Error of Asymptotic Formulae for Volume Approximation of Convex Bodies in $\mathbb{E}^d$

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Abstract. We estimate the error of asymptotic formulae for volume approximation of sufficiently differentiable convex bodies by circumscribed convex polytopes as the number of facets tends to infinity. Similar estimates hold for approximation with inscribed and general polytopes and for vertices instead of facets. Our result is then applied to estimate the minimum isoperimetric quotient of convex polytopes as the number of facets tends to infinity.

Mathematics Subject Classification 2000. 52A20, 52A21, 52A27, 52A38, 52A40; 53A15.

Dedicated to Professor Edmund Hlawka on the occasion of his 85[th] birthday.

1 Introduction and Statement of Results

1.1 Let C be a *convex body* in Euclidean d-space $\mathbb{E}^d$, that is a compact convex subset of $\mathbb{E}^d$ with non-empty interior and let $\delta(\cdot,\cdot)$ be a metric or some other measure of distance on the space of all convex bodies in $\mathbb{E}^d$. For $n = d+1, d+2, \ldots$, consider a family $\mathcal{P}_n$ of convex polytopes in $\mathbb{E}^d$, for example the families of all convex polytopes with n facets, k-faces or vertices, respectively, which may or may not be circumscribed or inscribed to C.

A main goal is to determine the quantity

$$\delta(C, \mathcal{P}_n) = \inf\{\delta(C, P) : P \in \mathcal{P}_n\}$$

and to describe the polytopes for which equality holds, the *best approximating polytopes* of C in $\mathcal{P}_n$. With the exception of trivial cases, such a goal is out of reach. For the metrics and other measures of distance commonly used in convex geometry, upper estimates for $\delta(C, \mathcal{P}_n)$ of the right order are comparatively easy to obtain. The proofs of asymptotic formulae for $\delta(C, \mathcal{P}_n)$ as $n \to \infty$ for sufficiently differentiable convex bodies, are more difficult. It is highly plausible that under additional differentiability assumptions one can obtain information on the error of the asymptotic formulae and even extend the asymptotic formulae to asymptotic series. For selected references see below. This topic, including initial results on the form of the best approximating polytopes, is surveyed in [12, 14].

1.2 In this article we will consider the symmetric difference metric $\delta^V(\cdot,\cdot)$ and the family $\mathcal{P}^c_{(n)} = \mathcal{P}^c_{(n)}(C)$ of all convex polytopes with at most n facets and circumscribed to C.

In [11, 13] it was shown that for C (the boundary of which is) of class $\mathcal{C}^2$ with Gauss curvature $\kappa_C > 0$, we have

$$(1) \qquad \delta^V(C,\mathcal{P}^c_{(n)}) \sim \frac{1}{2}\,\mathrm{div}_{d-1} A(C)^{\frac{d+1}{d-1}} \frac{1}{n^{\frac{2}{d-1}}} \text{ as } n \to \infty.$$

Here $A(C)$ or $A(\mathrm{bd}\,C)$ is the *equi-affine surface area* measure of (the boundary $\mathrm{bd}\,C$ of) C,

$$(2) \qquad A(C) = \int\limits_{\mathrm{bd}\,C} \kappa_C(x)^{\frac{1}{d+1}} d\sigma(x),$$

where σ is the ordinary surface area measure in $\mathbb{E}^d$. div_{d-1} is a constant introduced in [13], depending only on d. The only explicitly known values are $\mathrm{div}_1 = 1/12$ and $\mathrm{div}_2 = 5/18\sqrt{3}$. The case $d = 2$ of (1) was settled before by Fejes Tóth [8]; for an alternative proof see [23]. Fejes Tóth [8] also conjectured the case $d = 3$. Böröczky [3] showed that the assumption $\kappa_C > 0$ can be omitted.

For $d = 2$ Ludwig [22] specified the second term of an asymptotic series development of $\delta^V(C,\mathcal{P}^c_{(n)})$. The complete series was given by Tabachnikov [27] in the form of a result on periodic trajectories of the "dual" or "exterior" billiard determined by C.

For $d = 3$ and C of class $\mathcal{C}^3$ with $\kappa_C > 0$ the author [18] proved that

$$\delta^V(C,\mathcal{P}^c_{(n)}) = \frac{5A(C)^2}{36\sqrt{3}n} + O\Big(\frac{1}{n^{1+\frac{1}{4}}}\Big) \text{ as } n \to \infty,$$

using techniques which, at present, are available only for $d = 3$. Böröczky [4] informs us that for $C = B^3$, the solid Euclidean unit ball in $\mathbb{E}^3$, the error of the asymptotic formula for $\delta^V(B^3,\mathcal{P}^i_n)$ has a lower bound of the form $f(n)/n^2$ where $f(n) \to \infty$ as $n \to \infty$. Here $\mathcal{P}^i_n$ is the family of all convex polytopes with n vertices inscribed to B^3. This makes it plausible that under suitable differentiability assumptions on C, the asymptotic series for $\delta^V(C,\mathcal{P}^c_{(n)})$ in case $d = 3$ should have the form

$$\delta^V(C,\mathcal{P}^c_{(n)}) = \frac{5A(C)^2}{36\sqrt{3}n} + \frac{A_2(C)}{n^{\frac{3}{2}}} + \dots$$

with appropriate coefficients $A_2(C)$, etc.

For general d Böröczky [2] proved that for C of class $\mathcal{C}^3$ with $\kappa_C > 0$,

$$\delta^V(C,\mathcal{P}^c_{(n)}) = \frac{1}{2}\,\mathrm{div}_{d-1} A(C)^{\frac{d+1}{d-1}} \frac{1}{n^{\frac{2}{d-1}}} + O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{1}{8d^2}}}\Big) \text{ as } n \to \infty.$$

1.3 The aim of this article is to show the following result.

Theorem. *Let C be a convex body in $\mathbb{E}^d$ of class $\mathcal{C}^3$ with Gauss curvature $\kappa_C > 0$ and equi-affine surface area $A(C)$. Then, given $\varepsilon > 0$,*

$$\delta^V(C, \mathcal{P}^c_{(n)}) = \frac{1}{2}\operatorname{div}_{d-1} A(C)^{\frac{d+1}{d-1}} \frac{1}{n^{\frac{2}{d-1}}} + O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{1}{3(d-1)}-\varepsilon}}\Big) \text{ as } n \to \infty. \tag{3}$$

Actually, a slightly stronger result will be given, see (68) and (82). Scrutiny of the proof shows that it is sufficient to assume that C is of class $\mathcal{C}^2$ with Lipschitz second derivatives, as in Böröczky [2].

We remark that formulae of the type (3) also hold for the mean width distance and L^p metrics, for families of inscribed and general polytopes and with vertices instead of facets. The proofs are, in essence, the same.

We conjecture that under suitable differentiability assumptions on C there is an asymptotic series for $\delta^V(C, \mathcal{P}^c_{(n)})$ of the form

$$\delta^V(C, \mathcal{P}^c_{(n)}) = \frac{1}{2}\operatorname{div}_{d-1} A(C)^{\frac{d+1}{d-1}} \frac{1}{n^{\frac{2}{d-1}}} + A_2(C)\frac{1}{n^{\frac{3}{d-1}}} + \cdots \text{ as } n \to \infty,$$

and, similarly, in the other cases.

