1-opc lattices

Martin Goldstern
TU Wien

http://www.tuwien.ac.at/goldstern/

opc lattices

Definition: A lattice $\mathfrak{L} = (L, \vee, \wedge)$ is called "order-polynomially complete" (opc) iff

> For all n, every monotone function $f : \mathfrak{L}^n \to \mathfrak{L}$ is induced by a lattice polynomial.

$\mathfrak{L}$ is called 1-opc if

> Every *unary* monotone function $f : \mathfrak{L} \to \mathfrak{L}$ is induced by a lattice polynomial.

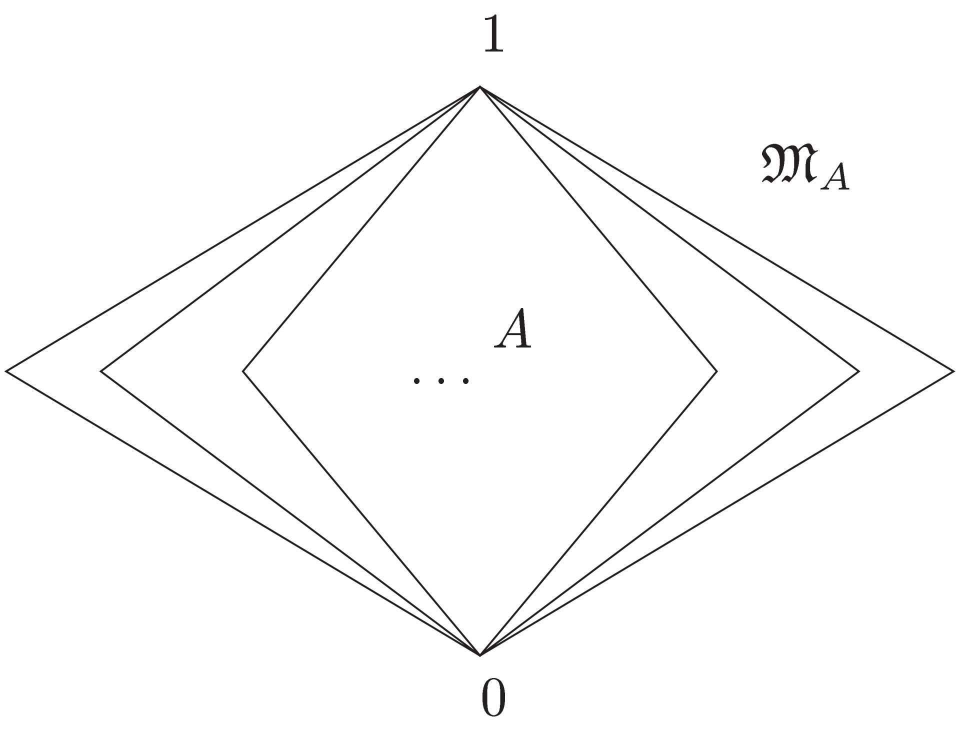
Finite opc lattices

Theorem A: (Schweigert 1972) If L is finite, then

$$\mathfrak{L} \text{ is 1-opc} \iff \mathfrak{L} \text{ is opc}$$

Proof: Lagrange interpolation. (Note: this yields a very *effective* or *explicit* proof)

Example: Let A be any finite set with ≥ 3 elements. Then $\mathfrak{M}_A$ is opc, where $\mathfrak{M}_A = (M_A, \leq)$, $M_A = A \dot\cup \{0, 1\}$, A an antichain between 0 and 1.



Infinite opc lattices

Theorem B: (G$*$+Shelah, 1997-98) There are no infinite opc lattices.

The proof of theorem B analyzes the properties that the *cardinality* of an infinite opc lattice must have: It cannot be countable, cannot be a successor cardinal (Erdős-Rado used here), cannot be a singular limit (more infinite combinatorics used here) and cannot be a limit of uncountable cofinality — hence does not exist.

This approach made many people unhappy.

