ANTICHAINS IN PRODUCTS OF LINEAR ORDERS

MARTIN GOLDSTERN AND SAHARON SHELAH

Abstract. We show that:

1. For many regular cardinals λ (in particular, for all successors of singular strong limit cardinals, and for all successors of singular ω-limits), for all $n \in \{2, 3, 4, \dots\}$: There is a linear order L such that L^n has no (incomparability-)antichain of cardinality λ, while L^{n+1} has an antichain of cardinality λ.
2. For any nondecreasing sequence $(\lambda_n : n \in \{2, 3, 4, \dots\})$ of infinite cardinals it is consistent that there is a linear order L such that, for all n: L^n has an antichain of cardinality λ_n, but no antichain of cardinality λ_n^+.

1. Introduction

Definition 1.1. For any partial ordering $(P, \leq)$ define $\mathrm{inc}(P)$ as

$$\mathrm{inc}(P) = \sup\{|A|^+ : A \subseteq P \text{ is an antichain}\}$$

Here, an antichain is a set of pairwise **incomparable** elements.

In other words, $\kappa < \mathrm{inc}(P)$ iff there is an antichain of cardinality κ.

Haviar and Ploscica in [2] asked: Can there be a linear ordering L such that $\mathrm{inc}(L^n) \neq \mathrm{inc}(L^k)$ for some natural numbers n, k? (Here, L^n and L^k carry the product, or pointwise, order.)

Farley [1] has shown that for any singular cardinal κ there is a linear order L of size κ such that $\mathrm{inc}(L^2) = \kappa$, $\mathrm{inc}(L^3) = \kappa^+$.

We will be mainly interested in this question for *regular* cardinals. First we show in ZFC that there are many successor cardinals λ (including $\aleph_{\omega+1}$) with the following property:

> For every $n \geq 2$ there is a linear order L of size λ such that $\mathrm{inc}(L^n) \leq \lambda$, $\mathrm{inc}(L^{n+1}) = \lambda^+$.

This proof is given in section 2. It uses a basic fact from pcf theory.

We then show that there are (consistently) many possible behaviours for the sequence $(\mathrm{inc}(L^n) : n = 2, 3, 4, \dots)$. More precisely, we show that for any nondecreasing sequence of infinite cardinals $(\lambda_n : 2 \leq n < \omega)$ there is a

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cardinal-preserving extension of the universe in which we can find a linear order L such that for all $n \in \{2, 3, \dots\}$: $\mathrm{inc}(L^n) = \lambda_n^+$.

For example, it is consistent that there is a linear order L such that L^2 has no uncountable antichain, while L^3 does.

Here we use forcing. The heart of this second proof is the well-known Δ-system lemma.

2. A ZFC proof

Let μ be a regular cardinal. We will write D^{bd}_μ for the filter of cobounded sets, i.e., the filter dual to the ideal $\{A \subseteq \mu : \exists i < \mu\ A \subseteq i\}$.

$\prod_{i<\mu} \lambda_i$ is the set of all functions f with domain μ satisfying $f(i) < \lambda_i$ for all i. The relation $f \sim_{D^{bd}_\mu} g \Leftrightarrow \{i : f(i) = g(i)\} \in D^{bd}_\mu$ is an equivalence relation. We call the quotient structure $\prod_i \lambda_i / D^{bd}_\mu$ (and we often do not distinguish between a function f and its equivalence class). $\prod_i \lambda_i / D^{bd}_\mu$ is partially ordered by the relation

$$f <_{D^{bd}_\mu} g \quad \text{iff} \quad \{i < \mu : f(i) < g(i)\} \in D^{bd}_\mu$$

For any partial order $(P, \leq)$ and any regular cardinal λ we say $\lambda = tcf(P)$ ("λ is the true cofinality of P") iff there is an increasing sequence $(p_i : i < \lambda)$ such that $\forall p \in P\, \exists i < \lambda : p \leq p_i$.

Theorem 2.1. *Assume that*

1. $(\lambda_i : i < \mu)$ *is an increasing sequence of regular cardinals*
2. *For each* $j < \lambda$, $\left|\prod_{i<j} \lambda_i\right| < \lambda_j$
3. λ *is regular and* $tcf(\prod \lambda_i / D^{bd}_\mu) = \lambda$,
4. $n \geq 2$.

