The Galois Connection between relations and automorphisms

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Setup: $A \ldots$ an infinite set

$\mathrm{Rel}_k(A) = \mathscr{P}(A^k) =$ k-ary relations on A

$\mathrm{Rel}(A) = \bigcup_{k\in\mathbb{N}} \mathrm{Rel}_k(A)$.

$\mathrm{Sym}(A) =$ all permutations of A:

Every $f \in \mathrm{Sym}A$ acts also on A^k (pointwise).

For $\rho \subseteq A^k$ we define $f[\rho]$ naturally.

The Galois connection sInv-Aut

For $f \in \mathrm{Sym}A$, $\rho \in \mathrm{Rel}A$ we write

$$f \in \mathrm{Aut}(\rho) \quad \text{or} \quad \rho \in \mathrm{sInv}(f)$$

iff $f[\rho] = \rho$. We say

"f is an automorphism of ρ"

"ρ is strongly invariant under f"

Definition:

$f \in \mathrm{Aut}(\rho) \Leftrightarrow \rho \in \mathrm{sInv}(f) \Leftrightarrow f[\rho] = \rho.$

For $F \subseteq \mathrm{Sym}(A)$ we let $\mathrm{Aut}(F) := \bigcap_{f \in F} \mathrm{sInv}(f)$

Dually: $\mathrm{Aut}(P) := \bigcap_{\rho \in P} \mathrm{Aut}(\rho)$ for $P \subseteq \mathrm{Rel}(A)$.

The Problem: *Characterize the Galois closed sets of relations.*

in other words:

- Which subsets of $\mathrm{Rel}(A)$ are of the form $\mathrm{sInv}(F)$ for some F?
- *(rephrased:)* describe the closure operator

$$P \mapsto \mathrm{sInv}\mathrm{Aut}(P)$$

Aside

The dual closure operator is easy to describe: For any set $F \subseteq \mathrm{Sym}(A)$*, the set* $\mathrm{Aut}\,\mathrm{sInv}(F)$ *is the smallest subset of* $\mathrm{Sym}(A)$

- *containing* F
- *closed under the group operations (composition, inverse)*
- *closed under finite interpolability.*
  (= closed in the Tychonoff topology on A^A*.*
  "$f \notin \mathrm{Aut}(\rho)$*" is an open=finitary property.)*

Thus, for infinite sets A *(of any cardinality) we have to combine an algebraic(=finitary) closure operator with a topological(=infinitary) closure operator.*

So we have to expect that the description of $P \mapsto \mathrm{sInv}\,\mathrm{Aut}(P)$ *also involves finitary and infinitary closure.*

Closure properties of $\mathrm{sInv}(F)$

Example 1: if ρ_1, ρ_2 are both in $\mathrm{sInv}(F)$ (for some F), and are of the same arity k, then also $\rho_1 \cup \rho_2$ and $A^k \setminus \rho_1$ must be in $\mathrm{sInv}(F)$.

Example 2: if $\rho \subseteq A \times A$ is in $\mathrm{sInv}(F)$, then also the projection of ρ,

$$\exists\rho := \{x \in A : \exists y \in A\,(x, y) \in \rho\}$$

is in $\mathrm{sInv}(F)$.

Conclusion: $\mathrm{sInv}(F)$ is closed under all *logical operations.* (=operations of first order logic). This is also called a "Krasner algebra" (of the second kind).

Moreover, $\mathrm{sInv}(F)$ is closed under arbitrary intersections.

Countable base sets

Theorem: If A is countable, $P \subseteq \mathrm{Rel}(A)$, then $\mathrm{sInvAut}(P)$ is the closure of P under logical operations and arbitrary intersections.

However, this is not true for uncountable sets:

E.g., if $\rho \subseteq A \times A$, then the relation $\exists^{>\aleph_0}\rho$, defined as

$$\{x : \text{ there are uncountably many } y{:}(x,y) \in \rho\}$$

is in $\mathrm{sInvAut}(\rho)$, but can in general not be obtained from ρ with logical operations and arbitrary intersections.

Invariant operations

The problem is that first order logic cannot distinguish different infinite cardinalities. So we add *all* operations from all higher order logics.

Definition: A map $h : \mathrm{Rel}_{k_1}(A) \times \cdots \times \mathrm{Rel}_{k_\ell}(A) \to \mathrm{Rel}(A)$ is called an *invariant operation*, if:

> For all $f \in \mathrm{Sym}(A)$ and all $\rho_1 \in \mathrm{Rel}_{k_1}$, $\ldots$, $\rho_\ell \in \mathrm{Rel}_{k_\ell}$ we have
>
> $$h[f(\rho_1, \ldots, \rho_\ell)] = f(h[\rho_1], \ldots, h[\rho_\ell])]$$

All logical operations, and all operations definable in higher order logic such as $\exists^{>\aleph_0}$ are invariant.

A plausible conjecture

We are looking for an internal description of the closure operator $P \mapsto \mathrm{Aut}\,\mathrm{sInv}P$. In many cases — in particular if the base set A is at most countable, or if the set P is finite — we have:

> $\mathrm{Aut}(\mathrm{sInv}(P))$ *is the smallest set containing* P *which is closed under*
>
> – *all invariant operations*
>
> – *arbitrary intersections.*

Is this true in general?

Answer: No!

A counterexample

There is an uncountable structure $\mathscr{A} = (A, \rho_1, \rho_2, \ldots)$ such that, letting

$$P := \text{ the first order definable relations of } \mathscr{A} \text{ (without parameters)}$$

$$\mathscr{A}\restriction n := (A, \rho_1, \ldots, \rho_n)$$

we have

1. $\mathscr{A}$ is $\aleph_0$-categorical. Hence:
   - P is **small** (countable)
   - P is closed under arbitrary $\bigcap$
2. $\forall n : \mathscr{A}\restriction n$ is homogeneous (every finite partial automorphism extends)

   This implies: P is closed under all invariant operations.
3. $\mathscr{A}$ is rigid.

   Hence, $\mathrm{Aut}(P) = \{\mathrm{id}\}$, and

   $\mathrm{sInv}\,\mathrm{Aut}P = \mathrm{Sym}(A)$ is **large** .

Characterization of $\mathrm{sInv}\,\mathrm{Aut}$

Definition: A map

$$h : \prod_{n\in\omega} \mathrm{Rel}_{k_n}(A) \to \mathrm{Rel}(A)$$

is called an *σ-invariant operation*, if:

For all $f \in \mathrm{Sym}(A)$ and all countable sequences $(\rho_1, \ldots)$ with $\rho_1 \in \mathrm{Rel}_{k_1}, \ldots,$ we have

$$h[f(\rho_1, \ldots, \rho_\ell, \ldots)] = f(h[\rho_1], \ldots, h[\rho_\ell], \ldots)]$$

Then we have for any (even uncountable) set A:

For any set P of relations on A, $\mathrm{Aut}\,\mathrm{sInv} P$ is the smallest set containing P which is closed under all invariant operations, and arbitrary intersections.