Asymptotic estimates for best and stepwise approximation of convex bodies IV

Peter M.Gruber

Abstract. In this article we first prove a stability theorem for coverings in $\mathbb{E}^2$ by congruent solid circles: if the density of such a covering is close to its lower bound $2\pi/\sqrt{27}$, then most of the centers of the circles are arranged in almost regular hexagonal patterns. A version of this result then is extended to coverings by geodesic discs in two-dimensional Riemannian manifolds.

Given a sufficiently differentiable convex body C in $\mathbb{E}^3$, the following two problems are closely related: (i) Approximation of C with respect to the Hausdorff metric, the Banach–Mazur distance and a notion of distance due to Schneider by inscribed or circumscribed convex polytopes. (ii) Covering of the boundary of C by geodesic discs with respect to suitable Riemannian metrics.

The stability result for Riemannian manifolds and the relation between approximation and covering yield rather precise information on the form of best approximating inscribed convex polytopes P_n of C with respect to the Hausdorff metric: if the number n of vertices is large, then most of the vertices are arranged in almost regular hexagonal patterns. Consequently, the majority of facets of P_n are almost regular triangles. Here 'regular' is meant with respect to the Riemannian metric of the second fundamental form. Similar results hold for circumscribed polytopes and also for the Banach–Mazur distance and Schneider's notion of distance.

1991 Mathematics Subject Classification: 41A60, 46B28, 52A15, 52A27, 52A41, 52A41, 52C15, 52C20, 53B21; 41A40.

1 Introduction and statement of results

1.1 The aim of this article is twofold. First, stability results for coverings in the Euclidean plane $\mathbb{E}^2$ by congruent solid circles and in two-dimensional Riemannian manifolds by geodesic discs will be proved. Second, given a convex body in $\mathbb{E}^3$, information on the form of best approximating inscribed or circumscribed convex polytopes will be obtained, where approximation is with respect to the Hausdorff metric, the Banach–Mazur distance and a notion of distance due to Schneider.

1.2 A classical result of L. Fejes Tóth [**?**] which refines an earlier result of Kershner [**?**] says the following: the density of a covering of a compact convex disc in $\mathbb{E}^2$ with non-empty interior by two or more congruent solid circles is greater than $2\pi/\sqrt{27}$. This lower bound is best possible. (By the *density* of a family of sets which cover a given set we mean the sum of the areas of the sets of the family divided by the area of the given set.) If the number of circles is large and the covering has minimum density, then L. Fejes Tóth [?], p. 61, indicated that the centers of the circles, in essence, are arranged almost hexagonally.

In recent years many stability problems have been investigated in convex geometry. These either belong to the area of geometric inequalities, or are of a more geometric type, see the survey of Groemer [?] and the references in Gruber [?]. Our first result lies somewhere between these two types.

Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{E}^2$. By $C(c,\varrho)$ we denote the solid circle with center c and radius ϱ in $\mathbb{E}^2$. In a set C in $\mathbb{E}^2$ a point $c \in C$ is the *center of a regular hexagon up to* $\delta > 0$ if there are a constant $\sigma > 0$, the *size* of the hexagon, and points $c_1, \dots, c_6 \in C$, its *vertices*, such that

$$| \, \|c_k - c\| - \sigma|, | \, \|c_{k+1} - c_k\| - \sigma| \le \sigma\delta \quad (c_7 = c_1),$$

$$C(c, \frac{3}{2}\sigma) \cap C = \{c, c_1, \dots, c_6\}.$$

Theorem 1. *Let S be a convex $3, 4, 5,$ or 6-gon and $\varepsilon > 0$ sufficiently small. Then for all coverings of S by, say, m congruent solid circles of sufficiently small radius and density less than*

$$\frac{2\pi}{\sqrt{27}}(1+\varepsilon)$$

holds: in the set of centers of these circles, each center, with a set of less than $50\varepsilon^{1/3}m$ exceptions, is the center of a regular hexagon up to $500\varepsilon^{1/3}$. All these hexagons have the same size.

Remark 1. In the proof explicit bounds for ε and the radius of the circles will be given. The choice of the exponents of ε and of the coefficients in the Theorem is somewhat arbitrary.

Remark 2. Several natural extensions of Theorem 1 suggest themselves. First, to coverings of the whole plane which is easy and left to the reader; compare the remark of L. Fejes Tóth, p. 61, for coverings of the plane of density $2\pi/\sqrt{27}$. Second, to Riemannian manifolds of dimension two. A pertinent result of this type is Theorem 2 below. Third, to higher dimensions. Since in this case not even the densities of the thinnest coverings of space by congruent balls are known, it seems at present to be out of reach, to extend Theorem 1 in a precise form to dimensions greater than two.

1.3 In recent years a series of results of Euclidean geometry have been extended to more general spaces. Our second result is a (weak) version of Theorem 1 in this context. We have chosen this version because of its applicability to approximation problems; see Theorem 3.

For exact definitions of the following notions compare section 3. Let M be a two-dimensional Riemannian manifold of class $\mathcal{C}^2$ with metric of class $\mathcal{C}^0$ and let γ_M and ω_M

be the corresponding geodesic metric and Jordan area measure on M. The geodesic disc $D_M(c,\varrho)$ with center c and radius ϱ in M is the set $\{x \in M : \gamma_M(c,x) \le \varrho\}$.

Consider a sequence of sets C_n in M such that the number $\#C_n$ of points of C_n is n. As $n \to \infty$, the set C_n is *asymptotically a regular hexagonal pattern* in M if the following hold: for each n there is a constant $\sigma_n > 0$ such that for all points $c \in C_n$, with a set of $o(n)$ exceptions, there are points $c_1, \dots, c_6 \in C_n$ such that

$$|\gamma_M(c,c_k) - \sigma_n|, |\gamma_M(c_{k+1},c_k) - \sigma_n| = \sigma_n \cdot o(1) \text{ as } n \to \infty \quad (c_7 = c_1),$$

$$D_M(c, 1.1\sigma_n) \cap C_n = \{c, c_1, \dots, c_6\}.$$

In [?] the following result was proved: Let $J \subset M$ be Jordan measurable with $\omega_M(J) > 0$ and consider for $n = 1,2,\dots,$ a covering of J by n geodesic discs of minimum radius. Then the densities of the coverings tend to $2\pi/\sqrt{27}$ as $n \to \infty$. The following result gives, in particular, information about the distribution of the centers of such coverings.

Theorem 2. *Let J be a Jordan measurable set in M with $\omega_M(J) > 0$. For each $n = 1,2,\dots,$ consider a covering of J by n geodesic discs of the same radius such that the densities of these coverings tend to $2\pi/\sqrt{27}$ as $n \to \infty$. Then, as $n \to \infty$, the set of centers of the nth covering is asymptotically a regular hexagonal pattern.*

Remark 3. At present it seems to be difficult to extend Theorem 2 to higher dimensions, see Remark 2.

