INTERPOLATION IN ORTHOLATTICES

MARTIN GOLDSTERN

Abstract. If $(L, \vee, \wedge, 0, 1, {}^\perp)$ is a complete ortholattice, $f : L^n \to L$ *any* partial function, then there is a complete ortholattice L^* containing L as a subortholattice, and an ortholattice polynomial p with coefficients in L^* such that $p(a_1, \dots, a_n) = f(a_1, \dots, a_n)$ for all $a_1, \dots, a_n \in L$.

Iterating this construction long enough yields a complete ortholattice in which every function can be interpolated by a polynomial on any set of small enough cardinality.

0. Introduction

In [2] and [3] we showed the following: Let L be any [bounded] lattice, then there is a lattice $\bar{L}$ extending L [with the same least and greatest element] such that every *monotone* function from L^n to L is represented by a polynomial with coefficients in $\bar{L}$.

It is clear that as long as we restrict ourselves to lattice polynomials we can only interpolate monotone functions. Here we consider the problem of interpolation on *ortholattices*, i.e., bounded lattices equipped with an "orthocomplement". Since the orthocomplement reverses order, there is no obvious monotonicity property that all orthopolynomials in an ortholattice will share.

The main theorem of this paper shows that indeed there are no restrictions on the behavior of orthopolynomials; more precisely: If L is an ortholattice, then *any* function $f : L^n \to L$ can be represented by a polynomial with coefficients in some suitable orthoextension $\bar{L}$.

By iterating the construction from the theorem we get, for every cardinal number κ, a lattice $\hat{L}$ with the property that every function from $\hat{L}^n$ to $\hat{L}$ can be interpolated on any set of size $\leq \kappa$.

We also show that we can construct $\hat{L}$ such that $\hat{L}$ will be complete (as a partial order). Moreover, assuming that the original ortholattice L is complete, we construct $\hat{L}$ such that L is a "convex" sublattice of $\hat{L}$.

1. Basic definitions

1.1. Notation. Lattices are denoted by L, L', L_1, etc. When we consider several lattices, we use the self-explanatory notation $\wedge_{L_1}$ or $\wedge_1$, $\leq_2$, etc. for the operations/relation in L_1, L_2, etc. We agree that the symbol $\wedge$ binds more tightly than $\vee$, i.e., $a \vee b \wedge c = a \vee (b \wedge c)$.

For any lattice L we let L^{dual} be the dual lattice (with the same underlying set): $x \leq^{L^{dual}} y$ iff $x \geq^L y$.

An ortholattice is a bounded lattice $(L, \vee, \wedge, 0, 1)$ with an additional unary operation $x \mapsto x^\perp$ which satisfies $x \leq y \Rightarrow x^\perp \geq y^\perp$, $(x^\perp)^\perp = x$, $x \vee x^\perp = 1$, $x \wedge x^\perp = 0$ for all x, y (and hence also the de Morgan laws $(x \vee y)^\perp = x^\perp \wedge y^\perp$, etc.).

1.2. Abuse of Notation. If $(L, \vee, \wedge, 0, 1)$ is a bounded lattice, $A \subseteq L$, then we say that A is "convex" in L iff

whenever $a, a' \in A$, $x \in L$, $0 < a \leq x \leq a' < 1$, then also $x \in A$

i.e., if $A \setminus \{0, 1\}$ is convex in L in the usual sense.

If $L_0 \subseteq L_1$, we say that L_0 is "downward closed" in L_1 iff: For all $z \in L_0 \setminus \{1\}$, for all $x \in L_1$, if $x \leq z$ then $x \in L_1$, i.e., if $L_0 \setminus \{1\}$ is downward closed in L_1 in the traditional sense.

1.3. Definition. Let L_0, L_1 be bounded lattices.

Date: August 14, 2000.

This research was supported by the Austrian Science Foundation (FWF), grant P13325-MAT.

This paper is available from www.arXiv.org, and also from my home page.