Then there is a linear order J of size λ such that

- J^{n+1} *has an antichain of size* λ
- J^n *has no antichain of size* λ

To show that there are indeed many cardinals λ which can appear as the true cofinality of a reduced product as in the assumptions of the above theorem, we use theorem II.1.5 from [3, page 50]:

Theorem 2.2. *If κ is singular, $cf\kappa = \mu$, then for some strictly increasing sequence $\langle \lambda_i : i < \mu \rangle$ of regular cardinals, $\kappa = \sum_{i<\mu} \lambda_i$ and $\left(\prod_{i<\mu} \lambda_i, <_{D^{bd}_\mu}\right)$ has true cofinality κ^+.*

It is clear that in the above theorem we can replace $(\lambda_i : i < \mu)$ by any increasing cofinal subsequence $(\lambda_{f(j)} : j < \mu)$.

Conclusion 2.3. *Whenever $\lambda = \kappa^+$ is the successor of a singular cardinal κ such that*

1. *Either κ is a strong limit cardinal*
2. *or at least* $\forall \kappa' < \kappa : (\kappa')^{<cf(\kappa)} < \kappa$

then we can find a sequence $(\lambda_i : i < cf(\kappa))$ as in the assumption of theorem 2.1. For example, if $\lambda = \aleph_{\omega+1}$, then there is an increasing sequence $(n_k : k \in \omega)$ of natural numbers such that $tcf(\prod_{k\in\omega} \aleph_{n_k}/D^{bd}_\omega) = \aleph_{\omega+1}$.

Proof. Let $\mu = cf(\kappa)$. We can start with a sequence $(\lambda_i : i < \mu)$ such that $\prod_{i<\mu} \lambda_i/D^{bd}_\mu$ has true cofinality $\lambda = \kappa^+$. Choose $f : \mu \to \mu$ increasing cofinal such that $\prod \lambda_{j<i}\lambda_{f(j)} < \lambda_i$ for all i, then $(\lambda_{f(j)} : j < \mu)$ is as required. $\square$

The proof of theorem 2.1 will occupy the rest of this section. We fix a sequence $(f_\alpha : \alpha < \lambda)$ as in the assumption of the theorem. We start by writing $\mu = \bigcup_{\ell=0}^n A_\ell$ as a disjoint union of $n+1$ many D^{bd}_μ-positive (i.e., unbounded) sets. For $\ell = 0, \ldots, n$ we define a linear order $<_\ell$ on λ as follows:

Definition 2.4. For any two distinct functions $f, g \in \prod_i \lambda_i$ we define

$$d(f,g) = \sup\{i < \mu : f{\restriction}i = g{\restriction}i\} = \max\{i < \mu : f{\restriction}i = g{\restriction}i\} \tag{1}$$

That is, if $f \neq g$ we have that $d(f,g) = \min\{j : f(j) \neq g(j)\}$ is the first point where f and g diverge.

For $\alpha, \beta \in \lambda$ we define $\alpha <_\ell \beta$ iff:

$$\begin{aligned} &\text{letting } i_{\alpha,\beta} := d(f_\alpha, f_\beta),\\ &\qquad \text{either } i_{\alpha,\beta} \in A_\ell \text{ and } f_\alpha(i_{\alpha,\beta}) < f_\beta(i_{\alpha,\beta})\\ &\qquad \text{or } i_{\alpha,\beta} \notin A_\ell \text{ and } f_\alpha(i_{\alpha,\beta}) > f_\beta(i_{\alpha,\beta}) \end{aligned} \tag{2}$$

We leave it to the reader to check that $<_\ell$ is indeed a linear order on λ.

We now define J to be the "ordinal sum" of all the orders $<_\ell$:

Definition 2.5. Let

$$J = \bigcup_{\ell=0}^{n} \{\ell\} \times (\lambda, <_\ell)$$

with the "lexicographic" order, i.e., $\langle \ell_1, \alpha_1\rangle < \langle \ell_2, \alpha_2\rangle$ iff $\ell_1 < \ell_2$, or $\ell_1 = \ell_2$ and $\alpha_1 <_{\ell_1} \alpha_2$.

Claim 2.6. J^{n+1} has an antichain of size λ.