1.4 A *convex body* C in $\mathbb{E}^d$ is a compact convex subset of $\mathbb{E}^d$ with non-empty interior. Given C, denote by $\mathcal{P}^i_n$ and $\mathcal{P}^c_{(n)}$ the spaces of inscribed and of circumscribed convex polytopes having at most n vertices, resp. facets. The *Hausdorff metric* δ^H on the space of all convex bodies in $\mathbb{E}^d$ is defined as follows: for convex bodies C, D let $\delta^H(C,D)$ be the maximum Euclidean distance which a point of one of the bodies can have from the other body. Call a polytope $P_n \in \mathcal{P}^i_n$ *best approximating* of C with respect to δ^H if

$$\delta^H(C,P_n) = \delta^H(C,\mathcal{P}^i_n) = \inf\{\delta^H(C,P) : P \in \mathcal{P}^i_n\}.$$

A basic problem is to determine or estimate $\delta^H(C,P_n)$ and to describe P_n as $n \to \infty$.

Let (the boundary bd C of) C be (a surface) of class $\mathcal{C}^2$ with Gauss curvature $\kappa_C > 0$. McClure and Vitale [?] showed for $d = 2$ that the set vertP_n of vertices of P_n is almost equally spaced along bd C as $n \to \infty$, if a suitable notion of length is used. Their result was substantially refined by Ludwig [?]. For general d Glasauer and Schneider [?] proved that vertP_n is uniformly distributed in bd C with respect to the density $\sqrt{\kappa_C}$ as $n \to \infty$ in the sense of uniform distribution theory. This result gives a rough idea about the distribution of vertP_n in bd C.

For $d = 3$ we will describe the (local) form of P_n and of the best approximating polytopes $Q_n \in \mathcal{P}^c_{(n)}$ for large n in a more precise way. This will be done using the Riemannian metric of the second fundamental form on bd C which in a natural sense corresponds to δ^H. Similar results hold also for the notions of distance δ^{BM} and δ^{SCH}.

The *Banach–Mazur distance* δ^{BM} is defined for convex bodies C, D which are symmetric with respect to the origin o:

$$\delta^{BM}(C, D) = \inf\{\lambda > 1 : C \subset l(D) \subset \lambda C \text{ for suitable linear } l : \mathbb{E}^d \to \mathbb{E}^d\}.$$

Let $\mathcal{P}_{s,2n}$ and $\mathcal{P}_{s,(2n)}$ be the spaces of convex polytopes which are o-symmetric and have at most $2n$ vertices, resp. facets. Given a convex body C which is symmetric in o, let $\mathcal{P}^i_{s,2n}$ be the set of those polytopes in $\mathcal{P}_{s,2n}$ which are inscribed into C; similarly for $\mathcal{P}^c_{s,(2n)}$. It is easy to see that

$$\delta^{BM}(C, \mathcal{P}_{s,2n}) = \delta^{BM}(C, \mathcal{P}^i_{s,2n}),\ \delta^{BM}(C, \mathcal{P}_{s,(2n)}) = \delta^{BM}(C, \mathcal{P}^c_{s,(2n)}).$$

The *Riemannian metric* on $\operatorname{bd} C$ corresponding to δ^{BM} is that *of central affine differential geometry.* See Gruber [**?**].

Given a convex body C and a convex polytope P contained in C, *Schneider's distance* $\delta^{SCH}(C, P)$ is the maximum volume of a cap of C determined by a halfspace which contains a facet of P in its boundary, but does not contain P. The *center of a cap* of C determined by a facet of P is the point of the cap with maximum Euclidean distance from the hyperplane containing the facet. Let $\text{capcenter}P$ denote the set of all centers of caps. The *Riemannian metric* on $\operatorname{bd} C$ which corresponds to Schneider's distance is that *of equiaffine differential geometry.* See Schneider [**?**].

Theorem 3. *Let C be a convex body in $\mathbb{E}^3$ of class $\mathcal{C}^2$ with positive Gauss curvature. Consider sequences $(P_n), \dots, (T_n)$ of best approximating polytopes*

$$\begin{aligned} &P_n \in \mathcal{P}^i_n,\ Q_n \in \mathcal{P}^c_{(n)} && \textit{with respect to } \delta^H, \\ &R_n \in \mathcal{P}^i_{s,2n},\ S_n \in \mathcal{P}^c_{s,(2n)} && \textit{with respect to } \delta^{BM}, \\ &T_n \in \mathcal{P}^i_{(n)} && \textit{with respect to } \delta^{SCH}, \end{aligned}$$

where for δ^{BM} we assume that C is symmetric in o. Then the sets

$$\begin{aligned} &\text{vert}P_n, (\operatorname{bd} Q_n) \cap C && \textit{for } \delta^H, \\ &\text{vert}R_n, (\operatorname{bd} S_n) \cap C && \textit{for } \delta^{BM}, \\ &\text{capcenter}T_n && \textit{for } \delta^{SCH} \end{aligned}$$

are asymptotically regular hexagonal patterns on $\operatorname{bd} C$ *with respect to the corresponding Riemannian metrics.*

Remark 4. A consequence of this result is that for large n most facets of P_n and R_n are almost regular triangles, approximately of the same size, and the facets of Q_n, S_n, and T_n are almost regular hexagons, approximately of the same size; the metrics being the corresponding Riemannian metrics on $\operatorname{bd} C$.

Remark 5. As may be seen from the proof, Theorem 3 holds also for the more general *asymptotically best approximating polytopes*, where, for example, in the case of the metric δ^H and for inscribed polytopes this means a sequence (U_n) where $U_n \in \mathcal{P}^i_n$ such that

$$\delta^H(C, U_n) \sim \delta^H(C, \mathcal{P}^i_n) \text{ as } n \to \infty.$$

2 Proof of Theorem 1

2.1 Choose

(2.1) $\quad 0 < \varepsilon < 10^{-9} \quad$ and $0 < \varrho < \dfrac{|S|}{2p(S)}\varepsilon.$

(Here $|\cdot|$ and $p(\cdot)$ stand for Euclidean area and perimeter in $\mathbb{E}^2$, respectively.) The proof of Theorem 1 is split into several steps.

2.2 In a first step two elementary properties of convex hexagons will be proved.

2.2.1

(2.2) $\quad$ Let D be a convex hexagon such that $C(\frac{\sqrt{3}}{2} - \varepsilon^{1/3}) \not\subset D \subset C(1)$. Then

$$|D| \le \frac{\sqrt{27}}{2}\Big(1 - \frac{\varepsilon^{2/3}}{3}\Big).$$

(Here $C(\varrho)$ denotes the solid circle in $\mathbb{E}^2$ with center o and radius ϱ.)