1. $L_0 \leq L_1$ means that L_0 is a $\{0,1\}$-sublattice of L_1 (i.e., L_0 is a sublattice of L_1 with the same maximal and minimal element)
2. $L_0 \trianglelefteq L_1$ means that
   (a) $L_0 \leq L_1$
   (b) For every $x \in L_1$ the set $\{z \in L_0 : z \leq_{L_1} x\}$ has greatest element (the "projection" of x to L_0, written $\pi^{L_1}_{L_0}(x)$ or $\pi^1_0(x)$)
   (c) L_0 is downward closed in L_1
3. $L_0 \trianglelefteq^{dual} L_1$ is the dual notion, i.e., $L_0^{dual} \trianglelefteq L_1^{dual}$.

2. Basic facts

In this section we collect a few easy facts for later reference. We also quote a theorem on interpolation of monotone functions in lattices.

2.1. Fact. Assume $L_0 \trianglelefteq L_1$, $L_0 \trianglelefteq^{dual} L_2$, $L_1 \cap L_2 = L_0$. Let $L = L_1 \cup L_2$, and let $\leq_L$ be the transitive closure of $(\leq_1) \cup (\leq_2)$. Then

- $x \leq_L y$ iff: $x \leq_1 y$ or $x \leq_2 y$ or there exists a $z \in L_0$ with $x \leq_2 z \leq_1 y$.
- $(L, \leq_L)$ is a lattice with

$$x \wedge_L y \;=\; \begin{cases} x \wedge_1 y & \text{if } x, y \in L_1 \\ x \wedge_2 y & \text{if } x, y \in L_2 \\ \pi^1_0(x) \wedge_2 y & \text{if } x \in L_1,\, y \in L_2 \end{cases}$$

  and similarly for $\vee_L$.

2.2. Fact. If $L_1 \trianglelefteq L_2 \trianglelefteq L_3$, then $L_1 \trianglelefteq L_3$.

2.3. Fact. L_1 and L_2 are complete lattices, $L_1 \leq L_2$, and $A \subseteq L_1$, then $\sup_{L_1} A \geq \sup_{L_2} A$.
However, if L_1 is convex in L_2 and $\sup_{L_1} A < 1$, then $\sup_{L_1} A = \sup_{L_2} A$.

2.4. Fact. Let $(I, \leq)$ be a linearly ordered set, and assume that $(L_i : i \in I)$ is a family of complete lattices such that:

- for all $i < j$, then L_i is a $\{0,1\}$-sublattice of L_j
- for all $i < j$: L_i is convex in L_j.

Then $L := \bigcup_i L_i$ is a complete lattice, and L_i is a convex $\{0,1\}$-sublattice of L for all $i \in I$.

Proof. It is clear that L is a lattice and that L_i is convex in L.

We now check that L is complete. Let $A \subseteq L$. We will show that $\sup_L A$ exists. Wlog we may assume that $A \neq \emptyset$, $1 \notin A$, $0 \notin A$. Let $a_0 \in A$.

We may also assume that 1 is not the least upper bound of A, so let $c < 1$ be some upper bound.

Fix i_0 such that $a_0, c \in L_{i_0}$.

Let $A' := \{a_0 \vee x : x \in A\}$. Since $a_0 \in A$, it is clear that A and A' have the same upper bounds. Now L_{i_0} is convex, and $a_0 \leq a_0 \vee x \leq c$ for all $x \in A$, hence $A' \subseteq L_{i_0}$.

Since L_{i_0} is complete, A' has a supremum c^* in L_{i_0}, and $c^* \leq c$. If d is any upper bound (in L) for A', then also $d \wedge c^*$ is an upper bound, and $d \wedge c^* \in L_{i_0}$ by convexity, so $d \wedge c^* = c^*$. Hence c^* is the supremum of A' not only in L_{i_0} but also in L.