1.4 The volume approximation of B^d by polytopes in $\mathcal{P}^c_{(n)}(B^d)$ has a natural application to the isoperimetric problem for convex polytopes with n facets. Early contributions of this type are Minkowski's [24] proof of a classical theorem of Lindelöf [21] and results of Fejes Tóth [7], see the extensive survey [9].

Our approximation result above can be applied in the following more general context: let $\mathbb{E}^d$ be endowed with a further norm. A natural choice for "volume" in the normed space thus obtained is the ordinary volume $V(\cdot)$. For the notion of "surface area" several natural definitions have been proposed by Busemann, Benson and Holmes-Thompson. These amount to the introduction of a convex body I with center at the origin o, a so-called *isoperimetrix* of the normed space. The *surface area* $S_I(C)$ of the convex body C then is defined by

$$S_I(C) = \lim_{\varepsilon \to +0} \frac{V(C + \varepsilon I) - V(C)}{\varepsilon}, \text{ where } C + \varepsilon I = \{x + \varepsilon y : x \in C, y \in I\}.$$

For more information and for references we refer to the book [28].

Assume now that P_n is a convex polytope in $\mathbb{E}^d$ having at most n facets and with minimum *isoperimetric quotient* $S_I(P_n)^d/V(P_n)^{d-1}$ among all convex polytopes in $\mathbb{E}^d$ with at most n facets. A result of Diskant [5] shows that after applying a suitable homothety to P_n, we may assume that P_n is circumscribed to I, i.e. $P_n \in \mathcal{P}^c_{(n)}(I)$. (Note that this is related to Wulff's [30] theorem on the form of crystals and that Lindelöf's theorem cited before is the Euclidean case of Diskant's result.) The definition of $S_I(P_n)$ now shows that $S_I(P_n)^d/V(P_n)^{d-1} = d^d V(P_n)$. Since the isoperimetric quotient is minimum, $P_n \in \mathcal{P}^c_{(n)}(I)$ is then a best approximating circumscribed polytope of I. Using this, it was shown in

[16, 17, 18] that in case $d = 3$ the polytope P_n has "asymptotically regular hexagonal facets" and there is an asymptotic formula for $S_I(P_n)^3/V(P_n)^2$ as $n \to \infty$ with a rather precise estimate of the error.

As a consequence of our Theorem we have the following result for general d.

Corollary. *Let I be an isoperimetrix in $\mathbb{E}^d$ related to a norm. Assume that I is of class $\mathcal{C}^3$ with Gauss curvature $\kappa_C > 0$ and equi-affine surface area $A(C)$. For $n = d+1, d+2, \dots,$ let $P_n \in \mathcal{P}^c_{(n)}(I)$ be a convex polytope with minimum isoperimetric quotient $S_I(P_n)^d/V(P_n)^{d-1}$. Then, for any $\varepsilon > 0$,*

$$\frac{S_I(P_n)^d}{V(P_n)^{d-1}} = d^d V(I) + \frac{d^d}{2} \operatorname{div}_{d-1} A(I)^{\frac{d+1}{d-1}} \frac{1}{n^{\frac{2}{d-1}}} + O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{1}{3(d-1)}-\varepsilon}}\Big) \text{ as } n \to \infty.$$

2 Proof of the Theorem

We first introduce needed notation. Let bd, int, relbd, relint, conv, diam, width, det, grad, #, $\|\cdot\|$, $|\cdot|$, $V(\cdot)$, S^{k-1} and B^k stand for boundary and interior, boundary and interior relative to the affine hull or the boundary of C, convex hull, diameter, minimum width (of a convex set), discriminant (of a positive definite quadratic form), gradient, cardinal number, Euclidean norm, $(d-1)$-dimensional (in one case $(d-2)$-dimensional) volume or ordinary surface area measure, volume, Euclidean unit sphere and solid unit ball in $\mathbb{E}^k$, respectively.

In slight contrast with the use above, we denote by $O(t)$ a function of the form αt for $t \geq 0$ where α is a positive constant and similarly for the other Landau symbols. If $O(\cdot)$ appears several times in some chain of inequalities, this does not mean that it denotes necessarily the same function. In general, small Greek letters denote constants. Unless stated otherwise, these constants and the constants in the Landau symbols depend only on C, possibly also on d or ε. In order not to run out of Greek letters, we use in some cases the same letter to denote different constants. This should cause no ambiguities.

When speaking of *squares, parallelograms* and *circular discs*, we mean cubes, parallelotopes and solid Euclidean balls of dimension $d-1$.

For $p \in \operatorname{bd} C$ let H_p be the (unique) support hyperplane of C at p. Support halfspaces are denoted H^+, where H is a support hyperplane. For terminology not explained we refer to [26].

2.1 Preliminaries

2.1.1 Metric Projection. The mapping "π" which maps each point x of $\mathbb{E}^d$ onto its unique nearest point x^π of C is the *metric projection* of $\mathbb{E}^d$ onto C. The following properties of $^\pi$ are well known:

(4) $\|x^\pi - y^\pi\| \leq \|x - y\|$ for $x, y \in \mathbb{E}^d$.

(5) Let $z \in \operatorname{bd} C$. Then $\{x \in \mathbb{E}^d : x^\pi = z\}$ is the exterior normal of $\operatorname{bd} C$ at z.

(6) $|M^\pi| \le |M|$ for any measurable set M in a convex polytopal or smooth surface in $\mathbb{E}^d$.

(7) Let Q be a convex polytope in $\mathbb{E}^d$ containing C. Then $^\pi$ maps $\operatorname{bd} Q$ homeomorphically onto $\operatorname{bd} C$.

Blaschke's rolling theorem and its dual say the following, see [1], § 24, [20], sect. 2: Since C is of class $\mathcal{C}^3$ with Gauss curvature $\kappa_C > 0$, there are constants $\varrho, \sigma > 0$ (depending only on C) such that for each $p \in \operatorname{bd} C$ there are a translate of ϱB^d contained in C and a translate of σB^d containing C, both with boundary point p. This implies the following two propositions:

(8) There are constants $\alpha > 0, \beta > 1$ such that for any $p \in \operatorname{bd} C$ and $x \in H_p$ with $\|x - p\| \le \alpha$ or $\|x^\pi - p\| \le \alpha$ hold

$$\frac{1}{\beta}\|x-p\|^2 \le \|x - x^\pi\| \le \beta \min\{\|x-p\|^2, \|x^\pi - p\|^2\}.$$

(9) There are constants $\gamma, \zeta > 0$ such that for any point $x \in \mathbb{E}^d \backslash C$ with $\|x - x^\pi\| \le \gamma$ the volume of the compact convex cone with apex x and base $\operatorname{conv}(\{x\} \cup C) \cap H_{x^\pi}$ is at least $\zeta \|x - x^\pi\|^{(d+1)/2}$.

A local version of *Steiner's formula for the volume of parallel bodies of a convex body* (see [26], ch. 4) yields the next remark:

(10) There is a constant $\eta > 0$ such that for any measurable set $M \subset \operatorname{bd} C$ the *local parallel set* $M_t = \{x \in \mathbb{E}^d : x^\pi \in M, \|x - x^\pi\| \le \eta\}$ is measurable and

$$V(M_t) \left\{ \begin{array}{c} \le 2t|M| \\ \ge t|M| \end{array} \right\} \text{ for } 0 \le t \le \eta.$$

2.1.2 Dissection of bd C. By a *dissection* of a measurable set M we mean a finite family of measurable subsets of M with boundaries of measure 0 such that M is the union of these subsets and any two distinct subsets have at most boundary points in common.