Elementary arguments show that $|D|$ is bounded above by the area of a convex hexagon with the following properties: all of its vertices are on $\operatorname{bd} C(1)$; the hexagon contains $C(\frac{\sqrt{3}}{2} - \varepsilon^{1/3})$; one of its edges is tangent to $C(\frac{\sqrt{3}}{2} - \varepsilon^{1/3})$ and the five other ones have equal length. Let

$$\frac{\pi}{3} + 2\varphi \text{ and } \frac{\pi}{3} - \frac{2\varphi}{5}, 0 < 2\varphi < \frac{\pi}{3},$$

be the angles under which the edges of the hexagon appear from o (note (2.1)). Then

$$\frac{\sqrt{3}}{2} - \varepsilon^{1/3} = \cos(\frac{\pi}{6} + \varphi) = \cos\frac{\pi}{6} - \sin(\frac{\pi}{6} + \xi\varphi)\cdot\varphi \ge \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\varphi \ge \frac{\sqrt{3}}{2} - \varphi$$

by Taylor's theorem, where $\xi \in (0,1)$ is chosen suitably. Thus

$$\varepsilon^{1/3} \le \varphi.$$

Then

$$\begin{aligned}
|D| &\le \sin(\frac{\pi}{6} + \varphi)\cos(\frac{\pi}{6} + \varphi) + 5\sin(\frac{\pi}{6} - \frac{\varphi}{5})\cos(\frac{\pi}{6} - \frac{\varphi}{5}) \\
&= \frac{1}{2}\sin(\frac{\pi}{3} + 2\varphi) + \frac{5}{2}\sin(\frac{\pi}{3} - \frac{2\varphi}{5}) \\
&= \frac{1}{2}\sin\frac{\pi}{3} + \frac{1}{2}\cos\frac{\pi}{3}\cdot 2\varphi - \frac{1}{4}\sin(\frac{\pi}{3} + 2\eta\varphi)\cdot(2\varphi)^2 \\
&+ \frac{5}{2}\sin\frac{\pi}{3} + \frac{5}{2}\cos\frac{\pi}{3}\cdot\frac{-2\varphi}{5} - \frac{5}{4}\sin(\frac{\pi}{3} - \frac{2\zeta\varphi}{5})\cdot(\frac{-2\varphi}{5})^2 \\
&\le \frac{3\sqrt{3}}{2} - \frac{1}{4}\sin\frac{\pi}{3}\cdot(2\varphi)^2 = \frac{3\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\varphi^2 \le \frac{\sqrt{27}}{2}(1 - \frac{\varepsilon^{2/3}}{3}),
\end{aligned}$$

where $\eta, \zeta \in (0,1)$ are chosen suitably, concluding the proof of (2.2).

2.2.2

(2.3) Let D be a convex hexagon such that $C(\frac{\sqrt{3}}{2}-\varepsilon^{1/3}) \subset D \subset C(1)$ and let $d_1,\dots,d_6$ be the mirror images of o in the lines containing the edges of D in counter clockwise ordering. Then

(i) $-2\varepsilon^{1/3} \leq \|d_k\| - \sqrt{3} \leq 22\varepsilon^{1/3}$,

(ii) $|\ \|d_{k+1}-d_k\| - \sqrt{3}| \leq 810\varepsilon^{1/3}$

and there are points $h_1,\dots,h_6$ forming the vertices of a regular hexagon of edgelength $\sqrt{3}$ and center o such that

(iii) $\|d_k - h_k\| \leq 405\varepsilon^{1/3}$.

Let

$$\frac{\pi}{6}+\varphi \quad \text{and} \quad \frac{\pi}{6}-\psi, 0\leq\varphi\leq\frac{\pi}{12},$$

be the maximum and the minimum angle with apex o between the exterior normal of an edge of D and an endpoint of this edge (note (2.1)). Then

(2.4) $0\leq\varphi\leq 2\varepsilon^{1/3},\ 0\leq\psi\leq 22\varepsilon^{1/3}$.

To see this note that by the assumption of (2.3) and the concavity of cos in $[0,\frac{\pi}{2}]$,

$$\frac{\sqrt{3}}{2}-\varepsilon^{1/3} \leq \cos(\frac{\pi}{6}+\varphi) \leq \cos\frac{\pi}{6} - \sin\frac{\pi}{6}\cdot\varphi = \frac{\sqrt{3}}{2}-\frac{\varphi}{2},$$

which implies the first statement in (2.4). The second one follows from

$$\frac{\pi}{6}-\psi \geq 2\pi - 11(\frac{\pi}{6}+\varphi) \geq \frac{\pi}{6} - 22\varepsilon^{1/3}.$$

The inclusion $C(\frac{\sqrt{3}}{2}-\varepsilon^{1/3}) \subset D$ yields the first inequality in (2.3)(i). To see the second inequality note that $D\subset C(1)$. The definition of ψ, the concavity of cos in $[0,\frac{\pi}{2}]$, and (2.4) then imply,

$$\|d_k\| \leq 2\cos(\frac{\pi}{6}-\psi) \leq 2\cos\frac{\pi}{6} - 2\sin\frac{\pi}{6}\cdot(-\psi) = \sqrt{3}+\psi \leq \sqrt{3}+22\varepsilon^{1/3}.$$

Next we prove (2.3)(iii). Let h_k be chosen as follows: h_1 is a positive multiple of d_1 and $\|h_1\| = \sqrt{3}$. $h_2,\dots,h_6$ are obtained from h_1 by rotations by angles $\frac{\pi}{3},\dots,\frac{5\pi}{3}$ about o. If φ_k is the angle with apex o between d_k and h_k, then the definitions of φ and ψ and (2.4) together yield,

$$|\varphi_k| \leq 44(k-1)\varepsilon^{1/3} \leq 220\varepsilon^{1/3}.$$

This combined with (2.3)(i), $\|h_k\| = \sqrt{3}$, and (2.1) shows that

$$\begin{aligned}\|d_k - h_k\| &\leq \max\{\|d_k\|,\|h_k\|\}\cdot|\varphi_k| + |\ \|d_k\| - \|h_k\|\ | \\ &\leq (\sqrt{3}+22\varepsilon^{1/3})220\varepsilon^{1/3} + 22\varepsilon^{1/3} \leq 405\varepsilon^{1/3}.\end{aligned}$$

This proves (2.3)(iii) which in turn yields (2.3)(ii).

2.3 Let a covering of S by m solid circles of radius ϱ with density less than

$$\frac{2\pi}{\sqrt{27}}(1+\varepsilon)$$

be given. Second, we will consider the intersections of the Dirichlet–Voronoi cells of the set C of centers with S and classify them according to their shape.

For each $c \in C$ let

$$D(c) = \{x : \|x-c\| \le \|x-d\| \text{ for each } d \in C\} \cap S.$$

If $D(c)$ has non-empty interior we call it a *cell with center* c. Since the Dirichlet–Voronoi cells of C form an edge-to-edge tiling of $\mathbb{E}^2$ (see e.g. [**?**]),

(2.5) the cells $D(c)$ form an edge-to-edge tiling of S.

Our aim is to show that for most $c \in C$ the cell $D(c)$ is close to a regular hexagon:

(2.6) Each $c \in C$, with a set of less than $4\varepsilon^{1/3}m$ exceptions, has the following properties:

(i) $D(c)$ is a hexagonal cell,

(ii) $C(c, (\frac{\sqrt{3}}{2} - \varepsilon^{1/3})\varrho) \subset D(c) \subset C(c,\varrho) \subset S$.

Such centers and the corresponding cells are called *good*, the others *bad*. The proof of (2.6) will be presented in subsections 2.3.1 – 2.3.5.

2.3.1 For $i = 3, 4, \ldots$, let m_i be the number of i-gons among the cells $D(c)$. Since S is a $3, 4, 5$, or 6-gon, proposition (2.5) and a simple consequence of Euler's polytope formula (see e.g. [**?**], p. 16) together imply that

$$3m_3 + 4m_4 + \ldots \le 6(m_3 + m_4 + \ldots) = 6m_0, \text{ say, where } m_0 \le m.$$

Thus

(2.7) $3m_3 + \ldots + 5m_5 + 7m_7 + \ldots \le 6(m_3 + \ldots + m_5 + m_7 + \ldots) = 6m_{\neq 6}$, say.