Problem 1:

Find a "purely algebraic" proof of theorem B.

Meta-Problem:

What does "purely algebraic" mean?

Problem 2:

Show that there are no infinite 1-opc lattices.

What is "purely algebraic"?

An approximation to "purely algebraic":

Let's use less set theory!

Definition: We call a proof "explicit", if it does not use the axiom of choice, i.e., if it can be carried out in ZF (Zermelo-Fraenkel axioms of set theory without the axiom of choice.)

Theorem "-B": (G$*$+Shelah, 1998)

There is no "explicit" proof of theorem B.

(Proof sketch see slide 9/11)

We can interprete this as a hint that there is no "purely algebraic" proof of theorem B.

opc vs 1-opc

Conjecture C:

There are no infinite 1-opc lattices.

Conjecture C':

Every (infinite) 1-opc lattice is opc.

Recall that for finite lattices: C=false, C'=true.

By theorems A and B, we have: C $\Leftrightarrow$ C'.

Conjecture C trivially implies theorem B, so:

By theorem -B, there is no explicit proof of C.

But perhaps it is easier to find a proof of C'?

Theorem -C': There is no explicit proof of C'.

Proof of theorems -B and -C' (sketch)

An infinite set A is called "amorphous" if all subsets of A are either finite or cofinite; equivalently, if all subsets of A are definable with parameters from A in predicate logic with no predicates except equality.

An infinite set A is called "strongly amorphous" if: for all n, all subsets of A^n are definable in predicate logic with equality,
or equivalently, if all subsets of A^n are in the Boolean algebra generated by the sets

$$\{(x_1, \ldots, x_n) : x_i = x_j\}\text{: } i, j \in \{1, \ldots, n\}$$

$$\{(x_1, \ldots, x_n) : x_i = a\}\text{: } i \in \{1, \ldots, n\}, a \in A.$$

Fact: If A is strongly amorphous, then all maps $f : M_A^n \to M_A$ (so in particular: all monotone maps $f : \mathfrak{M}_A^n \to \mathfrak{M}_A$) are definable in predicate logic with equality.

This can be used to show

> If A is strongly amorphous, then $\mathfrak{M}_A$ is infinite and opc.

Note: The existence of strongly amorphous sets is consistent with ZF (i.e., cannot be "explicitly" refuted).

Similarly, the existence of a an infinite set A with an equivalence relation E satisfying

(a) *All E-classes have exactly 3 elements*

(b) *All maps $f : A^n \to A$, and even all maps $f : M_A^n \to M_A$, can be defined in predicate logic using the predicates "E" and "$=$" only*

is consistent with ZF.

Now check that for such a set A, all unary monotone maps $f : \mathfrak{M}_A \to \mathfrak{M}_A$ are either almost constant or almost the identity, which in turn shows that they are represented by lattice polynomials. $\Rightarrow \mathfrak{M}_A$ is 1-opc

However, the binary map

$$f(x, y) = \begin{cases} 1 & \text{if } x = 1 \ \vee \ y = 1 \ \vee \ x \sim_E y \\ x & \text{otherwise} \end{cases}$$

is monotone and **not** represented by a lattice polynomial. $\Rightarrow \mathfrak{M}_A$ is not opc

Open Questions

Question: Show that there are no infinite 1-opc lattices.

(Equivalently, that 1-opc implies opc)

[We know that there is no explicit proof.]

The proof of theorem -C' tells us that really there is no explicit proof of 1-opc $\Rightarrow$ 2-opc.

Question:

Is there an explicit proof of 2-opc $\Rightarrow$ opc?

The proof of theorem -B really showed (explicitly) that $\mathfrak{M}_A$ is definably opc, i.e.: .

> Whenever $f : \mathfrak{M}_A^n \to \mathfrak{M}_A$ is a monotone function which is first order definable in the structure $(\mathfrak{M}_A, \leq)$,
> then f is represented by a polynomial.

Question: Find a lattice which is definably 1-opc but not definably opc.