$\square$

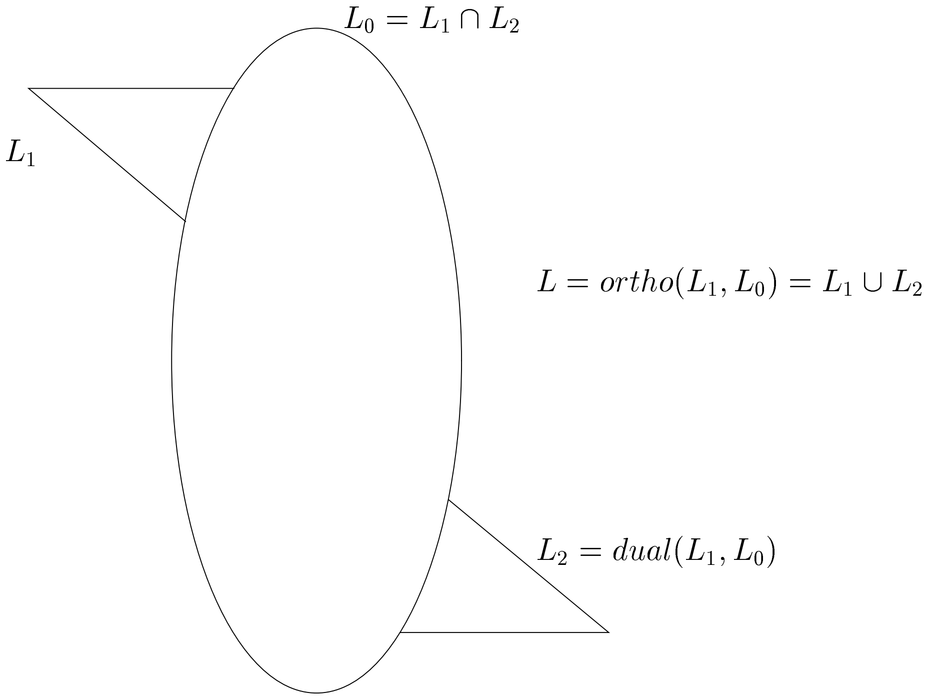
2.5. Fact. Let $L = L_1 \cup L_2$, where $L_1 \leq L$, $L_2 \leq L$. If L_1 and L_2 are both complete lattices, then also L is a complete lattice.

2.6. Theorem. *Let L be a complete lattice, $f : L \to L$ a partial monotone function.*

Then there is a complete lattice $\bar{L}$ and a polynomial $p(\mathrm{x}) \in \bar{L}[\mathrm{x}]$ with $p(a) = f(a)$ for all $a \in \mathrm{dom}(f)$. Moreover, we can choose $\bar{L}$ such that

- $L \trianglelefteq \bar{L}$.
- *$\bar{L}$ is complete.*

Proof. See [3] and [2]. (The "moreover" part is not stated there, but following the proof it is easy to see that the lattice $\bar{L}$ constructed in [3] satisfies $L \trianglelefteq \bar{L}$ and will be complete.) $\square$



3. From lattices to ortholattices

We describe a construction that allows us to extend a lattice L to an ortholattice, preserving the ortholattice structure of a given sublattice of L.

3.1. Construction. Let $(L_0, \vee, \wedge, 0, 1, ^\perp)$ be an ortholattice, $(L_1, \vee, \wedge, 0, 1)$ a bounded lattice with $L_0 \trianglelefteq L_1$.

We define two partial orders $L_2 = dual(L_1, L_0) \supseteq L_0$ and $L = ortho(L_1, L_0) = L_1 \cup L_2$ as follows:

Pick a set L_2 and a map ι satisfying the following:

- $\iota : L_1 \to L_2$ is a bijection.
- $L_1 \cap L_2 = L_0$.
- $\iota(z) = z^{\perp_0}$ for all $z \in L_0$.

(I.e., $L_2 \setminus L_0$ is just a disjoint copy of $L_1 \setminus L_0$.) We make L_2 into a lattice by requiring $\iota(x) \leq_2 \iota(y)$ iff $y \leq_1 x$, so that $\iota : L_1 \to L_2$ is a dual isomorphism.

