappa'} P(\mathfrak{A})$, and we have $L_\kappa P(\mathfrak{A}) = P(\mathfrak{A})$ whenever $\kappa > |A|$, but what else can be said about these various degrees of interpolability? For which κ can we have $F(A) = L_\kappa P(\mathfrak{A})$? In particular, are there algebras $\mathfrak{A}$ on which all functions are interpolable on all sets of cardinality smaller than A itself, i.e., $F(A) = L_{|A|} P(\mathfrak{A})$?

János Kollár in [4] has shown that the answer is "yes" iff $|A| = |A|^{<|A|}$. It is easy to see (e.g., from 6.2.4, 6.2.6 and 6.3.5), that $|A| = |A|^{<|A|}$ holds iff

- either $|A| = \aleph_0$,
- or $|A| = \kappa^+ = 2^\kappa$ for some infinite cardinal κ,
- or $|A|$ is a strongly inaccessible cardinal.

We give a criterion to decide, for regular κ, when there are algebras $\mathfrak{A}$ satisfying the κ-interpolation property (i.e., $F(A) = L_\kappa P(\mathfrak{A})$), and we show that a kind of "0-1-law" holds: If one algebra on the underlying set A has the κ-interpolation property, then "almost all" algebras have this property.

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For a fixed set A, a trivial counting argument (see 1.6) gives an upper bound $\kappa^*(A)$ for the set

$$K(A) := \{\kappa : \text{ there is an algebra } \mathfrak{A} \text{ on the set } A \text{ with the } \kappa\text{-IP}\}.$$

In 2.7 we show that this bound is sharp: For any $\kappa \le \kappa^*(A)$ there are algebras with the κ-IP. Moreover, for a fixed similarity type τ and a fixed set A we can define a topology on $\mathrm{Alg}_\tau(A)$, the set of algebras of type τ with underlying set A, which naturally induces an ideal on $\mathrm{Alg}_\tau(A)$, and we will show that except for a set in this ideal, all algebras in $\mathrm{Alg}_\tau(A)$ have the κ-IP but not the κ^+-IP. In particular, for any $\kappa \le \kappa^*(A)$ there are many $\mathfrak{A} \in \mathrm{Alg}_\tau(A)$ with $F(A) = L_\kappa P(\mathfrak{A}) \ne L_{\kappa^+} P(\mathfrak{A})$.

A variation of this argument (in 3.3) gives many algebras on A that have a majority function, and have the κ-IP but not the κ^+-IP. This stands in contrast to the case of finite k, where in the presence of a majority function the k-IP implies the $k+1$-IP (and hence even the $\aleph_0$-IP).

For the special case of a countable set A we show in 4.5 that the set of algebras on A with the IP is residual (=co-meager) in the Tychonoff topology. If A is uncountable then the set of algebras with the IP is dense but not co-meager in the Tychonoff topology.

Finally we investigate the case of unary algebras in section 5.

In an appendix we review basic notions from set theory.

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Setup 1.1. We fix a finite similarity type $\tau = (t_1, \dots, t_n)$, where the t_ℓ are finitely many natural numbers. ℓ will always range over $1, \dots, n$. We will assume that one of the t_ℓ, say t_1, is at least 2. (See section 5 for the case of unary algebras.)

To simplify the notation (but really without loss of generality) we assume $t_1 = 2$.

Notation 1.2. Let A be a nonempty (usually: infinite) set, t a natural number (usually > 0). A t-ary operation on A is a function from A^t into A. We write $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$ for the domain and range of f.

We let

$$\begin{aligned} F^t(A,B) &= \{f : \mathrm{dom}(f) = A^t,\ \mathrm{ran}(f) \subseteq B\} \\ F(A,B) &= \bigcup_{t=0}^{\infty} F^t(A,B) \end{aligned}$$

and we let $F^t(A) = F^t(A,A)$, $F(A) = F(A,A)$.

We will often have occasion to also consider partial operations, in particular partial operations with small domain:

$$\begin{aligned} \tilde{F}^t(A,B) &= \{f : \mathrm{dom}(f) \subseteq A^t,\ \mathrm{ran}(f) \subseteq B\} \\ \tilde{F}^t_\kappa(A,B) &= \{f \in \tilde{F}^t(A,B) : |\mathrm{dom}(f)| < \kappa\} \\ \tilde{F}(A,B) &= \bigcup_{t=1}^{\infty} \tilde{F}^t(A,B) \end{aligned}$$

$\tilde{F}(A) = \tilde{F}(A,A)$, etc.

A function is a set of ordered pairs. So if f and g are functions, then $f \subseteq g$ means that f is a restriction of g.

If f is a function, A a set, then $f\restriction A$ is the restriction of f to $A \cap \mathrm{dom}(f)$, i.e.,

$$f\restriction A := f \cap (A \times \mathrm{ran}(f))$$

For any set A we let

$$\mathrm{Alg}_\tau(A) := \{(A, f_1, \dots, f_n) : \forall \ell\, f_\ell \in F^{t_\ell}(A)\}$$

Since A is usually held constant, we identify $(A, f_1, \dots, f_n)$ with $(f_1, \dots, f_n)$.

Definition 1.3. If $\mathfrak{A} = (A, f_1, \dots, f_n)$ is in $\mathrm{Alg}_\tau(A)$, then we define the set of polynomial functions, $P(\mathfrak{A})$ or $P(A, f_1, \dots, f_n)$, as the smallest set of functions that contains all the constant functions and projections, and is closed under the (pointwise) operations f_ℓ. So $P(\mathfrak{A}) \subseteq F(A)$.

$\tilde{P}(\mathfrak{A}) \subseteq \tilde{F}(A)$ is the set of all partial functions which are restrictions of polynomial functions, $\tilde{P}^t(\mathfrak{A}) = \tilde{P}(\mathfrak{A}) \cap \tilde{F}^t(A)$, etc.

Definition 1.4. Let κ be an infinite cardinal. $\mathfrak{A} = (A, f_1, \dots, f_n)$ has the κ-IP if: For all $t \in \omega$, all $f \in F^t(A)$, all $B \subseteq A^t$:

If $|B| < \kappa$, then there is $g \in P^t(\mathfrak{A})$, $g\restriction B = f\restriction B$.

The next fact is just a reformulation of this definition.

