Two Models of $\neg\exists \mathrm{tcl}(x)$

Martin Goldstern

September 1998

Let $\mathrm{tcl}(x) = \{x\} \cup x \cup \bigcup x \cup \cdots$ be the transitive closure of x. The usual proof that $\mathrm{tcl}(x)$ exists invokes the replacement axiom.

To show that some form of replacement is necessary, we give a model of ZC (i.e., ZFC except for the replacement axiom), in which there is a set which has no transitive closure.

(Note that the "usual" model of ZC, the sets of rank $< \alpha$, satisfy $\forall x\, \exists tcl(x)$.)

First model: Let $\kappa = \sup\{\kappa_n : n \in \omega\}$, where $\kappa_0 \geq \aleph_0$ and $2^{\kappa_n} \leq \kappa_{n+1}$ for all n. Define $H(\kappa)$ as

$$H(\kappa) := \{x : \ \forall n \geq 0 \ \left|\bigcup^n x\right| < \kappa\}$$

where $\bigcup^0 x = x$, $\bigcup^{n+1} x = \bigcup(\bigcup^n x)$.

It is easy to see that $H(\kappa)$ satisfies ZC. Let $a := \{\kappa_0, \{\kappa_1\}, \{\{\kappa_2\}\}, \{\{\{\kappa_3\}\}\}$. Then $a \in H(\kappa)$, but $H(\kappa) \models$ "there is no transitive set containing a."

Second model: [suggested by Bernhard Bodenstorfer] For convenience we will work in the theory $ZFC \setminus \{\text{Foundation}\} + \exists A = \{a_n : n \in \mathbb{Z}\} \forall n : a_{n+1} = \{a_n\}$, all a_n distinct. (The argument can be translated into ZFC.)

Let $A = \{a_n : n \in \mathbb{Z}\}$ be as above. For every ordinal α we define sets R_α (the "pure" sets) and M_α:

- $R_0 = \emptyset$, $M_0 = A$.
- $R_{\alpha+1} = \mathfrak{P}(R_\alpha) = \{x : x \subseteq M_\alpha, x \cap (M_\alpha \setminus R_\alpha) = \emptyset\}$
- $M_{\alpha+1} = \{x : x \subseteq M_\alpha, x \cap (M_\alpha \setminus R_\alpha) \text{ is finite}\}$
- $M_\delta = \bigcup_{\alpha<\delta} M_\alpha$, $R_\delta = \bigcup_{\alpha<\delta} R_\alpha$ for δ a limit ordinal

Check that for any limit ordinal $\delta > \omega$ we have $M_\delta \models ZC$, and $M_\delta \models$ "a_0 has no transitive closure".

For example, to check the separation axiom, let $B \in M_{\alpha+1}$. We show that every $C \subseteq B$ is in $M_{\alpha+1}$:

Let $B_0 = B \cap R_\alpha$, $B_1 = B \setminus B_0$. So B_1 is finite. Let $C_0 = C \cap B_0$, $C_1 = C \cap B_1$. Now $C \subseteq R_\alpha$ and C_1 is a finite subset of $M_\alpha \setminus R_\alpha$, so $C \in M_{\alpha+1}$.

Each model M_α will satisfy the axiom of foundation. This can be proved by induction. (Note that $A \notin M_\alpha$, for all α)