Proof. Let $\vec{t}_\alpha = ((0,\alpha), \ldots, (n,\alpha)) \in J^{n+1}$.

For any $\alpha \neq \beta$ we have to check that $\vec{t}_\alpha$ and $\vec{t}_\beta$ are incomparable. Let $i^* = d(f_\alpha, f_\beta)$, and find ℓ^* such that $i^* \in A_{\ell^*}$. Wlog assume $f_\alpha(i^*) < f_\beta(i^*)$. Then $\alpha <_{\ell^*} \beta$, but $\alpha >_\ell \beta$ for all $\ell \neq \ell^*$, i.e., $\langle \ell^*, \alpha\rangle <_J \langle \ell^*, \beta\rangle$, but $\langle \ell, \alpha\rangle >_J \langle \ell, \beta\rangle$ for all $\ell \neq \ell^*$.

$\square$

Proof of 2.1. It remains to show that J^n does not have an antichain of size λ. Towards a contradiction, assume that $(\vec{t}_\beta : \beta < \lambda)$ is an antichain in J^m, $m \leq n$, and m as small as possible. Let $\vec{t}_\beta = (t_\beta(1), \ldots, t_\beta(m)) \in J^m$. For $k = 1, \ldots, m$ we can find functions ℓ_k, ξ_k such that

$$\forall \beta < \lambda\ \forall k :\ t_\beta(k) = \langle \ell_k(\beta), \xi_k(\beta)\rangle$$

Thinning out we may assume that the functions $\ell_1, \dots, \ell_m$ are constant. We will again write $\ell_1, \dots, \ell_m$ for those constant values.

We may also assume that for each k the function $\beta \mapsto \xi_k(\beta)$ is either constant or strictly increasing. If any of the functions ξ_k is constant we get a contradiction to the minimality of m, so all the ξ_k are strictly increasing. So we may moreover assume that $\beta < \gamma$ implies $\xi_k(\beta) < \xi_{k'}(\gamma)$ for all k, k', and in particular $\beta \le \xi_k(\beta)$ for all β, k.

Now define $g^+_\beta, g^-_\beta \in \prod_{i<\mu} \lambda_i$ for every $\beta < \lambda$ as follows:

$$\begin{aligned} g^+_\beta(i) &= \max(f_{\xi_1(\beta)}(i), \dots, f_{\xi_n(\beta)}(i)) \\ g^-_\beta(i) &= \min(f_{\xi_1(\beta)}(i), \dots, f_{\xi_n(\beta)}(i)) \end{aligned} \tag{3}$$

Subclaim. The set

$$C := \{i < \mu : \ \forall \beta \ \{g^-_\gamma(i) : \gamma > \beta\} \text{ is unbounded in } \lambda_i\} \tag{4}$$

is in the filter D^{bd}_μ, i.e., its complement

$$S := \{i < \mu : \exists \beta < \lambda \, \exists s < \lambda_i \ \ \{g^-_\gamma(i) : \gamma > \beta\} \subseteq s\}$$

is in the ideal dual to D^{bd}_μ (i.e., is a bounded set).

Proof of the subclaim. For each $i \in S$ let $\beta_i < \lambda$ and $h(i) < \lambda_i$ be such that $\{g^-_\gamma(i) : \gamma > \beta_i\} \subseteq h(i)$. Let $\beta^* = \sup\{\beta_i : i \in S\} < \lambda$, and extend h arbitrarily to a total function on μ. Since the sequence $(f_\alpha : \alpha < \lambda)$ is cofinal in $\prod_i \lambda_i / D^{bd}_\mu$, we can find $\gamma > \beta^*$ such that $h <_{D^{bd}_\mu} f_\gamma$.

We have $\gamma \le \xi_k(\gamma)$ for all $k \in \{1, \dots, m\}$, so the sets

$$X_k := \{i < \mu : h(i) < f_{\xi_k(\gamma)}(i)\}$$

are all in D^{bd}_μ. Now if S were positive mod D^{bd}_μ, then we could find $i_* \in S \cap X_1 \cap \dots \cap X_m$. But then $i^* \in X_1 \cap \dots \cap X_m$ implies

$$h(i_*) < g^-_\gamma(i_*),$$

and $i \in S$ implies

$$g^-_\gamma(i_*) < h(i_*),$$

a contradiction.