Note that $\leq_0$ coincides with the restriction of $\leq_2$ to L_0, so $L_0 \leq L_2$. From $L_0 \trianglelefteq L_1$ we conclude $L_0 \trianglelefteq^{dual} L_2$.

We let $L = ortho(L_1, L_0)$ be the (set-theoretic) union of $L_1 \cup L_2$. By fact 2.1 we see that we can make L into a lattice containing L_1 and L_2 as sublattices.

3.2. Lemma. *If $L_0 \trianglelefteq L_1$, $(L_0, \vee, \wedge, 0, 1, ^\perp)$ is an ortholattice, then there is an ortholattice L such that $L_0 \leq L_1 \leq L$, and L_0 is a subortholattice of L.*

Moreover, if L_1 is complete, then also L will be complete.

Proof. We let L_2, ι, and $L = ortho(L_1, L_0)$ as above.

Let $x^{\perp_L} = \iota(x)$ for $x \in L_1$ and $= \iota^{-1}(x)$ for $x \in L_2$.

It remains to show that the map $x \mapsto x^{\perp_L}$ is an orthocomplement for L. Clearly this map is well defined and an involution, and it agrees with the map $x \mapsto x^{\perp_0}$ on L_0. Also, we have $z \vee_L z^{\perp_L} = z \vee_0 z^{\perp_0} = 1$ for all $z \in L_0$.

Now let $x \notin L_0$, wlog $x \in L_1$. We will only check $x \vee_L x^{\perp_L} = 1$, leaving the dual to the reader. So let $y \geq_L x$, $y \geq_L x^{\perp_L} = \iota(x)$. Then y must be in L_1, and there is a $z \in L_0$ such that $\iota(x) \leq_2 z \leq_1 y$. Now $\iota(x) \leq_2 z = \iota(z^{\perp_0})$ implies $z^{\perp_0} \leq_1 x$, hence $y \geq_1 z \vee z^{\perp_0} = 1$.

□

3.3. Fact. The operation $ortho(\cdot, L_0)$ commutes with direct limits. In particular, if $(I, \leq)$ is a linear order, $(L_i : i \in I)$ an increasing family of lattices, $L \trianglelefteq L_i$ for all i, then

$$ortho(\bigcup_{i\in I} L_i, L_0) = \bigcup_{i\in I} ortho(L_i, L_0)$$

4. Theorems

We prove the two main theorems mentioned in the introduction. We conclude with an open question concerning the difference of unary and n-ary functions.

4.1. Theorem. *If L_0 is an ortholattice, $f : L_0 \to L_0$, then there is an ortholattice L^* extending L_0 such that f is the restriction of a polynomial function over L^*.*

Moreover, if L_0 is complete then we can have $L^* = ortho(L_1, L_0)$ for some complete L_1, $L_0 \trianglelefteq L_1$.

Proof. Since every ortholattice can be embedded into a complete ortholattice (the MacNeille completion; see, e.g., [1, 4.1], [5]) we may assume that L_0 is complete. Let $f : L_0 \to L_0$.

Let L_0' be horizontal sum of L_0 and $L_0 \times L_0$, i.e., assume that L_0 and $L_0 \times L_0$ have the same least and greatest elements (but are otherwise disjoint), and make $L_0' = L_0 \cup (L_0 \times L_0)$ into a lattice by taking

$$\leq_{L_0'} = (\leq_{L_0}) \cup (\leq_{L_0 \times L_0}).$$

Note that L_0' is a complete $\{0, 1\}$-lattice and $L_0 \trianglelefteq L_0'$.

Now consider the partial functions $\bar{f}$, g_1 and g_2, defined by

- $\bar{f}(\langle x, x^\perp\rangle) = f(x)$ for all $x \in L_0$.
- $g_1(x) = \langle x, 0\rangle$, $g_2(x) = \langle 0, x\rangle$, for all $x \in L_0$.