As a preparatory result we consider dissections of S^{d-1}:

(11) There is a constant $\vartheta > 1$ depending only on d with the following property: for $l = 1, 2, \dots,$ the sphere S^{d-1} can be dissected into l spherically convex sets $A_i, i = 1, \dots, l,$ such that A_i contains a cap of S^{d-1} of spherical radius $1/\vartheta l^{1/(d-1)}$ and is contained in a concentric cap of spherical radius $\vartheta / l^{1/(d-1)}$.

For the proof of (11) we may assume that $l \ge 2d2^{d-1}$. Choose an integer $m \ge 3$ such that $2d(m-1)^{d-1} \le l < 2dm^{d-1}$. Let K be a cube of edge length 2 circumscribed to S^{d-1}. Dissect each of the $2d$ facets $K_i, i = 1, \dots, 2d,$ of K into

m^{d-1} squares of edge length $2/m$. Order these m^{d-1} squares starting with a square containing a vertex of K_i. Then take the adjacent squares in any order. Next take the adjacent squares of the latter in any order, etc. For $i = 1, \dots, 2d$ consider the centers of the first l_i squares in K_i where l_i is chosen such that $(m-1)^{d-1} \le l_i \le m^{d-1}$ and $l_1 + \ldots + l_{2d} = l$. This gives a set of l points on $\mathrm{bd}\, K$ such that any two distinct points have distance at least $2/m \ge 1/l^{1/(d-1)}$ and any point on $\mathrm{bd}\, K$ has distance at most $3\sqrt{d-1}/m \le 12\sqrt{d-1}/l^{1/(d-1)}$ from the nearest point in this set. Here "distance" means Euclidean distance measured in $\mathrm{bd}\, K$. The radial projection of $\mathrm{bd}\, K$ onto S^{d-1} and its inverse both are Lipschitz with Lipschitz constants depending only on d. Thus there is a constant $\vartheta > 1$ depending only on d, such that the radial projection of our set of l points on $\mathrm{bd}\, K$ into S^{d-1} is a set of l points on S^{d-1} with the following properties: any two distinct of its points have spherical distance at least $2/\vartheta l^{1/(d-1)}$ and for any point on S^{d-1} the spherical distance to the nearest point of this set is at most $\vartheta/l^{1/(d-1)}$. To conclude the proof of (11) take for $A_i, i = 1, \dots, l$, the Dirichlet–Voronoi cells on S^{d-1} of the points of this set, using spherical distance.

Assume from now on that

$$o \in \mathrm{int} C.$$

For $p \in \mathrm{bd}\, C$ let H^p be a hyperplane which intersects the ray $I\!R^+ p$ orthogonally, but does not meet C. Let "$\,'\,$" denote the orthogonal projection into H^p. Choose a Cartesian coordinate system in H^p with origin p'. Together with the normal unit vector of H^p pointing to C it forms a *Cartesian coordinate system corresponding to p*. When we speak of the "lower side" of C, of "below" or "above" this is meant with respect to the last coordinate.

Next it will be shown that

(12) there is a constant $\iota > 1$ such that for all sufficiently large l the following hold: there are a dissection $C_i, i = 1, \dots, l$, of $\mathrm{bd}\, C$ into l sets and for each i a point $p_i \in C_i$ such that for a corresponding Cartesian coordinate system the projection C_i' is convex and

$$\frac{1}{\iota l^{\frac{1}{d-1}}} B^{d-1} \subset C_i' \subset \frac{\iota}{l^{\frac{1}{d-1}}} B^{d-1} \quad (\subset \frac{1}{3} C').$$

The radial projection of S^{d-1} onto $\mathrm{bd}\, C$ and its inverse both are Lipschitz with Lipschitz constants depending only on C where "distance" in $\mathrm{bd}\, C$ is Euclidean distance measured in $\mathrm{bd}\, C$. Similarly, for any $p \in \mathrm{bd}\, C$ the orthogonal projection of the piece on the lower side of C over $\frac{1}{2}C' (\subset H^p)$ onto $\frac{1}{2}C'$ and its inverse both are Lipschitz. These Lipschitz constants have an upper bound depending only on C (and not on the individual p). These remarks together with (11) yield (12).

2.1.3 Representation of bd C. We will apply (12). Let $\iota > 1$ be chosen as in (12) and let l be sufficiently large. For each i consider a Cartesian coordinate system corresponding to p_i and represent the lower side of C in the form

$$\{(s, f_i(s)) : s \in C'\},$$

where f_i is a suitable convex function. Since C is of class $\mathcal{C}^3$, a version of a remark of Schneider [25], (7), then shows that

(13) $f_i|\operatorname{relint} C'$ is a convex function of class $\mathcal{C}^3$ and there is a constant $\kappa > 0$ such that $|f_{i;j}|, |f_{i;jk}|, |f_{i;jkm}|, (1 + (\operatorname{grad} f_i)^2)^{\frac{1}{2}} \leq \kappa$ on $\frac{1}{2}C'$.

Here $f_{i;j}$, $f_{i;jk}$, $f_{i;jkm}$ are first, second and third partial derivatives of f_i. To each $u \in \frac{1}{2}C'$ we let correspond the positive definite quadratic form q_{iu} on $\mathbb{E}^{d-1}$ defined by

$$q_{iu}(s) = \sum_{j,k} f_{i;jk}(u) s^j s^k \text{ for } s = (s^1, \ldots, s^{d-1}) \in \mathbb{E}^{d-1}, u \in \frac{1}{2}C'.$$

(The positive definiteness of q_{iu} is a consequence of the assumption that $\kappa_C > 0$; see also the following remarks.) From Blaschke's rolling theorem and its dual (see 2.1.1) we deduce that

(14) there is a constant $\lambda > 1$ such that

$$\frac{1}{\lambda} \leq \frac{q_{iu}(s)}{\|s\|^2} \leq \lambda \text{ for } s \in \mathbb{E}^{d-1}, \neq o,\ u \in \frac{1}{2}C',$$

$$\frac{1}{\lambda} \leq (\det q_{iu})^{\frac{1}{d+1}} \leq \lambda \text{ for } u \in \frac{1}{2}C'.$$

If x is on the lower side of C and $x' = u \in \frac{1}{2}C'$, we write $\kappa_C(u)$ instead of $\kappa_C(x)$. Then

(15) $$\kappa_C(u) = \frac{\det q_{iu}}{(1 + (\operatorname{grad} f_i(u))^2)^{\frac{d+1}{2}}} \text{ for } u \in \frac{1}{2}C'.$$

(16) $\kappa_C(\cdot)$ is of class $\mathcal{C}^1$ and has bounded partial derivatives on $\frac{1}{2}C'$ where the bound depends only on C.

2.1.4 Inequalities and Infinite Products. The following versions of *Hölder's inequality* will be needed:

(17) $$\sum_i \frac{1}{a_i^{\frac{2}{d-1}}} \geq n^{\frac{d+1}{d-1}} \frac{1}{(\sum_i a_i)^{\frac{2}{d-1}}} \text{ for } a_1, \ldots, a_n > 0,$$

(18) $$\sum_i a_i^{\frac{d+1}{d-1}} b_i \geq (\sum_i a_i b_i)^{\frac{d+1}{d-1}} \frac{1}{(\sum_i b_i)^{\frac{2}{d-1}}} \text{ for } a_1, b_1, \ldots, a_n, b_n > 0,$$

compare [6] or [19], sect. 2.7, 2.8. For the next remark see [6] or [29], sect. 1.4:

(19) let $a, \varepsilon > 0$. Then

$$(1 + \frac{a}{t})(1 + \frac{a}{t^{1+\varepsilon}})(1 + \frac{a}{t^{(1+\varepsilon)^2}}) \ldots \sim 1 + \frac{a}{t} \text{ as } t \to \infty.$$

2.2 Facets and Dirichlet–Voronoi Cells

For $n = d+1, d+2, \dots$, let $P_n \in \mathcal{P}^c_{(n)}$ be a best approximating polytope of C.