This shows that C is indeed a set in the filter D^{bd}_μ. $\square$

We will now use the fact that $m < n + 1$. Let

$$\ell^* \in \{0, \dots, n\} \setminus \{\ell_1, \dots, \ell_m\}.$$

Since A_{ℓ^*} is positive mod D^{bd}_μ, we can pick

$$i^* \in A_{\ell^*} \cap C \tag{5}$$

Using the fact that $i^* \in C$ and definition (4) we can find a sequence $(\beta_\sigma : \sigma < \lambda_{i^*})$ such that

$$\forall \sigma < \sigma' < \lambda_{i^*} : \ g^+_{\beta_\sigma}(i^*) < g^-_{\beta_{\sigma'}}(i^*), \tag{6}$$

We now restrict our attention from $(\vec{t}_\beta : \beta < \lambda)$ to the subsequence $(\vec{t}_{\beta_\sigma} : \sigma < \lambda_{i^*})$; we will show that this sequence cannot be an antichain. For notational simplicity only we will assume $\beta_\sigma = \sigma$ for all $\sigma < \lambda_{i^*}$.

Recall that $\vec{t}_\sigma = (\langle \ell_1, \xi_1(\sigma)\rangle, \dots, \langle \ell_m, \xi_m(\sigma)\rangle)$. For each $\sigma < \lambda_{i^*}$ define $\vec{x}_\sigma := (f_{\xi_1(\sigma)} \restriction i^*, \dots, f_{\xi_n(\sigma)} \restriction i^*) \in \left(\prod_{j<i^*} \lambda_j\right)^m$. Since $\left|\prod_{j<i^*} \lambda_j\right| < \lambda_{i^*}$, there are only $< \lambda_{i^*}$ many possible values for $\vec{x}_\sigma$, so we can find $\sigma_1 < \sigma_2 < \lambda_{i^*}$ such that $\vec{x}_{\sigma_1} = \vec{x}_{\sigma_2}$.

Now note that by (3) and (6) we have

$$f_{\xi_k(\sigma_1)}(i^*) \le g^+_{\sigma_1}(i^*) < g^-_{\sigma_2}(i^*) \le f_{\xi_k(\sigma_2)}(i^*). \tag{7}$$

Hence $d(f_{\xi_k(\sigma_1)}, f_{\xi_k(\sigma_2)}) = i^*$ for $k = 1, \dots, m$.

Since $i^* \in A_{\ell^*}$ we have for all k: $i^* \notin A_{\ell_k}$. From (1), (2), (7) we get

$$\xi_k(\sigma_1) <_{\ell_k} \xi_k(\sigma_2) \qquad \text{for } k = 1, \dots, m.$$

Hence $\langle \ell_k, \xi_k(\sigma_1)\rangle < \langle \ell_k, \xi_k(\sigma_2)\rangle$ for all k, which means $\vec{t}_{\sigma_1} < \vec{t}_{\sigma_2}$. $\square$

3. Consistency

Theorem 3.1. *Assume $\aleph_0 \le \lambda_2 \le \lambda_3 \le \cdots$, $\lambda_n^{<\kappa} = \lambda_n$. $\kappa^{<\kappa} = \kappa$. Then there is a forcing notion $\mathbb{P}$ which satisfies the κ-cc and is κ-complete, and a $\mathbb{P}$-name $\underset{\sim}{I}$ such that*

$$\Vdash_P \underset{\sim}{I} \subseteq 2^\kappa, \mathrm{inc}(\underset{\sim}{I}^n) = \lambda_n^+$$

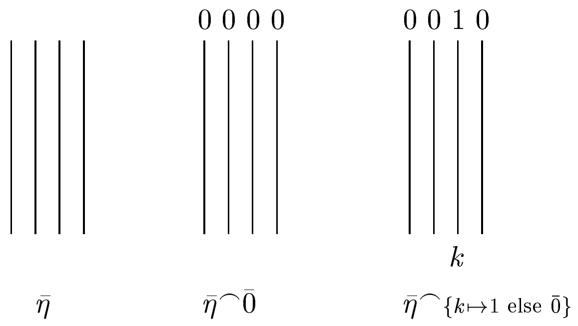
Remark 3.2. At first reading, the reader may want to consider the special case $\kappa = \aleph_0$, $\lambda_{n+2} = \aleph_n$.