Fact 1.5. *$\mathfrak{A} = (A, f_1, \dots, f_n)$ has the κ-IP iff $\tilde{F}_\kappa(A) \subseteq \tilde{P}(\mathfrak{A})$.*

If the underlying set A has size λ, then there are of course only λ many polynomials on A, whereas $|F(A)| = \lambda^\lambda = 2^\lambda > \lambda$. It is still possible for $\mathfrak{A}$ to have the κ-IP (for sufficiently small κ), since the same polynomial can serve for many partial functions. The following observation, a straightforward generalization of [4, 4.1], gives an upper bound on κ:

Lemma 1.6. *Assume that $\mathfrak{A} = (A, f_1, \dots, f_n)$ has the κ-IP, $|A| \geq \aleph_0$. Then $|A|^{<\kappa} = |A|$. (See 6.2)*

Proof. Clearly $\mathfrak{A}$ does not have the $|A|^+$-IP, so $\kappa \leq |A|$.

Let $\mu < \kappa$ be arbitrary, and fix a set $B \subseteq A$ of size μ. We have $|F^1(B, A)| = |A|^\mu$. Each $f \in F^1(B, A)$ is the restriction of some polynomial function $\bar{f} \in P^1(\mathfrak{A})$, and clearly $f \neq f'$ implies $\bar{f} \neq \bar{f}'$, so $|A| = |P^1(\mathfrak{A})| \geq |A|^\mu$. $\square$

Reformulation 1.7. *Let $\kappa^*(A) := \min\{\kappa : |A|^\kappa > |A|\}$. Then $\mathfrak{A}$ can have the κ-IP only if $\kappa \leq \kappa^*(A)$.*

In section 2 we will show that this estimate is best possible. That is: If $\kappa \leq \kappa^*(A)$, then "most" algebras on A will have the κ-IP. (Our notion of "most" will depend on κ, see 2.4.)

2. κ-TOPOLOGIES

We will consider universal algebras of a fixed type $\tau = (t_1, \dots, t_n)$, over a fixed (infinite) set A, i.e., structures of the form $\mathfrak{A} = (A, f_1, \dots, f_n)$, where $f_\ell \in F^{t_\ell}(A)$. We assume $t_1 = 2$.

Fact 2.1. *If A is an infinite set of cardinality λ, κ a cardinal $\leq \lambda$, then there is a continuous increasing chain $(A_\gamma : \gamma < \kappa)$ such that $\bigcup_{\gamma<\kappa} A_\gamma = A$ and $\forall \gamma < \kappa$: $|A_{\gamma+1} \setminus A_\gamma| = \lambda$, $|A_\gamma| = \lambda$. (See section 6.3)*

Proof. Since $\kappa \cdot \lambda = \lambda$, there is a bijection $f : \{(\alpha, \beta) : \alpha < \lambda, \beta < \kappa\} \to A$. Let $A_\alpha := \{f(\gamma, \beta) : \gamma < \lambda, \beta < \alpha\}$. $\square$

Definition 2.2. Assume that A, $(A_\gamma : \gamma < \kappa)$ satisfy

– $A = \bigcup_{\gamma<\kappa} A_\gamma$, $|A| = \lambda$
– $(A_\gamma : \gamma < \kappa)$ is continuous and increasing
– $\forall \gamma < \kappa : |A_{\gamma+1} \setminus A_\gamma| = \lambda$, $|A_\gamma| = \lambda$.

Then:

1. For any $\gamma < \kappa$ we let
$$\mathrm{Fun}(\gamma) := \{(\eta_1, \dots, \eta_n) : \text{each } \eta_\ell \text{ is in } F^{t_\ell}(A_\gamma, A)\}$$
2. For $\eta = (\eta_1, \dots, \eta_n) \in \mathrm{Fun}(\gamma)$ we define
$$\mathcal{B}_\eta := \{(f_1, \dots, f_n) \in \mathrm{Alg}_\tau(A) : \text{for each } \ell,\ \eta_\ell \subseteq f_\ell\}$$

Definition and Remark 2.3. The family $\{\mathcal{B}_\eta : \eta \in \mathrm{Fun}(\gamma), \gamma < \kappa\}$ is a basis for a Hausdorff topology on $\mathrm{Alg}_\tau(A)$.

For different choices of the family $(A_\gamma : \gamma < \kappa)$ we will of course get different topologies. We call each topology that is obtained in this way a *κ-topology.*

Definition 2.4. Let $(X, \mathcal{E})$ be a topological space. We say that $Y \subseteq X$ is κ-meager if Y can be covered by κ many nowhere dense sets (or equivalently, by κ many closed sets with empty interior). We say that $Y \subseteq X$ is κ-residual (= κ-co-meager) if $X \setminus Y$ is κ-meager.

Clearly the κ-meager sets form an ideal. (In the general case this ideal may be improper, e.g. if κ is too large. See 4.3 for a less trivial example.)

Terminology 2.5. Assume $\mathcal{J}$ is a proper ideal on a (nonempty) set X. Sets in $\mathcal{J}$ are considered to be "negligible" or "small", while sets in the dual filter are considered to be "large".

We express this intuitive notion of small vs large also in the following way: Let $p(\cdot)$ be any property. We say that "$p(x)$ holds for $\mathcal{J}$-almost all $x \in X$"(or "$\mathcal{J}$-almost every ($\mathcal{J}$-a.e.) x has property p") iff
$$\{x \in X : p(x) \text{ does not hold}\} \in \mathcal{J}.$$

This terminology is derived from measure theory: If μ is a measure, then the customary notion of "μ-a.e." is in our language "$\mathcal{N}_\mu$-a.e.", where $\mathcal{N}_\mu$ is the ideal of μ-null sets, i.e., $\mathcal{N}_\mu = \{Y \subseteq X : \mu(Y) = 0\}$.

If the definition of $\mathcal{J}$ is sufficiently "natural", then we may write "almost every" instead of "$\mathcal{J}$-almost every".

Of course such a terminology would not be justified if $\mathcal{J}$ were improper, i.e., if $X \in \mathcal{J}$. The following lemma, a straightforward generalization of Baire's theorem, shows that all our ideals are proper.

Lemma 2.6. *Let $\mathcal{E}$ be a κ-topology on $\mathrm{Alg}_\tau(A)$. Then $\mathrm{Alg}_\tau(A)$ is not κ-meager.*

Moreover, no nonempty open set in $\mathrm{Alg}_\tau(A)$ is κ-meager, i.e., the complement of any κ-meager set is still dense.

Proof. Let $\mathcal{U}$ be an open set, and let $(\mathcal{C}_\alpha : \alpha < \kappa)$ be a family of closed nowhere dense sets. Let $\mathcal{O}_\alpha := \mathrm{Alg}_\tau(A) \setminus \mathcal{C}_\alpha$. We have to show that $\mathcal{U} \cap \bigcap_{\alpha<\kappa} \mathcal{O}_\alpha$ is not empty.