Since among all convex i-gons in $C(\varrho)$ the regular ones have maximum area, it follows:

(2.8) if the cell $D(c)$ is an i-gon, then $\dfrac{|D(c)|}{\varrho^2} \le \dfrac{i}{2}\sin\dfrac{2\pi}{i}$.

Proposition (2.3) shows:

(2.9) if the cell $D(c)$ is a 6-gon and $C(c, (\frac{\sqrt{3}}{2} - \varepsilon^{1/3})\varrho) \not\subset D \subset C(c,\varrho)$, then

$$\frac{|D(c)|}{\varrho^2} \le \frac{\sqrt{27}}{2}\left(1 - \frac{\varepsilon^{2/3}}{3}\right).$$

By (2.5) and (2.8) we have,

$$(2.10)\quad \frac{|S|}{\varrho^2} = \sum_{i\geq 3}\{\sum\{\frac{|D(c)|}{\varrho^2} : c \in C \text{ the cell } D(c) \text{ is an } i\text{-gon }\}\}$$
$$\leq \frac{\sqrt{27}}{2} m_6 + \sum_{i\geq 3, \neq 6} m_i \frac{i}{2} \sin\frac{2\pi}{i}.$$

2.3.2 See (2.7) for the definition of $m_{\neq 6}$. We show that

(2.11) $m_{\neq 6} \leq 64\varepsilon m$.

Considering derivatives, it is easy to prove that the function f defined by

$$f(x) = \frac{x}{2}\sin\frac{2\pi}{x}, x \geq 3,$$

is concave and non-decreasing. Hence the function g whose graph is the polygon connecting the points

$$(3, f(3)), \ldots, (5, f(5)), (7, f(7)), \ldots,$$

in this order, is also concave and non-decreasing. Thus Jensen's inequality together with (2.7) implies,

$$(2.12)\quad \frac{1}{m_{\neq 6}} \sum_{i\geq 3,\neq 6} m_i \frac{i}{2}\sin\frac{2\pi}{i} = \sum_{i\geq 3,\neq 6} \frac{m_i}{m_{\neq 6}} g(i) \leq g(\frac{1}{m_{\neq 6}} \sum_{i\geq 3,\neq 6} m_i \cdot i)$$
$$\leq g(6) = 2.5570257\ldots < \frac{\sqrt{27}}{2}(1 - \frac{1}{64}) < \frac{\sqrt{27}}{2}.$$

From (2.10) and (2.12) we conclude that

$$\frac{|S|}{\varrho^2} \leq \frac{\sqrt{27}}{2} m_6 + \frac{\sqrt{27}}{2} m_{\neq 6} - \frac{\sqrt{27}}{128} m_{\neq 6} \leq \frac{\sqrt{27}}{2} m - \frac{\sqrt{27}}{128} m_{\neq 6} = \frac{\sqrt{27}}{2}(1 - \frac{m_{\neq 6}}{64m})m$$

(note that $m_6 + m_{\neq 6} = m_0 \leq m$). This yields a lower bound for the density of the given covering of S:

$$\frac{\varrho^2 \pi m}{|S|} \geq \frac{2\pi}{\sqrt{27}}(1 - \frac{m_{\neq 6}}{64m})^{-1} \geq \frac{2\pi}{\sqrt{27}}(1 + \frac{m_{\neq 6}}{64m}).$$

Since by assumption this density is less than

$$\frac{2\pi}{\sqrt{27}}(1 + \varepsilon),$$

we obtain (2.11).

2.3.3 Let

$$m_{nr6} = \#\{D(c) : \text{ the cell } D(c) \text{ is a 6-gon, } C(c, (\frac{\sqrt{3}}{2} - \varepsilon^{1/3})\varrho) \not\subset D(c) \subset C(c, \varrho)\},$$

where # means cardinal number. Then the following estimate holds:

(2.13) $m_{nr6} \leq 3\varepsilon^{1/3} m.$

Applying (2.10), (2.9) and (2.12) gives

$$\frac{|S|}{\varrho^2} \leq \frac{\sqrt{27}}{2}(m_6 - m_{nr6}) + \frac{\sqrt{27}}{2}(1 - \frac{\varepsilon^{2/3}}{3})m_{nr6} + \frac{\sqrt{27}}{2}m_{\neq 6} \leq \frac{\sqrt{27}}{2}(1 - \frac{\varepsilon^{2/3} m_{nr6}}{3m})m$$

and thus

$$\frac{\varrho^2 \pi m}{|S|} \geq \frac{2\pi}{\sqrt{27}}(1 + \frac{\varepsilon^{2/3} m_{nr6}}{3m}).$$

Since the density of our covering is less than

$$\frac{2\pi}{\sqrt{27}}(1+\varepsilon),$$

this yields (2.13).

2.3.4 Finally we estimate

$$m_{\mathrm{bd}} = \#\{D(c) : D(c) \text{ is a cell, } C(c,\varrho) \cap T = \emptyset\},$$

where T is the inner parallel set of S at distance 2ϱ:

(2.14) $m_{\mathrm{bd}} \leq 2\varepsilon m.$

Consider the $m_0 - m_{\mathrm{bd}}$ cells $D(c)$ where $C(c,\varrho) \cap T \neq \emptyset$, that is, $C(c,\varrho) \subset S$. Since then $D(c) \subset C(c,\varrho)$, it follows from (2.5) that the corresponding circles cover $T(\subset S)$. Thus L. Fejes Tóth's [**?**] theorem cited in 1.2 says that for the density of this family of circles we have that

$$\frac{\varrho^2 \pi (m_0 - m_{\mathrm{bd}})}{|T|} \geq \frac{2\pi}{\sqrt{27}}.$$

Noting that $p(S)$ is the perimeter of S, we obtain

$$|T| \geq |S| - 2\varrho p(S).$$

By assumption, the density of the given covering of S satisfies

$$\frac{\varrho^2 \pi m}{|S|} \leq \frac{2\pi}{\sqrt{27}}(1+\varepsilon).$$

Combining these three inequalities and (2.1) then yields (2.14):

$$\frac{\varrho^2 \pi (m - m_{\mathrm{bd}})}{|S| - 2\varrho p(S)} \geq \frac{\varrho^2 \pi (m_0 - m_{\mathrm{bd}})}{|T|} \geq \frac{2\pi}{\sqrt{27}} \geq \frac{\varrho^2 \pi m}{|S|}(1+\varepsilon)^{-1},$$

or

$$\frac{m_{\mathrm{bd}}}{m} \leq 1 - (1 - \frac{2\varrho p(S)}{|S|})(1-\varepsilon) \leq 1 - 1 + \frac{2\varrho p(S)}{|S|} + \varepsilon \leq 2\varepsilon.$$

2.3.5 Finally, (2.11), (2.12), (2.14) and (2.1) together yield (2.6):

$$m_{\neq 6} + m_{nr6} + m_{\mathrm{bd}} \leq (64\varepsilon + 3\varepsilon^{1/3} + 2\varepsilon)m \leq 4\varepsilon^{1/3} m.$$

2.4 Third, we choose from the set of good centers a still large set of centers having one additional property:

(2.15) Each $c \in C$, with a set of less than $50\varepsilon^{1/3}m$ exceptions, has the property that all centers in $C(c, 2\varrho) \cap C$ are good.