Notation 3.3. 1. We let $\lambda_1 = 0$, $\lambda_\omega = \sup\{\lambda_n : n < \omega\}$.

2. It is understood that 2^α is linearly ordered lexicographically, and $(2^\alpha)^m$ is partially ordered by the pointwise order.
3. For $\alpha \le \beta \le \kappa$, $\eta \in 2^\alpha$, $\nu \in 2^\beta$, we define
$$\eta \trianglelefteq \nu \quad \text{iff} \quad \nu \text{ extends } \eta, \text{ i.e., } \eta \subseteq \nu$$
4. For $\bar\eta \in (2^\alpha)^n$, $\bar\eta = (\eta(0), \dots, \eta(n-1))$, $\bar\nu \in (2^\beta)^n$, $\nu = (\nu(0), \dots, \nu(n-1))$, we let
$$\bar\eta \trianglelefteq \bar\nu \ \text{ iff } \ \eta(0) \trianglelefteq \nu(0), \dots, \eta(n-1) \trianglelefteq \nu(n-1).$$
5. For $\eta \in 2^\alpha$, $i \in \{0,1\}$ we write $\eta{}^\frown i$ for the element $\nu \in 2^{\alpha+1}$ satisfying $\eta \trianglelefteq \nu$, $\nu(\alpha) = i$.

Definition 3.4. Let $\bar\eta \in (2^\alpha)^m$, $k \in \{0, \dots, m-1\}$, $m \ge 2$. We define $\bar\eta{}^\frown\bar 1$, $\bar\eta{}^\frown\bar 0$, $\bar\eta{}^\frown\{k\mapsto 1 \text{ else } \bar 0\}$, $\bar\eta{}^\frown\{k\mapsto 0 \text{ else } \bar 1\}$ in $(2^{\alpha+1})^m$ as follows: All four are $\trianglelefteq$-extensions of $\bar\eta$, and:

- $\bar\eta{}^\frown\bar 0(n) = \eta(n){}^\frown 0$ for all $n < m$.
- $\bar\eta{}^\frown\bar 1(n) = \eta(n){}^\frown 1$ for all $n < m$.
- $\bar\eta{}^\frown\{k\mapsto 0 \text{ else } \bar 1\}(n) = \eta(n){}^\frown 1$ for all $n \ne k$, $\bar\eta{}^\frown\{k\mapsto 0 \text{ else } \bar 1\}(k) = \eta(n){}^\frown 0$.
- $\bar\eta{}^\frown\{k\mapsto 1 \text{ else } \bar 0\}(n) = \eta(n){}^\frown 0$ for all $n \ne k$, $\bar\eta{}^\frown\{k\mapsto 1 \text{ else } \bar 0\}(k) = \eta(n){}^\frown 1$.



Fact 3.5. 1. If $\alpha \le \beta \le \kappa$, $\bar\eta, \bar\eta' \in (2^\alpha)^n$ are incomparable, $\bar\nu, \bar\nu' \in (2^\beta)^n$, $\bar\eta \trianglelefteq \bar\nu$, $\bar\eta' \trianglelefteq \bar\nu'$, then also $\bar\nu$ and $\bar\nu'$ are incomparable.

2. $\bar\eta{}^\frown\bar 0 < \bar\eta{}^\frown\bar 1$.
3. $\bar\eta{}^\frown\{k\mapsto 0 \text{ else } \bar 1\}$ and $\bar\eta{}^\frown\{k\mapsto 1 \text{ else } \bar 0\}$ are incomparable.

Definition 3.6. We let $\mathbb{P}$ be the set of all "conditions"

$$p = (u^p, \alpha^p, (\bar\eta^p_\xi : \xi \in u^p))$$

satisfying the following requirements for all m:

- $u^p \in [\lambda_\omega]^{<\kappa}$
- $\alpha^p < \kappa$
- For all $\xi \in u^p \cap (\lambda_m \setminus \lambda_{m-1})$: $\bar\eta^p_\xi = (\eta^p_\xi(0), \dots, \eta^p_\xi(m-1)) \in (2^{\alpha^p})^m$.
- For all $\xi \neq \xi'$ in $u^p \cap (\lambda_m \setminus \lambda_{m-1})$, $\bar\eta^p_\xi$ and $\bar\eta^p_{\xi'}$ are incomparable in $(2^{\alpha^p})^m$.

