— Beispiel 3: Gleichverteilung/Maßtheorie —

The set $W := \{t : \exists x\, t \in B_x\}$ is a Σ^1_1 set, hence measurable. Assume that $\mu(W) > 0$. So there exists a Borel (closed) set A with $\mu(A) > 0$ such that $A \subseteq W$. Note that the statement

$$(*) \qquad \forall t : [t \in A \;\Rightarrow\; \exists x\; t \in B_x]$$

is a Π^1_2-statement, hence absolute between any two transitive universe with the same ordinals [Shoenfield].

Now let r be a random real over the universe V, $r \in A$, and work in $V[r]$. Random forcing is $\mathbb{N}^{\mathbb{N}}$-bounding, i.e., we have

$$\forall x \in \mathbb{N}^{\mathbb{N}}\, \exists x' \in \mathbb{N}^{\mathbb{N}} \cap V : x \leq x'$$

This implies $\forall x \in \mathbb{N}^{\mathbb{N}}\, \exists x' \in \mathbb{N}^{\mathbb{N}} \cap V : B_x \supseteq B_{x'}$, and so

$$\bigcup_{x \in \mathbb{N}^{\mathbb{N}}} B_x = \bigcup_{x' \in \mathbb{N}^{\mathbb{N}} \cap V} B_{x'}$$

Moreover,

$$\forall x' \in \mathbb{N}^{\mathbb{N}} \cap V : r \notin B_{x'}$$

(since r is a random real and each $B_{x'}$ is of measure 0), so together we get in $V[r]$:

$$\forall x \in \mathbb{N}^{\mathbb{N}} : r \notin B_x.$$

Since $(*)$ holds also in $V[r]$, we get $V[r] \models r \notin A$, a contradiction.