Such centers will be called *very good*.

By (2.6) the number of bad centers is less than $4\varepsilon^{1/3}m$. For each bad center consider all good centers at distance at most 2ϱ from it. For any of these good centers take the circle of radius $(\frac{\sqrt{3}}{2} - \varepsilon^{1/3})\varrho$ centered at it. Since these circles are contained in the corresponding cell (see (2.6)), they do not overlap. Comparing areas, we thus see that the number of good centers at distance at most 2ϱ from a given bad center is at most

$$\frac{(2 + \frac{\sqrt{3}}{2} - \varepsilon^{1/3})^2 \varrho^2 \pi}{(\frac{\sqrt{3}}{2} - \varepsilon^{1/3})^2 \varrho^2 \pi} \leq 11$$

by (2.1). Cancelling all bad centers and for each bad center at most 11 good centers amounts to the omission of less than

$$4\varepsilon^{1/3}m \cdot 12 \leq 50\varepsilon^{1/3}m$$

centers. Clearly, the remaining centers all are very good. The proof of (2.15) is complete.

2.2 We come to the final step of the proof.

(2.16) Let $c \in C$ be very good. Then there are $c_1, \ldots, c_6 \in C$ such that

(i) $|\|c_k - c\| - \sqrt{3}\varrho| \leq 22\varepsilon^{1/3}\varrho \leq 13\varepsilon^{1/3}\sqrt{3}\varrho$,

(ii) $|\|c_{k+1} - c_k\| - \sqrt{3}\varrho| \leq 500\varepsilon^{1/3}\sqrt{3}\varrho$,

(iii) $C(c, \frac{3}{2}\sqrt{3}\varrho) \cap C = \{c, c_1, \ldots, c_6\}$.

The definition of cells in 2.3, the fact that c is a very good and thus a good center, and propositions (2.6) and (2.3) show that there are centers $c_1, \ldots, c_6$, satisfying (2.16) (i), (ii). In addition we see that there are points $h_1, \ldots, h_6$ forming the vertices of a regular hexagon of edgelength $\sqrt{3}\varrho$ and center at o such that

(2.17) $\|c_k - (c + h_k)\| \leq 810\varepsilon^{1/3}\varrho \leq 500\varepsilon^{1/3}\sqrt{3}\varrho$.

For the proof of (2.16)(iii) we proceed as follows: the definition of cells in 2.2 and (2.6), (2.3) yield the following:

(2.18) Let d be a good center. Then $C(d, (\sqrt{3} - 2\varepsilon^{1/2})\varrho)$ contains no center except d.

The centers c_k satisfy (2.16)(i) and thus are contained in $C(c, 2\varrho)$. Since c is a very good center, each c_k is a good center, see (2.5). Clearly, c is good too. Now apply (2.18) to each of $c, c_1, \ldots, c_6$. Taking into account (2.17), we see that the union of the circles

$$C(c, (\sqrt{3} - 2\varepsilon^{1/3})\varrho), C(c + h_k, (\sqrt{3} - 812\varepsilon^{1/3})\varrho), \ k = 1, \ldots, 6,$$

contains precisely the centers $c, c_1, \ldots, c_6$. Noting (2.1), an elementary calculation then shows that the circle $C(c, \frac{3}{2}\sqrt{3}\varrho)$ is contained in this union. This proves (2.16)(iii), and thus concludes the proof of (2.16).

2.6 Having shown (2.15) and (2.16), the proof of Theorem 1 is complete.

3 Proof of Theorem 2

3.1 To make this article more self-contained, we repeat the relevant parts of the definitions in [**?**].

Let M be a *two-dimensional Riemannian manifold of class* $\mathcal{C}^2$ *with metric of class* $\mathcal{C}^0$. Then for any $p \in M$ there are a neighborhood U of p in M and a homeomorphism $h =$"$\,'\,$" of U onto an open solid circle $U' = h(U)$ in $\mathbb{E}^2$. To any $u \in U'$ there corresponds a positive quadratic form $q_u(s) = q_{p,u}(s)$ on $\mathbb{E}^2$ with continuous coefficients.

A curve in U is of class $\mathcal{C}^1$ if it has a parametrization $x : [\mu, \nu] \to U$ such that $u = h \circ x$ is of class $\mathcal{C}^1$. Its *length* is

$$\int_\mu^\nu q_{u(\tau)}(\dot{u}(\tau))d\tau.$$

If a curve is not contained in a single neighborhood, dissect it suitably. For $x, y \in M$, let $\gamma_M(x, y)$ be the infimum of the lengths of the curves of class $\mathcal{C}^1$ in M which connect x, y. The metric $\gamma_M(\cdot,\cdot)$ induces the original topology on M. The geodesic disc $D_M(c, \varrho)$ in M with center $c \in M$ and radius ϱ is the set $\{x \in M : \gamma_M(c, x) \le \varrho\}$.

A set $J \subset M$ is *Jordan measurable* if its closure cl$J$ is compact and for any p, U, h and any neighborhood V of p with cl$V \subset U$ for which V' is Jordan measurable in $\mathbb{E}^2$, also $(J \cap V)'$ is Jordan measurable in $\mathbb{E}^2$. If M is compact, then it is Jordan measurable and so are geodesic discs. Finite unions, intersections and differences of Jordan measurable sets are again Jordan measurable. If $J \subset U$ is Jordan measurable, then

$$\omega_M(J) = \int_{J'} (\det q_u)^{1/2} du \quad (du = du^1 du^2)$$

is its *Jordan area*; otherwise dissect J suitably. Clearly, ω_M may be extended to a Borel measure on M. Then, equivalently, a set in M may be defined Jordan measurable, if its closure is compact and its boundary has Borel measure 0.

3.2 Next, some tools are collected.

The definition of geodesic discs and the compactness of the closure of a Jordan measurable set yield the following well-known result:

(3.1) Let $K \subset M$ be Jordan measurable. Then $\omega_M(D_M(c, \varrho)) = \varrho^2\pi(1 + o(1))$ for $c \in K$ as $\varrho \to 0$, where $o(\cdot)$ may be chosen to be independent of c.

A consequence of Lemma 1 in [**?**] is the following:

(3.2) Let $K \subset M$ be Jordan measurable with $\omega_M(K) > 0$ and for $m = 1, 2, \dots,$ let $\sigma_m(K)$ be the minimum radius $\sigma > 0$ such that m suitable geodesic discs of radius σ cover K. Then

$$\frac{m\sigma_m(K)^2\pi}{\omega_M(K)} \to \frac{2\pi}{\sqrt{27}} \text{ as } m \to \infty.$$

3.3 Let J be the Jordan measurable set of Theorem 2 and let $\{D_M(c,\varrho_n) : c \in C_n\}$ be the corresponding coverings of J. This subsection contains two auxiliary results on J.

First,

(3.3) $\quad \sigma_n \leq \varrho_n \leq \sigma_n(1+o(1))$ as $n \to \infty$, where $\sigma_n = \sigma_n(J)$.