2.2.1 Diameters of the Facets of $\boldsymbol{P_n}$. We will show that

(20) there is a constant $\mu > 0$ such that $\max\{\operatorname{diam} F : F \text{ facet of } P_n\} < \dfrac{\mu}{n^{\frac{1}{d-1}}}$.

Since the proof of (20) is rather long it will be divided into several steps.

First, we make some preparations. It is not too difficult to prove the following complement of (6):

(21) Let $0 < \nu < 1$. Then for all sufficiently large n holds: $\nu|M| \leq |M^\pi|$ for any measurable set M in $\operatorname{bd} P_n$.

The next proposition is a consequence of (8) (consider the cases $\|x-p\| \leq \alpha$ and $\|x-p\| \geq \alpha$ separately):

(22) For any $\xi > 0$ there is $\varrho > 0$ such that for all sufficiently large n we have the following: for each $p \in \operatorname{bd} C$ and $x \in H_p$ with $\|x-p\| \geq \varrho/n^{1/(d-1)}$ holds
$$\|x - x^\pi\| \geq \frac{\xi}{n^{2/(d-1)}}.$$

Now the following will be shown:

(23) Let $\varrho > 0$ and let $\sigma > \varrho$ be so large that $((\sigma+\varrho)/(\sigma-\varrho))^{d-1} \leq 5/4$. Then for any compact convex set F in $\mathbb{E}^{d-1}$ with $o \in F$ and $\operatorname{diam} F \geq 2\sigma$,
$$|\{x \in F : \|x\| \geq \varrho\}| \geq \frac{3}{4}|F|.$$

Let $r \in F$ have maximum distance from o. Since $o \in F$ and $\operatorname{diam} F \geq 2\sigma$, we have that $\|r\| \geq \sigma$. Let L be the line through o and r and let H be the $(d-2)$-dimensional plane in $\mathbb{E}^{d-1}$ orthogonal to L which supports ϱB^{d-1} and separates it from r. Consider the unbounded convex cone with apex r generated by $F \cap H$. The convexity of F then shows that the part of the cone between r and H is contained in F and the part between H and $-H$ contains $F \cap \varrho B^{d-1}$. Thus,

$$\begin{aligned}
&|F| \geq (\|r\| - \varrho)|F \cap H|\frac{1}{d-1},\\
&|F \cap \varrho B^{d-1}| \leq (\|r\| - \varrho)|F \cap H|\frac{1}{d-1}\Big(\Big(\frac{\|r\|+\varrho}{\|r\|-\varrho}\Big)^{d-1} - 1\Big)\\
&\quad \leq |F|\Big(\Big(\frac{\sigma+\varrho}{\sigma-\varrho}\Big)^{d-1} - 1\Big) \leq \frac{|F|}{4},
\end{aligned}$$

concluding the proof of (23). Here $|F \cap H|$ means the $(d-2)$-dimensional volume of $F \cap H$. Clearly, (23) implies the following proposition:

(24) Let $\varrho > 0$. Then there is $\sigma > \varrho$ such that for any $p \in \operatorname{bd} C$ and any compact convex set F in H_p with $p \in F$ and $\operatorname{diam} F \geq 2\sigma/n^{1/(d-1)}$ holds

$$|\{x \in F : \|x - p\| \geq \frac{\varrho}{n^{\frac{1}{d-1}}}\}| \geq \frac{3}{4}|F|.$$

Second, after these preparations, we first show a weaker version of (20):

(25) There is a constant $\varsigma > 0$ such that for each n there is a facet F_n of P_n with

$$\operatorname{diam} F_n \leq \frac{\varsigma}{n^{\frac{1}{d-1}}}.$$

If (25) did not hold, then

(26) for any (arbitrarily large) $\sigma > 0$ there are infinitely many n such that

$$\min\{\operatorname{diam} F : F \text{ facet of } P_n\} \geq \frac{2\sigma}{n^{\frac{1}{d-1}}}.$$

Let $\nu = 2/3, \xi = 3\operatorname{div}_{d-1} A(C)^{(d+1)/(d-1)}/|\operatorname{bd} C|$ and choose ϱ corresponding to ξ as in (22) and σ corresponding to ϱ as in (24). Propositions (26), (24) and (22) then show that for an infinite set of (sufficiently large) n the following holds: there is a measurable set M_n in $\operatorname{bd} P_n$ (a union of pieces of the facets of P_n) such that $|M_n| \geq 3|\operatorname{bd} P_n|/4$ and $\|x - x^\pi\| \geq \xi/n^{2/(d-1)}$ for each $x \in M_n$. Then (21) and (6) imply that $|M_n^\pi| \geq 2|M_n|/3 \geq |\operatorname{bd} P_n|/2 \geq |\operatorname{bd} C|/2$ for an infinite set of (sufficiently large) n. Combining this with (10) we see that

$$\begin{aligned}\delta^V(C, P_n) &\geq V(\bigcup_{x \in M_n} [x, x^\pi]) \geq V(\{y \in \mathbb{E}^d : y^\pi \in M_n^\pi, \|y - y^\pi\| \leq \frac{\xi}{n^{\frac{2}{d-1}}}\}) \\ &\geq \frac{\xi |M_n^\pi|}{n^{\frac{2}{d-1}}} > \frac{3}{2}\operatorname{div}_{d-1} A(C)^{\frac{d+1}{d-1}} \frac{1}{n^{\frac{2}{d-1}}}\end{aligned}$$

for an infinite set of (sufficiently large) n.

Since this contradicts the asymptotic formula (1), the proof of (25) is complete.

Third, using the facets F_n, a polytope $Q_{n-1} \in \mathcal{P}^c_{(n-1)}$ will be constructed such that $V(Q_{n-1}) - V(P_n)$ is small. To begin with, note that the strict convexity of C (which is a consequence of $\kappa_C > 0$) together with the relation $\delta^V(C, P_n) \to 0$ as $n \to \infty$ implies that

(27) $\max\{\operatorname{diam} F : F \text{ facet of } P_n\} \to 0$ as $n \to \infty$.

P_n is the intersection of the support halfspaces of C determined by the facets of P_n. Delete from this intersection the halfspace determined by F_n (see (25)). By (27) this gives a polytope

(28) $Q_{n-1} \in \mathcal{P}^c_{(n-1)}$ for all sufficiently large n.