We define $p \leq q$ ("q is stronger than p") iff

- $u^p \subseteq u^q$
- $\alpha^p \leq \alpha^q$
- for all $\xi \in u^p$, $\bar\eta^p_\xi \trianglelefteq \bar\eta^q_\xi$

Fact 3.7. 1. For all $\alpha < \kappa$: The set $\{p \in \mathbb{P} : \alpha^p \geq \alpha\}$ is dense in $\mathbb{P}$.
2. For all $\xi < \lambda_\omega$: The set $\{p \in \mathbb{P} : \xi \in u^p\}$ is dense in $\mathbb{P}$.

Fact and Definition 3.8. We let $(\underset{\sim}{\bar\nu}_\xi : \xi < \lambda_\omega)$ be the "generic object", i.e., a name satisfying

$$\forall m \in \omega\ \forall p \in \mathbb{P}\ \forall \xi \in u^p \cap (\lambda_m \setminus \lambda_{m-1}) : p \Vdash_{\mathbb{P}} \underset{\sim}{\bar\nu}_\xi \in (2^\kappa)^m$$
$$\forall p \in \mathbb{P}\ \forall \xi \in u^p : p \Vdash \bar\eta^p_\xi \trianglelefteq \underset{\sim}{\bar\nu}_\xi$$

(This definition makes sense, by fact 3.7.)

Clearly, $\Vdash \xi, \xi' \in \lambda_m \setminus \lambda_{m-1} \Rightarrow \underset{\sim}{\bar\nu}_\xi, \underset{\sim}{\bar\nu}_{\xi'}$ incompatible.

We let $\Vdash \underset{\sim}{I} = \bigcup_{m=2}^\infty \{\nu_\xi(\ell) : \xi \in \lambda_m \setminus \lambda_{m-1}, \ell < m\}$.

Theorem 3.9. *Let $\mathbb{P}$, $\underset{\sim}{I}$ be as in 3.6 and 3.8.*
Then $\Vdash_{\mathbb{P}} \operatorname{inc}(\underset{\sim}{I}^m) = \lambda_m$.

It is clear that $\mathbb{P}$ is κ-complete, and κ^+-cc is proved by an argument similar to the Δ-system argument below. So all the λ_m stay cardinals.

We can show by induction that $\Vdash \alpha(\underset{\sim}{I}^m) > \lambda_m$, i.e., $\underset{\sim}{I}^m$ has an antichain of size λ_m: This is clear if $\lambda_m = \lambda_{m-1}$ (and void if $m = 0$); if $\lambda_m > \lambda_{m-1}$ then $(\bar\nu_\xi : \xi \in \lambda_m \setminus \lambda_{m-1})$ will be forced to be antichain.

It remains to show that (for any m) there is no antichain of size λ_m^+ in $\underset{\sim}{I}^m$.

Fix $m^* \in \omega$, and assume wlog that $\lambda_{m^*+1} > \lambda_{m^*}$.

[Why is this no loss of generality? If $\lambda_{m^*} = \lambda_\omega$, then the cardinality of $\underset{\sim}{I}$ is at most λ_{m^*}, and there is nothing to prove. If $\lambda_{m^*} = \lambda_{m^*+1} < \lambda_\omega$, then replace m^* by $\min\{m \geq m^* : \lambda_m < \lambda_{m+1}\}$]

Towards a contradiction, assume that there is a condition p and a sequence of names $\langle \underset{\sim}{\bar\rho}_\beta : \beta < \lambda^+_{m^*} \rangle$ such that

$$p \Vdash \langle \underset{\sim}{\bar\rho}_\beta : \beta < \lambda^+_{m^*} \rangle \text{ is an antichain in } \underset{\sim}{I}^{m^*}$$

Let $\underset{\sim}{\bar\rho}_\beta = (\underset{\sim}{\rho}_\beta(n) : n < m^*)$. For each $\beta < \lambda^+_{m^*}$ and each $n < m^*$ we can find a condition $p_\beta \geq p$ and

$$m(\beta,n) \in \omega \qquad \ell(\beta,n) < m(\beta,n) \qquad \xi_n(\beta) \in \lambda_{m(\beta,n)} \setminus \lambda_{m(\beta,n)-1}$$

such that

$$p_\beta \Vdash \rho_\beta(n) = \nu_{\xi_n(\beta)}(\ell(\beta,n))$$

We will now employ a Δ-system argument.