Let $\eta_0 = (\eta_{0,1}, \dots, \eta_{0,n}) \in \mathrm{Fun}(\gamma_0)$ be such that $\mathcal{B}_{\eta_0} \subseteq \mathcal{U}$. We will define sequences $(\eta_\alpha : \alpha < \kappa)$ and $(\gamma_\alpha : \alpha < \kappa)$, $\eta_\alpha = (\eta_{\alpha,1}, \dots, \eta_{\alpha,n}) \in \mathrm{Fun}(\gamma_\alpha)$, with the following properties:

1. $\forall \alpha \forall \beta : \alpha < \beta \Rightarrow \gamma_\alpha < \gamma_\beta$.
2. $\forall \ell \forall \alpha \forall \beta : \alpha < \beta \Rightarrow \eta_{\alpha,\ell} \subset \eta_{\beta,\ell}$ (hence $\mathcal{B}_{\eta_\alpha} \supset \mathcal{B}_{\eta_\beta}$).
3. $\forall \ell \forall \alpha$: If α is a limit ordinal, then $\eta_{\alpha,\ell} = \bigcup_{\beta<\alpha} \eta_{\beta,\ell}$ and $\gamma_\alpha = \sup\{\gamma_\beta : \beta < \alpha\}$.

4. $\forall\alpha$: $\mathcal{B}_{\eta_{\alpha+1}} \subseteq \mathcal{O}_\alpha$.

The construction of this sequence is straightforward. For limit steps, η_α is given by 3, and in successor steps we have to find $\gamma_{\alpha+1}$ and $\eta_{\alpha+1} \in \mathrm{Fun}(\gamma_{\alpha+1})$ satisfying $\mathcal{B}_{\eta_{\alpha+1}} \subseteq \mathcal{O}_\alpha \cap \mathcal{B}_{\eta_\alpha}$. This is possible because $\mathcal{O}_\alpha$ is dense open and $\mathcal{B}_{\eta_\alpha}$ is open.

Finally, let $f_\ell := \bigcup_{\alpha<\kappa} \eta_{\alpha,\ell}$. Note that $\sup_{\alpha<\kappa} \gamma_\alpha = \gamma$, so $\mathrm{dom}(f_\ell) = \bigcup_{\alpha<\kappa} A_\alpha^{t_\ell} = A^{t_\ell}$, so f_ℓ is a total function in $F^{t_\ell}(A)$. Clearly $(f_1, \dots, f_n) \in \bigcap_{\alpha<\kappa} \mathcal{O}_\alpha \cap \mathcal{U}$. ☐

The following theorem shows that the condition $|A| = |A|^{<\kappa}$ is not only necessary for an algebra on A to have the κ-IP, but will in "most" cases also be sufficient.

Theorem 2.7. *Assume that A is infinite, κ an infinite* **regular** *cardinal satisfying $|A|^{<\kappa} = |A|$. Then the set of algebras on A of type τ which have the κ-interpolation property is κ-residual in any κ-topology.*

For example, choosing a set A of size $2^{\aleph_0}$ and $\kappa = \aleph_1$ we conclude that in any $\aleph_1$-topology on A, "almost all" algebras on A have the property that all functions can be interpolated by a polynomial on any countable set.

Informally, the main point of the proof is the following:
Fix a basic set $\mathcal{B}_\nu$, $\nu \in \mathrm{Fun}(\alpha)$, $\nu = (\nu_1, \dots, \nu_n)$. We have to find an $\eta = (\eta_1, \dots, \eta_n)$ such that the basic set $\mathcal{B}_\eta$ is a subset of $\mathcal{B}_\nu$ and all algebras in $\mathcal{B}_\eta$ have the interpolation property with respect to sufficiently many partial functions. (For the details about bookkeeping see the formal proof below.) It turns out that choosing η_1 appropriately will be sufficient — $\eta_2, \dots, \eta_n$ can be chosen arbitrarily subject to the requirement that they extend $\nu_2, \dots, \nu_n$, respectively.

Assume first that we are only concerned with a single partial unary function σ. Recall that $\nu_1 : A_\alpha^2 \to A$. We can pick some element $c_\sigma \in A_{\alpha+1} \setminus A_\alpha$ and extend ν_1 to a function $\eta_1 : A_{\alpha+1}^2 \to A$ such that the function $x \mapsto \eta_1(c_\sigma, x)$ extends σ.
Now for any algebra $(A, f_1, f_2, \dots)$ with f_1 extending η_1 the partial function σ will be the restriction of the polynomial $f_1(c_\sigma, x)$.

We can deal in this way with many functions σ simultaneously, since $A_{\alpha+1} \setminus A_\alpha$ is quite large.

Now consider the case with a function σ which is binary: the domain of σ is a set of pairs. For each a which appears as a first coordinate in the domain of σ pick a (distinct) constant $c_{\sigma,a} \in A_{\alpha+1} \setminus A_\alpha$, and stipulate $\eta_1(c_{\sigma,a}, b) = \sigma(a, b)$ for all relevant b. Then pick a new constant $c_\sigma \in A_{\alpha+1} \setminus A_\alpha$ and stipulate $\eta_1(c, a) = c_{\sigma,a}$.
Clearly, then
$$f_1(f_1(c, a), b) = f_1(c_{\sigma,a}, b) = \sigma(a, b)$$
for all $(a, b) \in \mathrm{dom}(\sigma)$ whenever f_1 is any function extending η_1.

k-ary partial functions are treated similarly. This ends our informal description of the proof of theorem 2.7.

The proof of theorem 2.7 is based on the following lemma:

Lemma 2.8. *Assume $|A|^{<\kappa} = |A|$, and let $(A_\alpha : \alpha < \kappa)$ induce a κ-topology on $\mathrm{Alg}_\tau(A)$ as in 2.2.*

Let $\beta < \kappa$. Then the set
$$\mathcal{U}_\beta := \{(f_1, \dots, f_n) \in \mathrm{Alg}_\tau(A) : \tilde{F}_\kappa(A_\beta, A) \subseteq \tilde{P}(f_1, \dots, f_n)\}$$
contains a dense open set.

(Note: $\mathcal{U}_\beta$ is the set of all algebras $\mathfrak{A}$ such that, for any t, every partial t-ary function whose domain is a subset of A_β of cardinality $< \kappa$ is the restriction of a polynomial function over $\mathfrak{A}$.)

Proof. First we need to define some notation: If f is a binary function, then we define $f^{[2]} := f$ and $f^{[t+1]}(x_0, \dots, x_t) = f^{[t]}(f(x_0, x_1), x_2, \dots, x_t)$ for $t \geq 2$.