The left hand side inequality follows from the definition of σ_n (see (3.2)) and the assumption of Theorem 2. To see the right hand side inequality note that by the assumption of Theorem 2, and (3.1), (3.2), both applied for $K = J$,

$$\frac{n\varrho_n^2\pi}{\omega_M(J)} \to \frac{2\pi}{\sqrt{27}},\ \frac{n\sigma_n^2\pi}{\omega_M(J)} \to \frac{2\pi}{\sqrt{27}} \text{ as } n \to \infty.$$

Second,

(3.4) $\quad$ let $K \subset J$ be Jordan measurable with $\omega_M(K) > 0$ and let

$$n(K) = \#\{c \in C_n : D_M(c,\varrho_n) \cap K \neq \emptyset\}.$$

Then

$$\frac{n(K)\varrho_n^2\pi}{\omega_M(K)} \to \frac{2\pi}{\sqrt{27}} \text{ as } n \to \infty.$$

For assume not. Then the definition of $n(K)$ together with (3.2) shows that

$$\frac{n(K)\varrho_n^2\pi}{\omega_M(K)} \geq (1+\alpha)\frac{2\pi}{\sqrt{27}} \text{ for infinitely many } n,$$

where $\alpha > 0$ is chosen suitably. Next take a compact Jordan measurable set L in the interior $\mathrm{int}(J \backslash K)$ of $J \backslash K$ such that

$$(1+\alpha)\omega_M(K) + \omega_M(L) > \omega_M(J).$$

By our choice of L, the sets K and L have positive distance with respect to γ_M. Since $\varrho_n \to 0$ as $n \to \infty$ (see (3.3) and note that $\sigma_n \to 0$ by (3.2)), we thus have

$$n(K) + n(L) \leq n \text{ for sufficiently large } n,$$

by the definition of $n(\cdot)$ in (3.4). The definition of $n(L)$ and (3.2) imply:

$$\frac{n(L)\varrho_n^2\pi}{\omega_M(L)} \geq \frac{n(L)\sigma_{n(L)}(L)^2\pi}{\omega_M(L)} \to \frac{2\pi}{\sqrt{27}} \text{ as } n \to \infty.$$

Finally, (3.2) and (3.3) show that

$$\frac{n\varrho_n^2\pi}{\omega_M(J)} \to \frac{2\pi}{\sqrt{27}} \text{ as } n \to \infty.$$

We now apply these relations:

$$\frac{n\varrho_n^2\pi}{\omega_M(J)} \geq \frac{(n(K)+n(L))\varrho_n^2\pi}{\omega_M(J)} = \frac{n(K)\varrho_n^2\pi}{\omega_M(K)} \cdot \frac{\omega_M(K)}{\omega_M(J)} + \frac{n(L)\varrho_n^2\pi}{\omega_M(L)} \cdot \frac{\omega_M(L)}{\omega_M(J)},$$

and thus

$$\frac{2\pi}{\sqrt{27}} \geq (1+\alpha)\frac{2\pi}{\sqrt{27}}\frac{\omega_M(K)}{\omega_M(J)} + \frac{2\pi}{\sqrt{27}}\frac{\omega_M(L)}{\omega_M(J)},$$

which is a contradiction.

3.4 Let

(3.5) $0 < \varepsilon < 10^{-9}$,

(3.6) $\lambda > 1$ so small that $\lambda^7 < 1+\varepsilon, \quad 2(500\varepsilon^{1/3} + \lambda - 1)\lambda \leq \varepsilon^{1/4}, \quad \lambda - 1 < \varepsilon.$

Given $p \in M$, we may choose $U, h =$“ ′”, where U is so small that for $q = q_{p,p}$ the following hold:

$$\frac{1}{\lambda}q(x'-y')^{1/2} \leq \gamma_M(x,y) \leq \lambda q(x'-y')^{1/2} \quad \text{for } x,y \in U,$$
$$\frac{1}{\lambda}(\det q)^{1/2}|K'| \leq \omega_M(K) \leq \lambda(\det q)^{1/2}|K'| \quad \text{for Jordan measurable } K \subset U$$

(see section 2 in [**?**]). Let V be a Jordan measurable, open neighborhood of p with $\mathrm{cl}V \subset U$.

As p ranges over the compact set $\mathrm{cl}J$, the neighborhoods V form an open covering of $\mathrm{cl}J$. Thus there is a finite subcover. Hence we may choose points $p_l \in \mathrm{cl}J, l = 1, \dots, m$, say, and corresponding neighborhoods U_l, V_l, homeomorphisms $h_l =$“′”, and quadratic forms q_l, such that

(3.7) $\frac{1}{\lambda}q_l(x'-y')^{1/2} \leq \gamma_M(x,y) \leq \lambda q_l(x'-y')^{1/2}$ for $x,y \in U_l$,

(3.8) $\frac{1}{\lambda}(\det q_l)^{1/2}|K'| \leq \omega_M(K) \leq \lambda(\det q_l)^{1/2}|K'|$ for Jordan measurable $K \subset U_l$.

V_l is Jordan measurable and the inclusions $\mathrm{cl}V_l \subset \mathrm{int}U_l$, and $J = V_1 \cup \ldots \cup V_m$ hold. Clearly,

$$W_l = J \cap (V_l \backslash (V_1 \cup \ldots \cup V_{l-1})) \text{ is Jordan measurable,}$$
$$W_l \subset \mathrm{int}U_l, \text{ and } J \text{ is the disjoint union of } W_1, \dots, W_m.$$

Next choose sets $S_{li} \subset \mathrm{int}W_l \subset \mathrm{int}U_l, i = 1, \dots, i_l$, with the following properties:

(3.9) $S'_{li} \subset \mathrm{int}U'_l$ is a compact square; thus S_{li} is Jordan measurable,

(3.10) the sets S_{li} are pairwise disjoint,

(3.11) $\omega_M(T) < (\lambda - 1)\omega_M(J)$, where $T = J \backslash \bigcup_{l,i} S_{li}$.

3.5

(3.12) Let $C_{nli} = \{c \in C_n : D_M(c, \varrho_n) \cap S_{li} \neq 0\}$. Then for all sufficiently large n the $n(S_{li})$ geodesic discs $D_M(c, \lambda\varrho_n) : c \in C_{nli}$ are contained in U_l and the $n(S_{li})$ ellipses $\{s : q_l(s - c')^{1/2} \leq \lambda\varrho_n\} : c \in C_{nli}$ form a covering of S'_{li} of density less than

$$\frac{2\pi}{\sqrt{27}}(1 + \varepsilon).$$

Since S_{li} is compact, $S_{li} \subset \mathrm{int}U_l$, and $\varrho_n \to 0$ as $n \to \infty$, it follows from (3.7) that

$$\begin{aligned}(3.13) \qquad &\{s : q_l(s - c')^{1/2} \leq \frac{\varrho_n}{\lambda}\} \subset D_M(c, \varrho_n)' \subset \{s : q_l(s - c')^{1/2} \leq \lambda\varrho_n\} \\ &\subset \{s : q_l(s - c')^{1/2} \leq 2\lambda\varrho_n\} \subset U'_l \text{ for } c \in C_{nli} \text{ for all sufficiently large } n.\end{aligned}$$