Next it will be shown that

(29) there is a constant $\tau > 0$ such that for all sufficiently large n,

$$V(Q_{n-1}) - V(P_n) \le \frac{\tau}{n^{\frac{d+1}{d-1}}}.$$

Let p_n be the point where the facet F_n touches C and let r_n be the point where an adjacent facet of P_n, say G_n, touches C. Choose $y_n \in F_n \cap G_n$. By (25) $\|y_n - p_n\| \le \varsigma / n^{1/(d-1)}$. Hence (8) implies that

(30) $$\|y_n - y_n^\pi\| \le \beta \|y_n - p_n\|^2 \le \frac{\beta\varsigma^2}{n^{\frac{2}{d-1}}} \text{ for all sufficiently large } n.$$

Since $\operatorname{diam} G_n \to 0$ by (27) and thus $\|y_n - r_n\| \to 0$ as $n \to \infty$, a second application of (8) together with (30) shows that

(31) $$\|y_n - r_n\|^2 \le \beta \|y_n - y_n^\pi\| \le \frac{\beta^2\varsigma^2}{n^{\frac{2}{d-1}}} \text{ for all sufficiently large } n.$$

For the proof that

(32) there is a constant $\nu > 0$ such that for all sufficiently large n holds

$$\|x - x^\pi\| \le \frac{\nu}{n^{\frac{2}{d-1}}} \text{ for each } x \in H_{r_n} \text{ with } x^\pi \in F_n^\pi,$$

note that by (4), (25), and (31),

$$\begin{aligned} \|x^\pi - r_n\| &\le \|x^\pi - p_n\| + \|p_n - y_n\| + \|y_n - r_n\| \\ &\le \frac{\varsigma}{n^{\frac{1}{d-1}}} + \frac{\varsigma}{n^{\frac{1}{d-1}}} + \frac{\beta\varsigma}{n^{\frac{1}{d-1}}} \text{ for all sufficiently large } n. \end{aligned}$$

Now apply (8) with $p = r_n$ to obtain (32). Finally, (32), (25) and (10) together show that

$$V(Q_{n-1}) - V(P_n) \le \frac{2\nu}{n^{\frac{2}{d-1}}} |F_n^\pi| \le \frac{2\nu\varsigma^{d-1}|B^{d-1}|}{n^{\frac{2}{d-1}} n^{\frac{d-1}{d-1}}} \text{ for all sufficiently large } n,$$

concluding the proof of (29).

Fourth, we are now ready to prove (20). If (20) did not hold, then for any (arbitrarily large) $\phi > 0$ there are infinitely many n such that P_n has a facet of diameter at least $2\phi / n^{1/(d-1)}$. Taking into account (27) and (8) this shows the following:

(33) Let $\phi > 0$ (arbitrarily large). Then there are infinitely many (sufficiently large) n such that P_n has a vertex v_n with

$$\|v_n - v_n^\pi\| \ge \frac{\phi^2}{\beta n^{\frac{2}{d-1}}}.$$

Choose $\phi > 0$ so large that

(34) $$\frac{\phi}{\beta} > \varsigma,$$

(35) $$\frac{\phi^{d+1}\zeta}{\beta^{\frac{d+1}{2}}} > \tau.$$

For the rest of this subsection assume that n is as in (33). Then, if n is sufficiently large, (25), (8), (33) and (34) show that v_n is not a vertex of F_n. Hence v_n must be a vertex of Q_{n-1}. Since $\delta^V(C, P_n) \to 0$, (29) implies that $\delta^V(C, Q_{n-1}) \to 0$ as $n \to \infty$. This yields that $\|v_n - v_n^\pi\| \to 0$ as $n \to \infty$. Thus (9), (33) and (35) imply the following:

> There are infinitely many (sufficiently large) n such that $R_n = H^+_{v_n^\pi} \cap Q_{n-1}$ is a polytope in $\mathcal{P}^c_{(n)}$ for which
> $$V(Q_{n-1}) - V(R_n) \geq \frac{\phi^{d+1}\zeta}{\beta^{\frac{d+1}{2}} n^{\frac{d+1}{d-1}}} > \frac{\tau}{n^{\frac{d+1}{d-1}}}.$$

This together with (29) shows that for each n from an infinite set of (sufficiently large) n there is a polytope $R_n \in \mathcal{P}^c_{(n)}$ with $V(R_n) < V(P_n)$. This contradicts the fact that $P_n \in \mathcal{P}^c_{(n)}$ is best approximating of C and thus concludes the proof of proposition (20).

2.2.2 Width of the Facets of P_n. We next show the following:

(36) There is a constant $\varphi > 0$ such that $\min\{\text{width } F : F \text{ facet of } P_n\} \geq \dfrac{\varphi}{n^{\frac{1}{d-1}}}$.

First, two auxiliary propositions will be shown:

(37) There is a constant $\chi > 0$ such that for all sufficiently large n the following holds: let F be any facet of P_n and let Q_{n-1} be the intersection of the support halfspaces of C determined by the facets of P_n except for F. Then $Q_{n-1} \in \mathcal{P}_{(n-1)}$ and
$$V(Q_{n-1}) - V(P_n) \leq \frac{\chi|F|}{n^{\frac{2}{d-1}}}.$$

If in the proof of (28) and (29) the facet F_n is replaced by F and proposition (25) by the inequality $\operatorname{diam} F \leq \mu/n^{1/(d-1)}$ which follows from (20), we obtain (37).

(38) There is a constant $\psi > 0$ such that for all sufficiently large n the following holds: let Q_{n-1} be an arbitrary polytope in $\mathcal{P}^c_{(n-1)}$. Then there is a support halfspace H^+ of C such that $R_n = H^+ \cap Q_{n-1} \in \mathcal{P}^c_{(n)}$ and
$$V(Q_{n-1}) - V(R_n) > \frac{\psi^{d+1}\zeta}{\beta^{\frac{d+1}{2}} n^{\frac{d+1}{d-1}}}.$$

By (7) the metric projections of the facets of Q_{n-1} into $\operatorname{bd} C$ form a dissection of $\operatorname{bd} C$ into at most $n-1$ pieces. Hence there is a facet F of Q_{n-1} with $|F|(\geq |F^\pi| \geq |\operatorname{bd} C|/(n-1)) > |\operatorname{bd} C|/n$ by (6). Let p be the point where F touches C. The isoperimetric inequality in H_p then shows that the circular disc in H_p with center p and radius $\psi/n^{1/(d-1)}$, where $\psi = (|\operatorname{bd} C|/|B^{d-1}|)^{1/(d-1)}$, does not contain F. Therefore we may choose $x \in F$ with $\|x-p\| = \psi/n^{1/(d-1)}$. Thus, if n is sufficiently large, an application of (8) yields $\psi^2/\beta n^{2/(d-1)} \leq \|x - x^\pi\| \leq \beta\psi^2/n^{2/(d-1)}$. This shows that for sufficiently large n we may apply (9) to get proposition (38).

Second, after these preparations, (36) will be shown by contradiction. If (36) did not hold, then, taking into account (20),

(39) to any (arbitrarily small) $\omega > 0$ there correspond infinitely many n such that P_n has a facet F_n with $|F_n| \leq \omega/n^{(d+1)/(d-1)}$.

Choose $\omega > 0$ such that

$$\chi\omega < \frac{\psi^{d+1}\zeta}{\beta^{\frac{d+1}{2}}}.$$

For n which are so large that (37) and (38) hold and which correspond to the chosen ω as in (39) define $Q_{n-1} \in \mathcal{P}^c_{(n-1)}$ as follows: Q_{n-1} is the intersection of the support halfspaces of C determined by the facets of P_n except for F_n (see (38)). For such n,

$$Q_{n-1} \in \mathcal{P}^c_{(n-1)} \text{ and } V(Q_{n-1}) - V(P_n) \leq \frac{\chi\omega}{n^{\frac{d+1}{d-1}}}$$

by (37) and (39). Further, again for such n, (38) shows that there is

$$R_n \in \mathcal{P}^c_{(n)} \text{ with } V(Q_{n-1}) - V(R_n) > \frac{\psi^{d+1}\zeta}{\beta^{\frac{d+1}{2}} n^{\frac{d+1}{d-1}}}.$$

Concluding, we see that for $R_n \in \mathcal{P}^c_{(n)}$ holds $V(R_n) < V(P_n)$, in contradiction to the fact that $P_n \in \mathcal{P}^c_{(n)}$ is best approximating of C. The proof of (36) is complete.