The following easy observation is the crucial step in this proof:

> The elements of the set $\{\langle x, x'\rangle : x \in L_0\}$ are pairwise incomparable, so the function $\bar{f}$ is trivially monotone.

By theorem 2.6 we can find a lattice L_1, $L_0' \trianglelefteq L_1$ in which the functions $\bar{f}$, g_1 and g_2 are restrictions of polynomials p, q_1 and q_2, respectively. Now let $L = ortho(L_1, L_0)$, so L is an orthoextension of L_0.

Now $h(\mathtt{x}) = p(q_1(\mathtt{x}) \vee q_2(\mathtt{x}^\perp))$ is an orthopolynomial with coefficients in L, and clearly $h(x) = p(\langle x, 0\rangle \vee \langle 0, x^\perp\rangle) = p(\langle x, x^\perp\rangle) = \bar{f}(\langle x, x^\perp\rangle) = f(x)$ for all $x \in L_0$.

□

4.2. *Remark.* For every orthopolynomial $p(\mathtt{x})$ there is a lattice polynomial $p'(\mathtt{x}, \mathtt{y})$ such that (by de Morgan's laws) $p(\mathtt{x})$ is equivalent to $p'(\mathtt{x}, \mathtt{x}^\perp)$.

4.3. Definition. Let $\mathscr{F}$ a family of lattices. We say that $\mathscr{F}$ is square closed if:

> For every $S \in \mathscr{F}$ there is some $S' \in \mathscr{F}$ which is isomorphic to $S \times S$.

We say that L is κ-square closed if the family of sublattices of size $\leq \kappa$ is square closed.

4.4. Fact. If $\mathscr{F}$ is square closed and closed under the formation of sublattices, then: For every $S \in \mathscr{F}$ and every $n > 1$ there is some $S' \in \mathscr{F}$ which is isomorphic to S^n.

4.5. Theorem. *Let L be a complete ortholattice, and let κ be any cardinal. Then:*

1. *There is a complete ortholattice $\bar{L}$ extending L such that every function $f : L \to L$ is represented by an orthopolynomial of $\bar{L}$.*
   Moreover, $\bar{L}$ can be chosen to be of the form $ortho(L_1, L)$ with $L \trianglelefteq L_1$. In particular, L will be convex in $\bar{L}$.
   Moreover, $\bar{L}$ can be chosen to be κ-square closed.
2. *There is a complete ortholattice $\hat{L}$ extending L such that:*

For every natural number n, for every function $f : \hat{L}^n \to \hat{L}$ and for every set $A \subseteq \hat{L}^n$ of cardinality $\le \kappa$ there is an orthopolynomial $p(\mathtt{x}_1, \dots, \mathtt{x}_n)$ with coefficients in $\hat{L}$ that interpolates f on every point in A.

Proof. Choose a cardinal λ of cofinality $> \kappa$ such that there is a transfinite enumeration (not necessarily 1-1) $\{f_i : 0 \le i < \lambda\}$ of all functions $f : L \to L$. Define an increasing transfinite sequence $(L_i : i \le \lambda)$ of $\{0,1\}$-lattices satisfying

1. $L_0 = L$.
2. If $i < j \le \lambda$, then $L_i \trianglelefteq L_j$
3. For every $i < \lambda$ there is a lattice polynomial p_i with coefficients in L_{i+1} such that for all $z \in L$: $f_i(z) = p_i(z, z^{\perp_L})$.
4. For every $i < \lambda$, L_{i+1} contains an isomorphic copy of L_i^2.
5. If i is a limit stage, then L_i is the direct limit of $(L_j : j < i)$, (i.e., $L_i = \bigcup_{j<i} L_j$).

Finally, let $\bar{L} = \mathit{ortho}(L_\lambda, L)$. Note that $\bar{L}$ will contain an isomorphic copy of $\mathit{ortho}(L_i, L)$ for every i. This finishes the proof of the first statement.