The geodesic discs $D_M(c, \varrho_n) : c \in S_{nli}$ cover S_{nli}. Thus the ellipses $\{s : q_l(s - c')^{1/2} \leq \lambda\varrho_n\} : c \in C_{nli}$ cover S_{nli} by (3.13). We determine the density of this covering. By (3.8) and (3.13) we have:

$$\frac{1}{\lambda}(\mathrm{det}q_l)^{1/2}|\{s : q_l(s - c')^{1/2} \leq \frac{\varrho_n}{\lambda}\}| \leq \frac{1}{\lambda}(\mathrm{det}q_l)^{1/2}|D_M(c, \varrho_n)'| \leq \omega_M(D_M(c, \varrho_n)|$$
$$\text{for } c \in C_{nli} \text{ and all sufficiently large } n$$

and

$$\omega_M(S_{li}) \leq \lambda(\mathrm{det}q_l)^{1/2}|S'_{li}|.$$

Propositions (3.1), (3.4) yield,

$$\frac{\sum\{\omega_M(D_M(c, \varrho_n)) : c \in C_{nli}\}}{\omega_M(S_{nli})} < \frac{2\pi}{\sqrt{27}}\lambda \text{ for all sufficiently large } n.$$

(3.13) together with these inequalities and (3.6) gives the desired bound for the density of our covering of S'_{li} by ellipses:

$$\begin{aligned}&\frac{n(S_{li})|\{s : q_l(s - c')^{1/2} \leq \lambda\varrho_n\}|}{|S'_{li}|} \\ &= \frac{n(S_{nli})\frac{1}{\lambda}(\mathrm{det}q_l)^{1/2}|\{s : q_l(s - c')^{1/2} \leq \frac{\varrho_n}{\lambda}\}|}{\lambda(\mathrm{det}q_l)^{1/2}|S'_{li}|} \cdot \lambda^6 \\ &\leq \frac{\sum\{\omega_M(D_M(c, \varrho_n)) : c \in C_{nli}\}}{\omega_M(S_{li})} \cdot \lambda^6 < \frac{2\pi}{\sqrt{27}}\lambda^7 \leq (1 + \varepsilon)\frac{2\pi}{\sqrt{27}} \\ &\text{for all sufficiently large } n,\end{aligned}$$

concluding the proof of (3.12).

3.6 Next we show that with an obvious extension of terminology from $\mathbb{E}^2$ to M,

(3.14) for all sufficiently large n the following hold: each of the $n(S_{li})$ centers $c \in C_{nli}$, with a set of at most $50\varepsilon^{1/3}n(s_{li})$ exceptions, is contained in S_{li} and is the center of a regular hexagon with vertices $c_1, \ldots, c_6$, say, in C_{nli} which is regular up to $\varepsilon^{1/4}$ and has size $\sqrt{3}\varrho_n$. In addition, $D_M(c, 1.1\sqrt{3}\varrho_n) \cap C_n = \{c, c_1, \ldots, c_6\}$.

Consider the Euclidean norm $q_l^{1/2}$ in the plane. By (3.12) the $n(S_{li})$ circles (in the sense of the norm $q_l^{1/2}$) $\{s : q_l(s-c')^{1/2} \le \lambda\varrho_n\} : c \in C_{nli}$ form a covering of the square S'_{li} of density less than $\frac{2\pi}{\sqrt{27}}(1+\varepsilon)$ for all sufficiently large n. Thus by Theorem 1 and its proof we conclude that for all sufficiently large n all $n(S_{li})$ centers in C'_{nli}, with a set of less than $50\varepsilon^{1/3}n(S_{li})$ exceptions, are very good. Let $c \in C_{nli}$ be such that c' is very good and thus good.

By (2.6)(ii) and (3.13) $c' \in D_M(c, \varrho_n)' \subset \{s : q_l(s-c')^{1/2} \le \lambda\varrho_n\} \subset S'_{li}$ and thus

(3.15) $c \in D_M(c, \varrho_n) \subset S_{li}$.

Note (2.16). Let $c_1, \dots, c_6 \in C_{nli}$ be such that $c'_1, \dots, c'_6$ form a hexagon with enter c' which is regular up to $500\varepsilon^{1/3}$ of size $\sqrt{3}\lambda\varrho_n$ and such that

(3.16) $\{s : q_l(s-c')^{1/2} \le 2\lambda\varrho_n\} \cap C'_{nli} = \{c', c'_1, \dots, c'_6\}$.

Then

(3.17)
$$\begin{aligned}
&|\gamma_M(c_k, c) - \sqrt{3}\varrho_n| \\
&\quad \le |\gamma_M(c_k, c) - q_l(c'_k - c')^{1/2}| + |q_l(c'_k - c')^{1/2} - \sqrt{3}\lambda\varrho_n| + \sqrt{3}(\lambda - 1)\varrho_n \\
&\quad \le (\lambda - 1)q_l(c'_k - c')^{1/2} + 500\varepsilon^{1/3}\sqrt{3}\lambda\varrho_n + \sqrt{3}(\lambda - 1)\varrho_n \\
&\quad \le (\lambda - 1)(500\varepsilon^{1/3} + 1)\sqrt{3}\lambda\varrho_n + 500\varepsilon^{1/3}\sqrt{3}\lambda\varrho_n + \sqrt{3}(\lambda - 1)\varrho_n \\
&\quad \le ((\lambda - 1)(500\varepsilon^{1/3} + 1) + 500\varepsilon^{1/3} + \lambda - 1)\lambda\sqrt{3}\varrho_n \le \varepsilon^{1/4}\sqrt{3}\varrho_n, \\
&|\gamma_M(c_{k+1}, c_n) - \sqrt{3}\varrho_n| \le \dots \le \varepsilon^{1/4}\sqrt{3}\varrho_n \quad \text{for all sufficiently large } n
\end{aligned}$$

by (3.7) and (3.6). By (3.7) again and (3.13) we have,

(3.18) $D_M(c, 2\varrho_n)' \subset \{s : q_l(s-c')^{1/2} \le 2\lambda\varrho_n\}$.

(3.15), (3.17), (3.5), the definition of C_{nli} in (3.12), (3.18) and (3.16) together yield

$$\{c, c_1, \dots, c_6\} \subset D_M(c, 1.1\sqrt{3}\varrho_n) \cap C_n \subset D_M(c, 2\varrho_n) \cap C_{nli} \subset \{c, c_1, \dots, c_6\}$$
for all sufficiently large n.

The proof of (3.14) is complete.

3.7 Since $\varrho_n \to 0$ as $n \to \infty$ and the compact sets S_{li} are disjoint by (3.10), the definition of the sets C_{nli} in (3.12) shows that the sets C_{nli} are disjoint if n is sufficiently large. Since $n(S_{li}) = \#C_{nli}$, we thus see that

(3.19) $\sum_{l,i} n(S_{li}) \le n$ for all sufficiently large n.

In order to prove that

(3.20) $n(T) \le \varepsilon n$ for all sufficiently large n

note that

$$\frac{n(T)}{n} = \frac{n(T)\varrho_n^2\pi}{\omega_M(T)} \cdot \frac{\omega_M(J)}{n\varrho_n^2\pi} \cdot \frac{\omega_M(T)}{\omega_M(J)} < \lambda - 1 \text{ for all sufficiently large } n$$

by (3.4) and (3.11). Now apply (3.6).