We define a family $(\zeta^\beta : \beta < \lambda^+_{m^*})$ of functions as follows: Let i_β be the order type of u^{p_β}, and let

$$u^\beta = u^{p_\beta} = \{\zeta^\beta(i) : i < i_\beta\} \text{ in increasing enumeration}$$

By 3.7.2 may assume $\xi_n(\beta) \in u^\beta$, say $\xi_n(\beta) = \zeta^\beta(i(\beta,n))$.

By thinning out our alleged antichain $\langle \underset{\sim}{\bar\rho}_\beta : \beta < \lambda^+_{m^*}\rangle$ we may assume

- For some $i^* < \kappa$, for all β: $i_\beta = i^*$
- For some $\alpha^* < \kappa$, for all β: $\alpha^{p_\beta} = \alpha^*$
- For each $i < i^*$ there is some $m_{\langle i\rangle}$ such that for all β: $\zeta^\beta(i) \in \lambda_{m_{\langle i\rangle}} \setminus \lambda_{m_{\langle i\rangle}-1}$
- For each $i < i^*$ there is some $\bar\eta_{\langle i\rangle} \in (2^{\alpha^*})^{m_{\langle i\rangle}}$ such that for all β: $\bar\eta^{p_\beta}_{\zeta^\beta(i)} = \bar\eta_{\langle i\rangle}$. (Here we use $\lambda_m^{<\kappa} = \lambda_m$.)
- the family $\langle u^\beta : \beta < \lambda^+_{m^*}\rangle$ is a Δ-system, i.e., there is some set u^* such that for all $\beta \neq \gamma$: $u^\beta \cap u^\gamma = u^*$.
- Moreover: there is a set $\Delta \subseteq i^*$ such that for all β: $u^* = \{\zeta^\beta(i) : i \in \Delta\}$. Since ζ^β is increasing, this also implies $\zeta^\beta(i) = \zeta^\gamma(i)$ for $i \in \Delta$.)
- The functions $(\beta,n) \mapsto \ell(\beta,n)$, and $(\beta,n) \mapsto i(\beta,n)$ do not depend on β, i.e. there are $(\ell_n : n < m^*)$ and $(i_n : n < m^*)$ such that $i(\beta,n) = i_n$, $\ell(\beta,n) = \ell_n$ for all β.

Note that for $i \in i^* \setminus \Delta$ all the $\zeta^\beta(i)$ are distinct elements of $\lambda_{m_{\langle i\rangle}}$, hence:

$$i \notin \Delta \text{ implies } \lambda^+_{m^*} \leq \lambda_{m_{\langle i\rangle}}, \text{ hence } m_{\langle i\rangle} > m^*.$$

Now pick $k^* \leq m^*$ such that $k^* \notin \{\ell_n : n < m^*\}$. Pick any distinct $\beta, \gamma < \lambda^+_{m^*}$. We will find a condition q extending p_β and p_γ, such that $q \Vdash \bar\rho_\beta \leq \bar\rho_\gamma$.

We define q as follows:

- $u^q := u_\beta \cup u_\gamma = u^* \mathbin{\dot\cup} \{\zeta^\beta(i) : i \in i^* \setminus \Delta\} \mathbin{\dot\cup} \{\zeta^\gamma(i) : i \in i^* \setminus \Delta\}$.
- $\alpha^q = \alpha^* + 1$.
- For $\xi \in u^*$, say $\xi = \zeta^\beta(i) = \zeta^\gamma(i)$, recall that $\bar\eta^{p_\beta}_\xi = \bar\eta_{\langle i\rangle} = \bar\eta^{p_\gamma}_\xi$. We let $\eta^q_\xi = \eta_{\langle i\rangle}{}^\frown\bar 0$ (see 3.3).
- For $\xi = \zeta^\beta(i)$, $i \in i^* \setminus \Delta$, we have $\bar\eta^{p_\beta}_\xi = \bar\eta_{\langle i\rangle} \in (2^{\alpha^*})^{m_{\langle i\rangle}}$, where $m_{\langle i\rangle} > m^*$. Hence $\bar\eta_{\langle i\rangle}{}^\frown\{\ell\mapsto 1 \text{ else } \bar 0\}$ is well-defined. We let

$$\bar\eta^q_\xi = \bar\eta_{\langle i\rangle}{}^\frown\{\ell^*\mapsto 1 \text{ else } \bar 0\}$$