To prove the second claim, apply the conclusion from the first claim κ^+ many times to get the conclusion for all unary functions, and then use fact 4.6 to take care of all n-ary functions. $\square$

4.6. Fact. Fix a cardinal number κ. We call a set A "small" if the cardinality of A is $< \kappa$.

Assume that L κ-square closed, and that (a) or (b) holds:

(a) L is a lattice, and for every small $A \subseteq L$ and for every *monotone* $f : A \to L$ there is a lattice polynomial $p \in L[\mathtt{x}]$ such that $f(a) = p(a)$ for all $a \in A$

(b) L is an ortholattice, and for every small $A \subseteq L$ and for *every* $f : A \to L$ there is an orthopolynomial $p \in L[\mathtt{x}]$ such that $f(a) = p(a)$ for all $a \in A$

Then also (a') or (b'), respectively, holds:

(a') For every n, for every small $A \subseteq L^n$, for every monotone $g : A \to L$ there is a lattice polynomial $p \in L[\mathtt{x}_1, \dots, \mathtt{x}_n]$ such that

$$g(a_1, \dots a_n) = p(a_1, \dots, a_n) \text{ for all } (a_1, \dots, a_n) \in A.$$

(b') Like (a'), but again replace "every monotone" by "every", and "lattice polynomial" by "orthopolynomial".

Proof. Let $g : L^n \to L$, $A \subseteq L$ small. Let S be the sub(ortho)lattice generated by A. Since L is κ-square closed, we can find a sublattice $S' \le L$ and an isomorphism $\iota : S^n \to S'$. For $\ell = 1, \dots, n$ let $\iota_\ell : S \to S'$ be defined by

$$\iota_\ell(s) = \iota(0, \dots, 0, s, 0, \dots, 0) \qquad \text{[here, } s \text{ appears in the } \ell\text{-th coordinate]}$$

Note that

$$(*) \qquad \iota(s_1, \dots, s_n) = \iota_1(s_1) \vee \dots \vee \iota_n(s_1)$$

Define a partial function $f : L \to L$ as follows:

$$(**) \qquad f(\iota(s_1, \dots, s_n)) = g(s_1, \dots, s_n)$$

and $f(t) =$ undefined if $t \notin S'$. Note that f is monotone if g is monotone.

Now note that g, ι_1, $\dots$, ι_n are unary functions from L to L, so they are all represented by polynomials, and so by $(*)$ and $(**)$ also g is represented by a polynomial. $\square$

Note that the construction we used in theorem 4.1 will automatically guarantee that the resulting structure will be κ-square closed (see definition 4.3), so by fact 4.6 there will be no difference between unary and n-ary interpolation. This motivates the following questions:

4.7. *Question.* 1. Are there infinite ortholattices L where every function $f : L \to L$ can be interpolated by an orthopolynomial on every (say) countable set, but not every function $f : L^2 \to L$?

2. Are there infinite lattices L where every monotone function $f : L \to L$ can be interpolated by a lattice polynomial on every (say) countable set, but not every monotone function $f : L^2 \to L$?

References

[1] Günter Bruns. Free ortholattices. *Can. J. Math.*, 28:977–985, 1976.
[2] Martin Goldstern. Interpolation of Monotone Functions in Lattices. *Algebra Universalis*, 36:108–121, 1996.
[3] Martin Goldstern. Interpolation of monotone functions in $\{0,1\}$-lattices. In *Contributions to General Algebra 10*. Heyn Verlag, 1998.
[4] George Grätzer. *General lattice theory. 2nd ed.* Birkhaeuser, Basel, 1998.
[5] Gudrun Kalmbach. *Orthomodular lattices*, volume 18 of *London Mathematical Society Monographs*. Academic Press Inc., London etc., 1983.

Technische Universität Wien, Wiedner Hauptstrasse 8–10/118.2, A-1040 Wien
E-mail address: `Martin.Goldstern@tuwien.ac.at`
URL: `http://info.tuwien.ac.at/goldstern/`