2.2.3 Diameter and Width of Dirichlet–Voronoi Cells. Given a parallelogram T in and a positive definite quadratic form q on $\mathbb{E}^{d-1}$, define $V_m(T,q)$ for $m = 1, 2, \dots,$ by

$$(40) \qquad V_m(T,q) = \inf_{p_1,\dots,p_m \in \mathbb{E}^{d-1}} \{ \int_T \min_{i=1,\dots,m} \{q(s-p_i)\} ds \}$$

This infimum is attained for m suitable points in T, say $p_i = p_{mi}, i = 1, \dots, m$. Then

$$V_m(T,q) = \sum_i \int_{D_i} q(s-p_i) ds$$

where the *Dirichlet–Voronoi cells* $D_i = D_{mi}, i = 1, \dots, m$ are defined by

$$D_i = \{s \in T : q(s-p_i) \leq q(s-p_j) \text{ for } j = 1, \dots, m\}.$$

It is thus appropriate to call $V_m(T,q)$ a *sum of moments*.

In the following we state two equivalent propositions. Their proofs are similar to the proofs of (20) and (36) but technically simpler and thus will not be given.

(41) Let $\lambda > 1$. Then there is a constant $\nu > 1$ depending only on d, λ with the following property: let S be a square in and q a positive definite quadratic form on $\mathbb{E}^{d-1}$ such that

$$\frac{1}{\lambda} \leq \frac{q(s)}{\|s\|^2} \leq \lambda \text{ for } s \in \mathbb{E}^{d-1}, \neq o.$$

For $m = 1, 2, \ldots,$ choose points $p_i = p_{mi} \in \mathbb{E}^{d-1}, i = 1, \ldots, m,$ such that

$$V_m(S,q) = \int\limits_S \min_{i=1,\ldots,m} \{q(s-p_i)\} ds$$

and define Dirichlet–Voronoi cells $D_i = D_{mi}, i = 1, \ldots, m,$ by

$$D_i = \{s \in S : q(s-p_i) \leq q(s-p_j) \text{ for } j = 1, \ldots, m\}.$$

Then the following inequalities hold:

$$\frac{|S|^{\frac{1}{d-1}}}{\nu m^{\frac{1}{d-1}}} \leq \min_{i=1,\ldots,m} \{\operatorname{width} D_i\} \leq \max_{i=1,\ldots,m} \{\operatorname{diam} D_i\} \leq \frac{\nu |S|^{\frac{1}{d-1}}}{m^{\frac{1}{d-1}}}.$$

(42) Let $\vartheta > 1$. Then there is a constant $\nu > 1$ depending only on d, ϑ with the following property: let T be a parallelogram in $\mathbb{E}^{d-1}$ which contains a square of edge length e, say, and is contained in a concentric square of edge length ϑe. For $m = 1, 2, \ldots,$ choose points $p_i = p_{mi} \in \mathbb{E}^{d-1}, i = 1, \ldots, m,$ such that

$$V_m(T, \|\cdot\|^2) = \int\limits_T \min_{i=1,\ldots,m} \{\|s-p_i\|^2\} ds$$

and define Dirichlet–Voronoi cells $D_i = D_{mi}, i = 1, \ldots, m,$ by

$$D_i = \{s \in T : \|s-p_i\| \leq \|s-p_j\| \text{ for } j = 1, \ldots, m\}.$$

Then the following inequalities hold:

$$\frac{e}{\nu m^{\frac{1}{d-1}}} \leq \min_{i=1,\ldots,m} \{\operatorname{width} D_i\} \leq \max_{i=1,\ldots,m} \{\operatorname{diam} D_i\} \leq \frac{\nu e}{m^{\frac{1}{d-1}}}.$$

2.3 Sums of Moments

The constant div_{d-1} considered in sect. 1 above is defined by

(43) $$\mathrm{div}_{d-1} = \lim_{m\to\infty} m^{\frac{2}{d-1}} V_m([0,1]^{d-1}, \|\cdot\|^2),$$

where $[0,1]^{d-1}$ is the unit square in $\mathbb{E}^{d-1}$, see [13]. A particular case of Lemma 1 of Glasauer and the author [10] is the following:

(44) Let T be a parallelogram in $\mathbb{E}^{d-1}$. Then

$$\mathrm{div}_{d-1}|T|^{\frac{d+1}{d-1}} = \lim_{m\to\infty} m^{\frac{2}{d-1}} V_m(T, \|\cdot\|^2).$$

The next observation is easy to show:

(45) Let T be a parallelogram in $\mathbb{E}^{d-1}$. Then

$$V_m(\varrho T, \|\cdot\|^2) = \varrho^{d+1} V_m(T, \|\cdot\|^2) \text{ for } \varrho > 0.$$

2.3.1 Lower Estimate of $\boldsymbol{V_m}$. Here the objective is to show the following proposition:

(46) Let S be a square in $\mathbb{E}^{d-1}$ and q a positive definite quadratic form on $\mathbb{E}^{d-1}$. Then

$$V_k(S,q) \geq \mathrm{div}_{d-1}|S|^{\frac{d+1}{d-1}} (\det q)^{\frac{1}{d-1}} \frac{1}{k^{\frac{2}{d-1}}} \text{ for } k = 1, 2, \dots .$$

For the proof of (46) it is sufficient to confirm the following equivalent version of it:

(47) Let T be a parallelogram in $\mathbb{E}^{d-1}$. Then

$$V_k(T, \|\cdot\|^2) \geq \mathrm{div}_{d-1}|T|^{\frac{d+1}{d-1}} \frac{1}{k^{\frac{2}{d-1}}} \text{ for } k = 1, 2, \dots .$$

Given k, choose $p_i = p_{ki} \in T, i = 1, \dots, k$, such that

(48) $$V_k(T, \|\cdot\|^2) = \int_T \min_{i=1,\dots,k} \{\|s - p_i\|^2\} ds.$$

For $l = 1, 2, \dots$, consider a covering of the unit square $[0,1]^{d-1}$ by a minimum number of translates of $(1/l)T$. For this number n_l, say, holds

(49) $$\frac{l^{d-1}}{|T|} \leq n_l \leq \frac{l^{d-1} + O(l^{d-2})}{|T|},$$ where the constant in $O(\cdot)$ depends only on T.