If $\mathrm{dom}(g) \subseteq A^t$, $x \in A$, then we define the $(t-1)$-ary (partial) function $g_{[x]}$ by

$$g_{[x]}(x_2, \dots, x_t) := g(x, x_2, \dots, x_t) \text{ whenever } (x, x_2, \dots, x_t) \in \mathrm{dom}(g).$$

We will prove lemma 2.8 by finding, for each basic open set $\mathcal{B}_\nu$, a basic open set $\mathcal{B}_\eta$ (i.e., we construct some appropriate $\eta = \eta_\nu$ from ν) such that $\mathcal{B}_\eta \subseteq \mathcal{B}_\nu \cap \mathcal{U}_\beta$. (This is clearly sufficient, as $\bigcup_\nu \mathcal{B}_{\eta_\nu}$ is open, contained in $\mathcal{U}_\beta$, and meets every open set $\mathcal{B}_\nu$.)

So fix $\nu = (\nu_1, \dots, \nu_n)$ with $\mathrm{dom}(\nu_\ell) = A_\alpha^{n_\ell}$. Wlog $\alpha \geq \beta$. ν_1 is a function from A_α^2 into A. We will extend the function ν_1 to a function η_1 on $A_{\alpha+1}^2$ such that

(**) for every t and for every t-ary partial function $\sigma \in \tilde{F}_\kappa^t(A_\beta, A)$
there is a c in $A_{\alpha+1} \setminus A_\alpha$ such that:
for all $(x_1, x_2, \dots, x_t) \in \mathrm{dom}(\sigma)$: $\sigma(x_1, \dots, x_t) = \eta_1^{[t+1]}(c, x_1, \dots, x_t)$.

Before we construct ν_1, let us notice that property (**) is sufficient to ensure $\mathcal{B}_\eta \subseteq \mathcal{U}_\beta$: Indeed, let $(f_1, \dots, f_n) \in \mathcal{B}_\eta$. We have to show $(f_1, \dots, f_n) \in \mathcal{U}_\beta$, i.e., $\tilde{F}_\kappa(A_\beta, A) \subseteq \tilde{P}(f_1, \dots, f_n)$.

So let $\sigma \in \tilde{F}_\kappa(A_\beta, A)$. Then (**) guarantees that we can find some c such that $\sigma(x_1, \dots, x_t) = \eta_1^{[t+1]}(c, x_1, \dots, x_t)$ for all $(x_1, x_2, \dots, x_t) \in \mathrm{dom}(\sigma)$. Now the function $f_1^{[t+1]}$ extends the partial function $\eta_1^{[t+1]}$, and clearly the function

$$(x_1, \dots, x_n) \mapsto f_1^{[t+1]}(c, x_1, \dots, x_n)$$

is a polynomial function. This shows that (**) will imply $\mathcal{B}_\eta \subseteq \mathcal{U}_\beta$.

Now we begin to define η_1: We will "code" partial functions in $\tilde{F}_\kappa(A_\beta, A)$ by elements of $A_{\alpha+1} \setminus A_\alpha$. From the assumption $|A|^{<\kappa} = |A|$ we get

$$|\tilde{F}_\kappa(A_\beta, A)| \leq |A|^{<\kappa} = |A| = |A_{\alpha+1} \setminus A_\alpha|$$

(for the first inequality see 6.2.7, the last equality is true by our choice of the sets A_α, see 2.2).

So we can find a 1-1 function

$$T : \tilde{F}_\kappa(A_\beta, A) \to A_{\alpha+1} \setminus A_\alpha$$

There is a a function $\eta_1 : A_{\alpha+1}^2 \to A$ satisfying the following properties:

(a) η_1 extends ν_1.
(b) $\eta_1(T(\sigma), a) = \sigma(a)$ whenever $\sigma \in \tilde{F}_\kappa^1(A_\alpha, A)$, $a \in \mathrm{dom}(\sigma)$.
(c) $\eta_1(T(\sigma), a) = T(\sigma_{[a]})$ if $\sigma \in \bigcup_{t \geq 2} \tilde{F}_\kappa^t(A_\beta, A)$.
(d) $\eta_1(a, b) =$ arbitrary, otherwise.

(There are no contradictions between the requirements (b) and (c), since T is 1-1. There are no contradictions between the requirements (a) and (b) (or (a) and (c)), because $\mathrm{dom}(\nu_1) \subseteq A_\alpha^2$, and $T(\sigma) \notin A_\alpha$, so $(T(\sigma), a) \notin \mathrm{dom}(\nu_1)$.)

Define $\eta_2, \dots, \eta_n$ arbitrarily extending $\nu_2, \dots, \nu_n$. To prove property (**), we show that for all $t \geq 1$, all $\sigma \in \tilde{F}_\kappa^t(A_\beta, A)$:

$$\eta_1^{[t+1]}(T(\sigma), a_1, \dots, a_t) = \sigma(a_1, \dots, a_t) \text{ for all } (a_1, \dots, a_t) \in \mathrm{dom}(\sigma).$$

The proof is by induction on t: For $t = 1$ we have

$$\eta_1^{[t+1]}(T(\sigma), a_1) = \eta_1(T(\sigma), a_1) = \sigma(a_1).$$

For $t > 1$ we have:

$$\begin{aligned} \eta_1^{[t+1]}(T(\sigma), a_1, \dots, a_t) &= \eta_1^{[t]}(\eta_1(T(\sigma), a_1), \dots, a_t) = \\ \text{(by definition of } \eta_1) &= \eta_1^{[t]}(T(\sigma_{[a_1]}), a_2, \dots, a_t) = \\ \text{(by induction hypothesis)} &= \sigma_{[a_1]}(a_2, \dots, a_t) = \sigma(a_1, a_2, \dots, a_t). \end{aligned}$$

□

The following fact explains the role that the regularity of κ plays:

Fact 2.9. *If $(A_\alpha : \alpha < \kappa)$ is increasing, $A = \bigcup_{\alpha<\kappa} A_\alpha$, and $B \subseteq A^t$ has cardinality $< cf(\kappa)$, then there is an $\alpha < \kappa$ such that $B \subseteq A^t_\alpha$.*

In particular, if κ is regular then every set set $B \subseteq A^t$ of size $< \kappa$ is "bounded", i.e. $B \subseteq A^t_\alpha$ for some $\alpha < \kappa$.