3.8 Combining (3.14), (3.19), (3.20) and noting (3.6), the following is obtained:

(3.21) Let $\varepsilon > 0$ be sufficiently small. Then for all sufficiently large n hold: each $c \in C_n$, with a set of at most $51\varepsilon^{1/3}n$ exceptions, is the center of a hexagon in C_n which is regular up to $\varepsilon^{1/4}$, has size $\sqrt{3}\varrho_n$ and $D_M(c, 1.1\sqrt{3}\varrho_n) \cap C_n$ consists precisely of c and the vertices of the hexagon.

Let k be so large that for $\varepsilon = j^{-12}, j = k, k+1, \dots,$ proposition (3.21) holds for $n \geq n(j)$, say. We clearly may assume that $n(j) \to \infty$ as $j \to \infty$. Now define $f, g : \mathbb{N} \to \mathbb{R}^+$ by

$$f(n) = n,\ g(n) = 1 \text{ for } 1 \leq n \leq n(k),$$
$$f(n) = \frac{51}{j^4}n,\ g(n) = \frac{1}{j^3} \text{ for } n(j) \leq n < n(j+1), j = k, \dots .$$

Then

each $c \in C_n$, with a set of at most $f(n)$ exceptions, is the center of a hexagon in C_n which is regular up to $g(n)$, has size $\sqrt{3}\varrho_n$ and $D_M(c, 1.1\sqrt{3}\varrho_n) \cap C_n$ consists precisely of c and the vertices of the hexagon.

Since $f(n) = o(n), g(n) = o(1)$ as $n \to \infty$, this concludes the proof of Theorem 2.

4 Proof of Theorem 3

4.1 The following is the proof for P_n.

Let $M = \mathrm{bd}\, C$ be endowed with the Riemannian metric γ_C of the second fundamental form and let ω_C be the corresponding Jordan area measure. Then $\mathrm{bd}\, C$ is a Riemannian manifold of dimension two of class $\mathcal{C}^2$ with metric of class $\mathcal{C}^0$.

The following proposition was first proved by Schneider [?]. Under the present assumptions it is due to the author [?], (5.2).

(4.1) For $n = 4, 5, \dots,$ let ϱ_n be the minimum radius such that the geodesic discs $D_C(v, \varrho_n) : v \in \mathrm{vert} P_n$ cover $\mathrm{bd}\, C$. Then

$$\varrho_n = (2\delta^H(C, P_n))^{1/2}(1 + o(1)) \text{ as } n \to \infty.$$

Conversely, if for $n = 4, 5, \dots, \sigma_n$ is the minimum radius such that n suitable geodesic discs $D_C(w_1, \sigma_n), \dots, D_C(w_n, \sigma_n)$, say, cover $\mathrm{bd}\, C$, then

$$\delta^H(C, \mathrm{conv}\{w_1, \dots, w_n\}) \leq \frac{\sigma_n^2}{2}(1 + o(1)) \text{ as } n \to \infty.$$

(By conv the convex hull is meant.) As a consequence of (4.1) we have,

(4.2) $\sigma_n \leq \varrho_n \leq \sigma_n(1 + o(1))$ as $n \to \infty$.

Applying propositions (3.1) and (3.2) with $K = M = \mathrm{bd}\, C$, yields the following:

(4.3) $\omega_C(D_C(v, \varrho_n)) = \varrho_n^2\pi(1 + o(1))$ as $n \to \infty$, where $o(1)$ is independent of v, (4.4) $\quad \dfrac{n\sigma_n^2\pi}{\omega_C(\operatorname{bd} C)} \to \dfrac{2\pi}{\sqrt{27}}$ as $n \to \infty$.

Combining (4.1) – (4.4) shows that

> the density of the covering of $\operatorname{bd} C$ by the geodesic discs $D_C(v, \varrho_n) : v \in \operatorname{vert} P_n$ converges to $2\pi/\sqrt{27}$ as $n \to \infty$.

An application of Theorem 2 then yields Theorem 3 for P_n.

4.2 The proofs for $Q_n, \ldots, T_n$, in essence, are the same, where in place of (4.1) use is made of analogous propositions; see [**?**] (5.4), [**?**] (3.5), (3.6), [**?**] (3.11), (3.13) and [**?**].

Final remark

In a subsequent article best approximating polytopes with respect to the symmetric difference metric will be considered. The result is based on a stability version of L. Fejes Tóth's [**?**], p. 81, so-called moment lemma which has also other applications.

Acknowledgements

In the course of the proof of Theorem 3 I was looking for a stability version of L. Fejes Tóth's circle covering result cited in 1.2 and asked his son G. Fejes Tóth for references. The latter informed me that, surprisingly, no such results were known and suggested to prove a stability theorem using ideas from the classical proofs of L. Fejes Tóth. This led to Theorem 1. Theorem 2 was the natural extension needed for the proof of Theorem 3. I am obliged to G. Fejes Tóth for his advice and to F. J. Schnitzer for his many helpful remarks. Part of this article was written while the author visited the Mathematical Sciences Research Institute at Berkeley in April 1996. Thanks for this invitation are due to Professors Milman and Thurston.

References

[1] Fejes Tóth,G., Kuperberg,W.: Packing and covering with convex sets. In: P.M.Gruber, J.M.Wills, eds.: Handbook of convex geometry B, 799–860. North-Holland, Amsterdam 1993

[2] Fejes Tóth,L.: Lagerungen in der Ebene, auf der Kugel und im Raum. 2nd ed., Springer-Verlag, Berlin – Göttingen – Heidelberg 1972

[3] Glasauer,S., Schneider,R.: Asymptotic approximation of smooth convex bodies by polytopes. Forum Math. **8** (1996) 363–377

[4] Groemer,H.: Stability of geometric inequalities. In: P.M.Gruber, J.M.Wills, eds.: Handbook of convex geometry A, 125–150. North-Holland, Amsterdam 1993

[5] Gruber,P.M.: Asymptotic estimates for best and stepwise approximation of convex bodies I. Forum Math. **5** (1993) 281–297

[6] Gruber,P.M.: Aspects of approximation of convex bodies. In: P.M.Gruber, J.M.Wills, eds.: Handbook of convex geometry A, 319–345. North-Holland, Amsterdam 1993

[7] Gruber,P.M.: Stability of Blaschke's characterization of ellipsoids and Radon norms, Discrete Comput. Geometry **17** (1997) 411–427

[8] Kershner,R.: The number of circles covering a set. Amer. Math. J. **61** (1939) 578–581

[9] Ludwig,M.: Asymptotic approximation of convex curves. Arch. Math. **63** (1994), 377–384

[10] McClure,D.E., Vitale,R.A.: Polygonal approximation of plane convex bodies. J. Math. Anal. Appl. **51** (1975) 326–358

[11] Schneider,R.: Zur optimalen Approximation konvexer Hyperflächen durch Polyeder. Math. Ann. **256** (1981) 289–301

[12] Schneider,R.: Affine-invariant approximation by convex polytopes. Studia Sci. Math. Hungar. **21** (1986) 401–408

[13] Schneider,R.: Convex bodies: the Brunn–Minkowski theory. Cambridge Univ. Press, Cambridge 1993

Peter M. Gruber, Abteilung für Analysis, Technische Universität Wien, Wiedner Hauptstraße 8–10/1142, A-1040 Vienna, Austria, pmgruber@pop.tuwien.ac.at