The union of the corresponding translates of the set $\{(1/l)p_i, i = 1, \dots, k\}$ then consists of

(50) $\quad k_l \leq kn_l \leq \dfrac{k(l^{d-1} + O(l^{d-2}))}{|T|}$

points, say of the points $r_j, j = 1, \dots, k_l$. Then (48), (45), the covering of $[0,1]^{d-1}$ by n_l translates of $(1/l)T$ and the definition of the points r_j, (49), (50) and the definition of V_{k_l} (see (40)) imply the following:

$$
\begin{aligned}
V_k(T, \|\cdot\|^2) &= \int_T \min_{i=1,\dots,k}\{\|s - p_i\|^2\} ds = l^{d+1} \int_{\frac{1}{l}T} \min_{i=1,\dots,k}\{\|t - \frac{1}{l}p_i\|^2\} dt \\
&= \frac{l^{d+1}}{n_l} n_l \int_{\frac{1}{l}T} \min_{i=1,\dots,k}\{\|t - \frac{1}{l}p_i\|^2\} dt \geq \frac{l^{d+1}}{n_l} \int_{[0,1]^{d-1}} \min_{j=1,\dots,k_l}\{\|u - r_j\|^2\} du \\
&\geq \frac{l^{d+1}|T|}{l^{d-1}(1 + O(\frac{1}{l}))} \frac{k_l^{\frac{2}{d-1}}}{k_l^{\frac{2}{d-1}}} \int_{[0,1]^{d-1}} \min_{j=1,\dots,k_l}\{\|u - r_j\|^2\} du \\
&\geq \frac{l^2|T|}{1 + O(\frac{1}{l})} \frac{|T|^{\frac{2}{d-1}}}{l^2(1 + O(\frac{1}{l}))^{\frac{2}{d-1}} k^{\frac{2}{d-1}}} k_l^{\frac{2}{d-1}} \int_{[0,1]^{d-1}} \min_{j=1,\dots,k_l}\{\|u - r_j\|^2\} du \\
&\geq \frac{|T|^{\frac{d+1}{d-1}}}{(1 + O(\frac{1}{l}))k^{\frac{2}{d-1}}} k_l^{\frac{2}{d-1}} V_{k_l}([0,1]^{d-1}, \|\cdot\|^2).
\end{aligned}
$$

Now, letting l tend to ∞, it follows that $k_l (\geq n_l \geq l^{d-1}/|T|)$ tends also to ∞ and (43) yields the inequality (47).

2.3.2 Upper Estimate of V_m. We will show the following result:

(51) Let $\varepsilon > 0, \lambda > 1$. Let S be a square in $\mathbb{E}^{d-1}$ and q a positive definite quadratic form on $\mathbb{E}^{d-1}$ such that

$$\frac{1}{\lambda} \leq \frac{q(s)}{\|s\|^2} \leq \lambda \text{ for } s \in \mathbb{E}^{d-1}, \neq o.$$

Then the following inequality holds:

$$V_k(S, q) \leq \operatorname{div}_{d-1} |S|^{\frac{d+1}{d-1}} (\det q)^{\frac{1}{d-1}} \frac{1}{k^{\frac{2}{d-1}}} + O\Big(\frac{1}{k^{\frac{2}{d-1}+\omega}}\Big),$$

where $\omega = (1-\varepsilon)/2(d-1)$ and the constant in $O(\cdot)$ depends only on d, ε, λ.

For the proof of (51) it is sufficient to show the following equivalent assertion:

(52) Let $\varepsilon > 0$ and $\vartheta > 1$. Let T be a parallelogram in $\mathbb{E}^{d-1}$ which contains a square of edge length e, say, and is contained in a concentric square of edge length ϑe. Then the following inequality holds:

$$V_k(T, \|\cdot\|^2) \leq \operatorname{div}_{d-1} |T|^{\frac{d+1}{d-1}} \frac{1}{k^{\frac{2}{d-1}}} + O\Big(\frac{1}{k^{\frac{2}{d-1}+\omega}}\Big),$$

where $\omega = (1-\varepsilon)/2(d-1)$ and the constant in $O(\cdot)$ depends only on $d, \varepsilon, \vartheta$.

The proof will be divided into six parts.

First, a weaker version of (52) will be shown:

(53) Let $\vartheta > 1$ and let T be a parallelogram in $\mathbb{E}^{d-1}$ which contains a square of edge length e, say, and is contained in a concentric square of edge length ϑe. Then

$$V_m(T, \|\cdot\|^2) \leq |T|^{\frac{d+1}{d-1}} O\Big(\frac{1}{m^{\frac{2}{d-1}}}\Big),$$

where the constant in $O(\cdot)$ depends only on d, ϑ.

Given m, choose a positive integer l such that $l^{d-1} \leq m < (l+1)^{d-1}$. Dissect the square of edge length ϑe which contains T into $l^{d-1} (\leq m)$ squares of edge length $\vartheta e/l$. Clearly, these squares cover T. Thus the minimum edge length of a square with the property that m suitable congruent copies of it cover T is at most $\vartheta e/l (\leq 2\vartheta/m^{1/(d-1)})$. This yields (53):

$$\begin{aligned} V_m(T, \|\cdot\|^2) &\leq m\Big(\frac{2\vartheta e}{m^{\frac{1}{d-1}}}\sqrt{d-1}\Big)^2 \Big(\frac{2\vartheta e}{m^{\frac{1}{d-1}}}\Big)^{d-1} = e^{d+1} \frac{2^{d+1}\vartheta^{d+1}(d-1)}{m^{\frac{2}{d-1}}} \\ &\leq |T|^{\frac{d+1}{d-1}} O\Big(\frac{1}{m^{\frac{2}{d-1}}}\Big). \end{aligned}$$

Second, we make some further preparations for the later steps of the proof of (52):

(54) Given k, let $l = \lceil k^{\frac{1+\varepsilon}{2(d-1)}} \rceil$, $m = l^{2(d-1)}$.

Choose $p_i = p_{mi} \in \mathbb{E}^{d-1}, i = 1, \dots, m$, such that

$$V_m(T, \|\cdot\|^2) = \int_T \min\{\|s - p_i\|^2\} ds$$

and let $D_i = D_{mi}, 1 = 1, \dots, m$, be the Dirichlet–Voronoi cells

(55) $D_i = \{s \in T : \|s - p_i\| \leq \|s - p_j\| \text{ for } j = 1, \dots, m\}$.

Proposition (42), the assumptions on T in (52) and (54) yield the next statement:

(56) There is a constant $\nu > 1$ depending only on d, ϑ such that the following inequalities hold:

$$\frac{e}{\nu l^2} \leq \min_{i=1,\dots,m}\{\text{width}\, D_i\} \leq \max_{i=1,\dots,m}\{\text{diam}\, D_i\} \leq \frac{\nu e}{l^2}.$$

Dissect T into l^{d-1} translates of $(1/l)T$, order these translates in some way and for $j = 1, \dots, l^{d-1}$, let T_j be a translate of $((1/l) - (2\nu/l^2))T$ which is concentric with the jth translate of $(1/l)T$. Let

(57) $m_j = \#\{i \in \{1, \dots, m\} : D_i \cap T_j \neq \phi\}, j = 1, \dots, l^{d-1}$.

By our choice of T_j, the assumption that T contains a square of edge length e and (56) we see that

(58) $D_i \cap T_j \neq \phi \Rightarrow D_i$ is contained in the jth translate of $(1/l)T$.

Since the m Dirichlet–Voronoi cells D_i form a dissection of T, (57) and (58) imply

(59) $m_1 + \ldots + m_{l^{d-1}} \leq m$

and (56)–(58) yield the estimate

(60) $m_j \leq \dfrac{|\frac{1}{l}T|}{\min\limits_{i=1,\dots,m}\{|D_i|\}} \leq O(l^{d-1})$,

where the constant in $O(\cdot)$ depends only on d, ϑ.

Third, we deal with good indices, where an index $j \in \{1, \dots, l^{d-1}\}$ is called *good* if $m_j \geq \lceil l^{(d-1)/2} \rceil$. We will show that

(61) $\#\{j \in \{1, \dots, l^{d-1}\} : j \text{ good}\} \geq l^{d-1}\Big(1 - O\Big(\dfrac{1}{l}\Big)\Big)$,

where the constant in $O(\cdot)$ depends only on d, ϑ.