Proof. Define a function $f : B \to \mathfrak{O}_{<\kappa}$ by

$$f(b_1, \dots , b_t) = \min\{\beta : \{b_1, \dots , b_t\} \subseteq A_\beta\}$$

Then $\operatorname{ran}(f)$ cannot be cofinal in $\mathfrak{O}_{<\kappa}$, so there is $\beta < \kappa$ such that $\forall b \in B : f(b) < \beta$. Thus, $\exists\beta < \kappa\, \forall(b_1, \dots , b_t) \in B\ b_1, \dots , b_t \in A_{f(b)} \subseteq A_\beta$, i.e., $B \subseteq A_\beta$. □

Proof of theorem 2.7. Let $(\mathcal{U}_\beta : \beta < \kappa)$ be the sets defined in 2.8. The set $\bigcap_{\beta<\kappa} \mathcal{U}_\beta$ is κ-residual, so it is enough to show that every algebra in $\bigcap_{\beta<\kappa} \mathcal{U}_\beta$ has the κ-IP.

Let $(A, f_1, \dots , f_n) \in \bigcap_{\beta<\kappa} \mathcal{U}_\beta$, and let $\sigma \in \tilde{F}^t_\kappa(A)$. Since $|\operatorname{dom}(\sigma)| < \kappa$, we can use the regularity of κ to conclude that there is an α such that $\operatorname{dom}(\sigma) \subseteq A^t_\alpha$.

Hence $\sigma \in \tilde{F}^t_\kappa(A_\alpha, A) \subseteq \tilde{P}(f_1, \dots , f_n)$. □

3. The Baker-Pixley Theorem

It is known that the k-IP and the k'-IP can coincide for natural numbers $k \neq k'$. For example (see [1])

Theorem 3.1. *Let $m : A^3 \to A$ be a majority function, i.e.:*

$$\forall x, y \in A : \quad m(x, x, y) = m(x, y, x) = m(y, x, x) = x$$

Then:

If $\mathfrak{A} := (A, f_1, \dots , f_n, m)$ has the 3-IP, then $\mathfrak{A}$ has the $\aleph_0$-IP.

We now show that such an implication κ-IP $\Rightarrow$ κ'-IP does not hold for infinite cardinals. (Except for the trivial cases: $\kappa \geq \kappa'$, or $\kappa > \kappa^*(A)$.)

We will need the following concept:

Definition 3.2. Let $(A_\gamma : \gamma < \kappa)$ be as in 2.2, $f : A^t \to A$. We say that $\alpha < \kappa$ is a *ceiling* for f if for all $b \in A_{\alpha+1} \setminus A_\alpha$ and for all $a_2, \dots , a_t \in A_{\alpha+1}$ we have:

$$\{f(b, a_2, \dots , a_t), f(a_2, b, \dots , a_t), \dots , f(a_2, a_3, \dots , b)\} \subseteq A_{\alpha+1}.$$

We say that α is a ceiling for $\mathfrak{A} = (A, f_1, \dots , f_n)$ if α is a ceiling for each f_ℓ. We let $cl(\mathfrak{A})$ be the set of all ceilings for $\mathfrak{A}$.

Theorem 3.3. *Let $\tau = (t_1, \dots , t_n)$ be a similarity type, $t_1 = 2$.*

1. *In any κ-topology on $\operatorname{Alg}_\tau(A)$ the set of algebras which have the κ-IP but not the κ^+-IP is κ-residual.*
2. *Moreover: Let $(A_\gamma : \gamma < \kappa)$ be as in 2.2, $t_0 \in \omega$. Let $m : A^{t_0} \to A$ be any function satisfying $m[A^{t_0}_\alpha] \subseteq A_\alpha$ for all α (i.e., $m(a_1, \dots , a_t) \in A_\alpha$ whenever $a_1, \dots , a_t \in A_\alpha$).*
   (It is easy to find a majority function $m : A^3 \to A$ with this property.)
   Let $\kappa \leq \kappa^*(A)$ (see 1.7). Then the set

   $$\big\{(f_1, \dots , f_n) \in \operatorname{Alg}_\tau(A) : (A, m, f_1, \dots , f_n) \text{ has the } \kappa^+\text{-IP}\big\}$$

   is κ-meager.

Proof. By 2.7, (2) implies (1).

First note that $cl(A, m, f_1, \dots, f_n) = cl(A, f_1, \dots, f_n)$ by our assumption on m.

We have:

1. For every $\alpha < \kappa$, the set $\{\mathfrak{A} : \alpha \in cl(\mathfrak{A})\}$ is clopen.
   [Why? If $\mathfrak{A} = (f_1, \dots, f_n)$ and $\mathfrak{A}' = (f'_1, \dots, f'_n)$ agree up to α, i.e., $f_\ell \restriction A_\alpha^{t_\ell} = f'_\ell \restriction A_\alpha^{t_\ell}$ for all ℓ, then $\alpha \in cl(\mathfrak{A})$ iff $\alpha \in cl(\mathfrak{A}')$.]
2. The sets $\mathcal{D}_\alpha := \{\mathfrak{A} : \exists \beta > \alpha \ \ \beta \in cl(\mathfrak{A})\}$ are open and dense.
   [Given $\eta \in \mathrm{Fun}(\gamma)$, it is easy to find an extension η' such that $\mathcal{B}_{\eta'} \subseteq \mathcal{D}_\alpha$]
3. The set $\{\mathfrak{A} : cl(\mathfrak{A}) \text{ is bounded in } \kappa\}$ is κ-meager in $\mathrm{Alg}_\tau(A)$.
   [Its complement is equal to $\bigcap_{\alpha<\kappa} \mathcal{D}_\alpha$.]

Now let $cl(\mathfrak{A})$ be unbounded. We will show that $\mathfrak{A}$ cannot have the κ^+-interpolation property:

For each $\alpha < \kappa$ pick $a_\alpha \in A_{\alpha+1} \setminus A_\alpha$.

Let f be the partial function that maps each a_α to $a_{\alpha+1}$.

If p is a nonconstant polynomial with coefficients in A_β, then we can find $\alpha > \beta$ in $cl(\mathfrak{A})$. Clearly we must then have $p(a_\alpha) \in A_{\alpha+1}$, but $a_{\alpha+1} \in A_{\alpha+2} \setminus A_{\alpha+1}$, so $p(a_\alpha) \neq a_{\alpha+1}$.