- For $\xi = \zeta^\gamma(i)$, $i \in i^* \setminus \Delta$, we let

$$\bar\eta^q_\xi = \bar\eta_{\langle i\rangle}{}^\frown\{k^*\mapsto 0 \text{ else } \bar 1\}$$

We claim that q is a condition. The only nontrivial requirement is the incompatibility of all $\bar\eta^q_\xi$: Let $\xi,\xi' \in u^q$, $\xi \neq \xi'$.

If $\xi,\xi' \in u^\beta$, then the incompatibility of $\bar\eta^q_\xi$ and $\bar\eta^q_{\xi'}$ follows from the incompatibility of $\bar\eta^{p_\beta}_\xi$ and $\bar\eta^{p_\beta}_{\xi'}$. The same argument works for $\xi,\xi' \in u_\gamma$.

So let $\xi \in u_\beta \setminus u^*$, $\xi' \in u_\gamma \setminus u^*$. Say $\xi = \zeta^\beta(i)$, $\xi' = \zeta^\gamma(i')$.

If $i \neq i'$, then $\bar\eta_{\langle i\rangle} = \bar\eta^{p_\beta}_{\zeta^\beta(i)} = \bar\eta^{p_\gamma}_{\zeta^\gamma(i)}$ and $\bar\eta_{\langle i'\rangle} = \bar\eta^{p_\gamma}_{\zeta^\gamma(i')}$ are incompatible. From $\bar\eta_{\langle i\rangle} \trianglelefteq \bar\eta^q_\xi$ and $\bar\eta_{\langle i'\rangle} \trianglelefteq \bar\eta^q_{\xi'}$ we conclude that also $\bar\eta^q_\xi$ and $\bar\eta^q_{\xi'}$ are incompatible.

Finally, we consider the case $i = i'$.

We have

$$\bar\eta^q_\xi = \bar\eta_{\langle i\rangle}{}^\frown\{k^*\mapsto 0 \text{ else } \bar 1\} \qquad \bar\eta^q_{\xi'} = \bar\eta_{\langle i\rangle}{}^\frown\{k^*\mapsto 1 \text{ else } \bar 0\}$$

so by 3.7.3, $\bar\eta^q_\xi$ and $\bar\eta^q_{\xi'}$ are incompatible.

This concludes the construction of q. We now check that $q \Vdash \bar\rho_\beta \le \bar\rho_\gamma$, i.e., $q \Vdash \rho_\beta(n) \le \rho_\gamma(n)$ for all n. Clearly, $q \Vdash \rho_\beta(n) = \nu_{\zeta^\beta(i_n)}(\ell_n) \trianglerighteq \bar\eta^q_{\zeta^\beta(i_n)} = \eta_{\langle i_n\rangle}{}^\frown\bar 0$. Here we use the fact that $k^* \neq \ell_n$. Similarly, $q \Vdash \rho_\gamma(n) = \nu_{\zeta^\beta(i_n)}(\ell_n) \trianglerighteq \eta_{\langle i_n\rangle}{}^\frown\bar 1$.

Hence $q \Vdash \bar\rho_\beta \le \bar\rho_\gamma$.

This concludes the proof of theorem 3.1

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Institut für Algebra, Technische Universität Wien, Wiedner Hauptstrasse 8-10/118.2, A-1040 Wien, Austria

E-mail address: Martin.Goldstern@tuwien.ac.at

URL: http://www.tuwien.ac.at/goldstern/

Department of Mathematics, Hebrew University of Jerusalem, Givat Ram, 91904 Jerusalem, Israel

E-mail address: shelah@math.huji.ac.il

URL: http://math.rutgers.edu/~shelah/