□

4. The Tychonoff Topology

Although the topologies presented in the previous section are very convenient to work with, they are rather unnatural since they depend on a specific arbitrary representation $A = \bigcup_\alpha A_\alpha$. In this section we will investigate $\mathrm{Alg}_\tau(A)$ with the natural topology:

Definition 4.1. Consider A as a discrete topological space. The product (or "Tychonoff") topology on $\mathrm{Alg}_\tau(A)$ is generated by sets of the form

$$\mathcal{B}_\eta := \{(f_1, \dots, f_n) : \forall \ell \ f_\ell \supseteq \eta_\ell\}$$

where η ranges over the set

$$\mathrm{Fun} := \{\eta : \eta = (\eta_1, \dots, \eta_n), \forall \ell \ \eta_\ell \in F_{\aleph_0}^{t_\ell}(A)\},$$

i.e., over all sequences of the form $\eta = (\eta_1, \dots, \eta_n)$, where each η_ℓ is a partial finite t_ℓ-ary function.

A set is *meager* (also called: of the first category) if it can be covered by countably many [closed] nowhere dense sets, and a set is *residual* (or co-meager) if it contains a countable intersection of dense open sets.

The following well-known fact can be proved similarly to 2.6.

Fact 4.2. *If A is any (infinite) set, then no nonempty open set in $\mathrm{Alg}_\tau(A)$ is meager in the Tychonoff topology.*

However, κ-meager sets are often trivial:

Remark 4.3. *If the continuum hypothesis holds, then for any countable set A we have $|\mathrm{Alg}_\tau(A)| = \aleph_1$, so $\mathrm{Alg}_\tau(A)$ is trivially $\aleph_1$-meager.*

If A has cardinality $\aleph_1$, then irrespective of the underlying cardinal arithmetic, $\mathrm{Alg}_\tau(A)$ is $\aleph_1$-meager in the Tychonoff topology.

Proof. The first remark is clear.

To prove the second remark, let A be of size $\aleph_1$. So we can write A as an increasing continuous union $A = \bigcup_{\alpha<\omega_1} A_\alpha$, where each A_α is countable.

For $\alpha < \omega_1$ let

$$\mathcal{C}_\alpha := \{(f_1, \dots, f_n) \in \mathrm{Alg}_\tau(A) : A_\alpha \text{ is a subalgebra of } (A, f_1, \dots, f_n)\}$$

We have to check that

1. $\mathcal{C}_\alpha$ is closed.
2. $\mathcal{C}_\alpha$ does not contain any nonempty open set, hence is nowhere dense.
3. $\bigcup_{\alpha<\omega_1} \mathcal{C}_\alpha = \mathrm{Alg}_\tau(A)$.

Proof of 1: Let $(f_1, \dots, f_n) \notin \mathcal{C}_\alpha$. So there are ℓ and $a_1, \dots, a_{t_\ell} \in A_\alpha$ with $f_\ell(a_1, \dots, a_{t_\ell}) \notin A_\alpha$. Define $\eta \in \mathrm{Fun}$ by letting η_ℓ have domain $\{(a_1, \dots, a_{t_\ell})\}$ and range $\{f_\ell(a_1, \dots, a_{t_\ell})\}$, and let $\eta_i = \emptyset$ for $i \neq \ell$. Then $\mathcal{B}_\eta \cap \mathcal{C}_\alpha = \emptyset$ and $(f_1, \dots, f_n) \in \mathcal{B}_\eta$, so we have found a basic neighborhood of $(f_1, \dots, f_n)$ disjoint from $\mathcal{C}_\alpha$. This shows that the complement of $\mathcal{C}_\alpha$ is open.

Proof of 2: We have to show that there is no η such that $\mathcal{B}_\eta \subseteq \mathcal{C}_\alpha$. Easy.

Proof of 3: Let $\mathfrak{A} = (A, f_1, \dots, f_n) \in \mathrm{Alg}_\tau(A)$. We have to find α such that $\mathfrak{A} \in \mathcal{C}_\alpha$. For each $\alpha < \omega_1$ and each $\ell \in \{1, \dots, n\}$ we can find an ordinal $\beta = \beta(\alpha, \ell) < \omega_1$ such that $\mathrm{ran}(f{\restriction}A_\alpha^{t_\ell}) \subseteq A_\beta$ (since $\mathrm{ran}(f{\restriction}A_\alpha^{t_\ell})$ is countable). Let $\beta(\alpha) := \max(\alpha+1, \beta(\alpha,1), \dots, \beta(\alpha,n))$. Define a countable sequence $\gamma_0 < \gamma_1 < \cdots < \omega_1$ by requiring $\gamma_{n+1} = \beta(\gamma_n)$. Let $\gamma = \sup\{\gamma_0, \gamma_1, \dots\}$, then $\gamma < \omega_1$ since ω_1 is regular. Now for each n we have $\mathrm{ran}(f_\ell{\restriction}A_{\gamma_n}^{t_\ell}) \subseteq A_{\gamma_{n+1}} \subseteq A_\gamma$, so $\mathrm{ran}(f_\ell{\restriction}A_\gamma^{t_\ell}) \subseteq A_\gamma$, which means $\mathfrak{A} \in \mathcal{C}_\gamma$.

□

So how many algebras will have the κ-interpolation property? From our result in the previous section we can deduce the following corollary:

Observation 4.4. *Using the Tychonoff topology on* $\mathrm{Alg}_\tau(A)$*, we have:*

1. *Let* $|A|^{<\kappa} = |A|$*. Then the set of algebras in* $\mathrm{Alg}_\tau(A)$ *that have the* κ*-IP is dense.*
2. *However, if* A *is uncountable, then the set of algebras with the* 2*-IP is not residual.*

Proof. (1) The set of algebras with the κ-IP is dense in any κ-topology, so also dense in the (coarser) Tychonoff topology.

(2) Let $\mathcal{M}$ be a meager set in $\mathrm{Alg}_\tau(A)$, then we can find a family $(\mathcal{M}_i : i \in \omega)$ of closed sets such that $\mathcal{M} \subseteq \bigcup_i \mathcal{M}_i$. Define a sequence $(\eta_i : i \in \omega)$, $\eta_i \in \mathrm{Fun}$, $\eta_i = (\eta_{i,1}, \dots, \eta_{i,n})$, such that for all i:

- $\mathcal{B}_{\eta_i} \cap \mathcal{M}_i = \emptyset$
- $\eta_{i,\ell} \subseteq \eta_{i+1,\ell}$, for $\ell = 1, \dots, n$ (i.e., $\mathcal{B}_{\eta_i} \supseteq \mathcal{B}_{\eta_{i+1}}$).

[Given η_i, it is possible to find the required η_{i+1} because $\mathcal{M}_{i+1}$ is nowhere dense.]

Now let $B \subseteq A$ be a countable set such that for all $i \in \omega$, all $\ell \in \{1, \dots, n\}$ the range of $\eta_{i,\ell}$ is contained in B. We can find an algebra $\mathfrak{A} = (f_1, \dots, f_n)$ such that for all $\ell = 1, \dots, n$:

- For all $i \in \omega$: $\eta_{i,\ell} \subseteq f_\ell$, i.e., $\mathfrak{A} \in \bigcap_{i\in\omega} \mathcal{B}_{\eta_i}$.
- $\mathrm{ran}(f_\ell) \subseteq B$.

Now since B is countable and A is uncountable, $B \neq A$. But any nonconstant polynomial in $P(\mathfrak{A})$ takes only values in B, so clearly $\mathfrak{A}$ does not have the IP; in fact, there are functions that cannot be interpolated on 2 points. By construction, $\mathfrak{A} \notin \mathcal{M}$.

□

For countable sets A the situation is different:

Theorem 4.5. *Let A be a countable set. Then the set of algebras on A which have the interpolation property is residual in the Tychonoff topology on* $\mathrm{Alg}_\tau(A)$.

Proof. Let $A = \bigcup_j A_j$, where the A_j form an increasing sequence of finite sets satisfying

$$|A_{j+1} \setminus A_j| > |F^{\leq j}(A_j)|$$

(where $F^{\leq j}(A_j) = \bigcup_{m \leq j} F^m(A_j)$). Similarly to 2.8 we can now show that each set

$$\mathcal{U}_j := \{(f_1, \ldots , f_n) \in \mathrm{Alg}_\tau(A) : \tilde{F}^{\leq j}(A_j) \subseteq \tilde{P}(f_1, \ldots , f_n)\}$$

contains a dense open set, so the complement of $\bigcap_j \mathcal{U}_j$ is meager. Now let $\mathfrak{A} \in \bigcap_j \mathcal{U}_j$, $\sigma \in F^t_i(A)$, $i < \omega$. Then there is some n such that $\sigma \in \tilde{F}^{\leq j}(A_j)$, so $\sigma \in \tilde{P}(\mathfrak{A})$. □

Remark 4.6. *One can argue that this topology does not capture the "right" notion of smallness. After all, many interesting sets, for example* $\mathrm{Alg}_\tau(A) \cap \mathbb{V}$, *for any nontrivial variety* $\mathbb{V}$, *will be (closed and) nowhere dense.*

5. Unary algebras

If $t_1 = \cdots = t_n = 1$, then for any algebra of type $\tau = (t_1, \ldots , t_n)$, all polynomials will depend at most on one variable, so the best one can hope for is $F^1(A) \subseteq P(\mathfrak{A})$.

Since there are, irrespective of the size of A, at most countably many (unary) polynomial functions that are not constant, we will consider only countably infinite algebras.

Definition 5.1. We say that $a \in A$ is a fixpoint of $\mathfrak{A} = (A, f_1, \ldots , f_n)$ iff $f_\ell(a) = a$ for $\ell = 1, \ldots , n$.

Observation 5.2. *Let A be countable, $\tau = (1, \ldots , 1)$. Then the set of algebras with a fixpoint is open dense in the Tychonoff topology. Hence the set of algebras with the interpolation property is nowhere dense.*

However, if we consider algebras with at least one injective function without cycles, then the picture changes: Let $\tau = (1, \ldots , 1)$, with $n \geq 2$ many 1's.

Definition 5.3. Let $f_* : A \to A$. We define

$$\mathrm{Alg}_\tau(A, f_*) := \{(A, f_*, f_2, \ldots , f_n) : f_2, \ldots f_n \in F^1(A)\}.$$

As a subset of $\mathrm{Alg}_\tau(A)$, $\mathrm{Alg}_\tau(A, f_*)$ carries the induced topology. Since we keep A and f_* fixed, we identify $(A, f_*, f_2, \ldots , f_n)$ with $(f_2, \ldots , f_n)$.

Theorem 5.4. *Let $f_* : A \to A$ be 1-1 and without cycles. Then the set of algebras in $\mathrm{Alg}_\tau(A, f_*)$ for which the inclusion $\tilde{F}^1_{\aleph_0}(A) \subseteq P^1(\mathfrak{A})$ holds is residual in* $\mathrm{Alg}_\tau(A, f^*)$.
In other words: Almost all algebras have the $\aleph_0$-IP with respect to unary functions.

Proof. For any $\sigma \in \tilde{F}^1_{\aleph_0}(A)$ let

$$\mathcal{U}(\sigma) := \{(f_2, \ldots , f_n) : \sigma \notin \tilde{P}(A, f_*, f_2, \ldots , f_n)\}.$$

Note that $\mathcal{U}(\sigma)$ is closed and that there are only countably many possible values for σ. Hence it is enough to show that $\mathcal{U}(\sigma)$ does not contain an open set.

So fix $\nu_2, \ldots , \nu_n$. We will find $(f_2, \ldots , f_n) \in \mathcal{B}_{\nu_2,\ldots,\nu_n} \setminus \mathcal{U}(\sigma)$.

We will write $f^{(j)}$ for the j-th iteration of f: $f^{(1)} = f$, $f^{(j+1)} = f \circ f^{(j)}$. First find j such that $\forall a \in \mathrm{dom}(\sigma) : f^{(j)}_*(a) \notin \mathrm{dom}(\nu_2)$. Let $\eta_2 := \nu_2 \cup \{(f^{(j)}_*(a), \sigma(a)) : a \in \mathrm{dom}(\sigma)\}$. η_2 is still a function, since $f^{(j)}_*$ is 1-1. Clearly, for any $f_2 \supseteq \eta_2$ we have $f_2 \circ f^{(j)}_* \supseteq \sigma$, and $f_2 \circ f^{(j)}_*$ is a polynomial function. □

6. Appendix: Set-theoretical Background

We collect here a few notations, definitions and results from basic set theory that can be helpful to understand the paper. The reader is referred to the first chapters of [3] and [5] for more information. We assume that the reader is familiar with the concepts of cardinals and ordinals, as discussed in basic algebra books (e.g. [2, 0.4]).

6.1. **Ordinals, cardinals and cofinality.**

1. We let $\mathfrak{O}$ be the class of all ordinals, $\mathfrak{O}_{<\alpha} = \{\beta \in \mathfrak{O} : \beta < \alpha\}$. We reserve the variables α, β, γ, δ, ε to range over the class of ordinals (e.g., "$\forall\alpha\,\exists\beta : \ldots$" is an abbreviation for $\forall\alpha \in \mathfrak{O}\,\exists\beta \in \mathfrak{O} : \ldots$")
2. For every set A there is an ordinal α and a bijection from A onto $\mathfrak{O}_{<\alpha}$. We let $|A|$, the cardinality of A, be the least α with this property. An ordinal κ is called a cardinal if $\kappa = |A|$ for some A. All infinite cardinals are limit ordinals. We will reserve the variables κ, λ, μ for infinite cardinals.
3. If κ is a cardinal, then we let κ^+ be the smallest cardinal $> \kappa$ (the "cardinal successor") of κ:
$$\kappa^+ = \min\{\lambda : \lambda \text{ is a cardinal}, \lambda > \kappa\}$$
4. λ is called a "successor cardinal" if $\lambda = \kappa^+$, for some (uniquely determined) cardinal κ, and "limit cardinal" otherwise.
5. All the finite natural numbers are cardinals. $\aleph_0$ is the smallest infinite cardinal. We let $\aleph_1 = (\aleph_0)^+$, $\aleph_2 = (\aleph_1)^+$, etc. We may write ω_1 for $\aleph_1$. We write ω for the set of natural numbers.
6. For any partial order $\mathfrak{A} = (A, \leq)$ we call a set $B \subseteq A$ "cofinal" if $\forall a \in A\,\exists b \in B : a \leq b$.
7. For any (wlog limit) ordinal α we let $cf(\alpha)$ be the cofinality of $\mathfrak{O}_{<\alpha}$:
$$cf(\alpha) = \min\{|B| : B \subseteq \mathfrak{O}_{<\alpha} \text{ is cofinal}\}$$
Equivalently, $cf(\alpha)$ is the smallest cardinal κ such that there is a ((strictly) monotone) function $f : (\mathfrak{O}_{<\kappa}, \leq) \to (\mathfrak{O}_{<\alpha}, \leq)$ whose range is cofinal in $\mathfrak{O}_{<\alpha}$.
8. We call an infinite cardinal κ "regular" if $cf(\kappa) = \kappa$, and "singular" otherwise. Equivalently, κ is singular iff there is $\lambda < \kappa$ and a family $(A_\alpha : \alpha < \lambda)$ such that $|A_\alpha| < \kappa$ for all α, and $|\bigcup_\alpha A_\alpha| = \kappa$.
9. $\aleph_0$ and all successor cardinals (such as $\aleph_1$) are regular.

6.2. **Cardinal Arithmetic.**

1. Recall that for any sets A, B we defined $F^1(A, B)$ to be the set of all functions from A into B.
2. For (finite or infinite) cardinals κ, λ, we let κ^λ be the cardinality of the set $F^1(\mathfrak{O}_{<\lambda}, \mathfrak{O}_{<\kappa})$. Equivalently, choose any sets K, L of cardinalities κ and λ, respectively, and let $\kappa^\lambda = |F^1(L, K)|$.
3. We let $\kappa^{<\lambda} = \sup\{\kappa^\mu : \mu < \lambda\}$ (In particular, $\kappa^{<\lambda^+} = \kappa^\lambda$.)
4. Cantor's theorem tells us $2^\lambda > \lambda$, and König's theorem says that even $\lambda^{cf(\lambda)} > \lambda$ (for infinite λ). In particular, $\lambda^{<\lambda} > \lambda$ if λ is singular.
5. If n is a finite cardinal, κ an infinite cardinal, then $\kappa^n = \kappa$, so $\kappa^{<\omega} = \kappa$.
6. Let $\mathfrak{P}(X)$ be the power set of X. It is easy to see that $|\mathfrak{P}(X)| = 2^{|X|}$. If κ is infinite, we have moreover $2^\kappa = \kappa^\kappa$ and even $2^\kappa = (\kappa^+)^\kappa$ (see also 6.3.4).
7. If $|L| = \lambda$ and $|M| = \mu$ are infinite, t a natural number > 0, then
$$|F(L, M)| = |F^t(L, M)| = |F^1(L, M)| = \mu^\lambda$$

In particular, $|F(L)| = \lambda^\lambda = 2^\lambda$. For $\kappa \le \lambda^+$ we have $|\tilde{F}_\kappa(L,M)| = \lambda^{<\kappa} \cdot \mu^{<\kappa}$, in particular $|\tilde{F}_\kappa(L)| = \lambda^{<\kappa}$.

8. κ is a limit cardinal iff $\lambda^+ < \kappa$ is true for any $\lambda < \kappa$. We call a cardinal κ a "strong limit cardinal" iff $2^\lambda < \kappa$ is true for any $\lambda < \kappa$, or equivalently, iff $2^{<\kappa} = \kappa$.
9. We call κ "strongly inaccessible" iff κ is an uncountable regular strong limit cardinal.

6.3. **Increasing unions.**

1. We say that a sequence $(A_\alpha : \alpha < \delta)$ of sets is increasing iff $\alpha < \beta < \delta$ implies $A_\alpha \subseteq A_\beta$. We call this sequence "continuous" iff $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ whenever β is a limit ordinal $< \delta$.
2. If $(A_\alpha : \alpha < \kappa)$ is increasing, $A = \bigcup_{\alpha<\kappa} A_\alpha$, and $B \subseteq A$ has cardinality $< cf(\kappa)$, then there is an $\alpha < \kappa$ such that $B \subseteq A_\alpha$. (Since this fact is essential for our main theorem, we give the [easy] proof in the main part of the paper. See 2.9.)
3. If $cf(\kappa) > \lambda$, then $\kappa^\lambda = \sup\{\mu^\lambda : \mu < \kappa\}$.
Proof: Let $|K| = \kappa$, then we can write K as an increasing union of κ many "small" sets: $K = \bigcup_{i<\kappa} K_i$, $|K_i| < \kappa$. Let $|L| = \lambda$. By 6.3.2, the range of every function $f : L \to K$ is contained in some K_i, so $F^1(L,K) = \bigcup_{i<\kappa} F(L,K_i)$. Hence $\kappa^\lambda = |F^1(L,K)| = \sup\{\kappa_i^\lambda : i < \kappa\} = \sup\{\mu^\lambda : \mu < \kappa\}$.
4. Consequently, $(\kappa^+)^\kappa = \kappa^\kappa$ for all κ, as κ^+ is always regular.
5. If κ is a regular strong limit cardinal, then $\kappa^{<\kappa} = \kappa$.
Proof: For all $\mu < \kappa$ and all $\lambda < \kappa$ we have $\mu^\lambda < \kappa$, as κ is a strong limit. Hence for every $\lambda < \kappa$ we have $\kappa^\lambda = \sup\{\mu^\lambda : \mu < \kappa\} \le \kappa$, so $\kappa^{<\kappa} \le \kappa$.

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