Let $n = \#\{j \in \{1, \dots, l^{d-1}\} : j \text{ not good}\}$. Then (48), (55), (57), (58), the definition of T_j and non good j s and (45) show that

$$V_m(T, \|\cdot\|^2) \geq \sum_{j \text{ not good}} V_{m_j}(T_j, \|\cdot\|) \geq n\Big(1 - \frac{2\nu}{l}\Big)^{d+1} \frac{1}{l^{d+1}} V_{\lceil l^{(d-1)/2}\rceil}(T, \|\cdot\|^2).$$

Now apply (53), (47) and note (54). This gives

$$n \leq l^{d-1} O\Big(\frac{1}{l}\Big) \text{ where the constant in } O(\cdot) \text{ depends only on } d, \vartheta.$$

Since the quantity considered in (61) is $l^{d-1} - n$, the proof of (61) is complete.

Fourth, it will be shown that

$$V_{m_j}(T,\|\cdot\|^2)m_j^{\frac{2}{d-1}} < \Big(1+\frac{1}{l^{\frac{1-\varepsilon}{1+\varepsilon}}}\Big)V_m(T,\|\cdot\|^2)m^{\frac{2}{d-1}} \tag{62}$$

for each sufficiently large k (depending on ε) and suitable corresponding good index j.

If (62) did not hold, then

$$V_{m_j}(T,\|\cdot\|^2)m_j^{\frac{2}{d-1}} \geq \Big(1+\frac{1}{l^{\frac{1-\varepsilon}{1+\varepsilon}}}\Big)V_m(T,\|\cdot\|^2)m^{\frac{2}{d-1}} \tag{63}$$

for infinitely many k and any good j.

For k as in (63), (48), (55), (57), (58), the definition of T_j, (45), (63), (17), (61) and (59) yield the following:

$$\begin{aligned}
V_m(T,\|\cdot\|^2) &\geq \sum_{j\text{ good}} V_{m_j}(T_j,\|\cdot\|^2) = \sum_{j\text{ good}} \Big(1-\frac{2\nu}{l}\Big)^{d+1}\frac{1}{l^{d+1}}V_{m_j}(T,\|\cdot\|^2)\\
&\geq \frac{1}{l^{d+1}}\Big(1-O\Big(\frac{1}{l}\Big)\Big)\Big(1+\frac{1}{l^{\frac{1-\varepsilon}{1+\varepsilon}}}\Big)V_m(T,\|\cdot\|^2)m^{\frac{2}{d-1}}\sum_{j\text{ good}}\frac{1}{m_j^{\frac{2}{d-1}}}\\
&\geq \frac{1}{l^{d+1}}\Big(1-O\Big(\frac{1}{l}\Big)+\frac{1}{l^{\frac{1-\varepsilon}{1+\varepsilon}}}\Big)V_m(T,\|\cdot\|^2)m^{\frac{2}{d-1}}\Big(\sum_{j\text{ good}}1\Big)^{\frac{d+1}{d-1}}\frac{1}{\Big(\sum\limits_{j\text{ good}} m_j\Big)^{\frac{2}{d-1}}}\\
&\geq \frac{1}{l^{d+1}}\Big(1+\frac{1}{l^{\frac{1-\varepsilon}{1+\varepsilon}}}-O\Big(\frac{1}{l}\Big)\Big)l^{d+1}\Big(1-O\Big(\frac{1}{l}\Big)\Big)^{\frac{d+1}{d-1}}\frac{m^{\frac{2}{d-1}}}{m^{\frac{2}{d-1}}}V_m(T,\|\cdot\|^2)\\
&\geq \Big(1+\frac{1}{l^{\frac{1-\varepsilon}{1+\varepsilon}}}-O\Big(\frac{1}{l}\Big)\Big)V_m(T,\|\cdot\|^2),
\end{aligned} \tag{64}$$

where the constants in the $O(\cdot)$ symbols depend only on d,ϑ.

If k and thus by (54) also l is sufficiently large, the last expression in (64) is larger than the first one. This contradiction concludes the proof of (62).

Fifth, we will prove the following estimate:

$$V_k(T,\|\cdot\|^2)k^{\frac{2}{d-1}} \leq \Big(1+O\Big(\frac{1}{l^{\frac{1-\varepsilon}{1+\varepsilon}}}\Big)\Big)V_{l^{2(d-1)}}(T,\|\cdot\|^2)l^4,\ l=\lceil k^{\frac{1+\varepsilon}{2(d-1)}}\rceil \tag{65}$$

for each sufficiently large k (depending on ε), where the constant in $O(\cdot)$ depends only on d,ε,ϑ.

Let k be so large that (62) holds and choose a corresponding good j. Let p be a positive integer such that

$$p^{d-1}m_j \leq k < (p+1)^{d-1}m_j. \tag{66}$$

Then (60) and (54) show that

$$\frac{1}{p+1} < \Big(\frac{m_j}{k}\Big)^{\frac{1}{d-1}} = O\Big(\frac{1}{l^{\frac{1-\varepsilon}{1+\varepsilon}}}\Big), \tag{67}$$

where the constant in $O(\cdot)$ depends only on d,ϑ.

Proposition (65) now is a consequence of (66), a tiling of T by translates of $(1/p)T$, (45), (66), (67), (62) and (54):

$$
\begin{aligned}
V_k(T, \|\cdot\|^2)k^{\frac{2}{d-1}} &\le V_{p^{d-1}m_j}(T, \|\cdot\|^2)k^{\frac{2}{d-1}} \le p^{d-1}V_{m_j}(\frac{1}{p}T, \|\cdot\|^2)k^{\frac{2}{d-1}}\\
&= \frac{p^{d-1}}{p^{d+1}}V_{m_j}(T, \|\cdot\|^2)(p+1)^2 m_j^{\frac{2}{d-1}} = \Big(1+\frac{1}{p}\Big)V_{m_j}(T, \|\cdot\|^2)m_j^{\frac{2}{d-1}}\\
&< \Big(1+O\Big(\frac{1}{l^{\frac{1-\varepsilon}{1+\varepsilon}}}\Big)\Big)\Big(1+\frac{1}{l^{\frac{1-\varepsilon}{1+\varepsilon}}}\Big)V_{l^{2(d-1)}}(T, \|\cdot\|^2)l^4
\end{aligned}
$$

for each sufficiently large k, where the constant in $O(\cdot)$ depends only on d, ϑ.

Sixth, applying (65) repeatedly and using (54), (44) and (19) we obtain (52):

$$
\begin{aligned}
V_k(T, \|\cdot\|^2)k^{\frac{2}{d-1}} &\le \Big(1+O\Big(\tfrac{1}{l^{\frac{1-\varepsilon}{1+\varepsilon}}}\Big)\Big)V_{l^{2(d-1)}}(T, \|\cdot\|^2)l^4\\
&\le \Big(1+O\Big(\frac{1}{k^{\frac{1-\varepsilon}{2(d-1)}}}\Big)\Big)V_{k_1}(T, \|\cdot\|^2)k_1^{\frac{2}{d-1}}, k_1 = l^{2(d-1)} \ge k^{1+\varepsilon},\\
&\le \Big(1+O\Big(\frac{1}{k^{\frac{1-\varepsilon}{2(d-1)}}}\Big)\Big)\Big(1+O\Big(\frac{1}{k_1^{\frac{1-\varepsilon}{2(d-1)}}}\Big)\Big)V_{k_2}(T, \|\cdot\|^2)k_2^{\frac{2}{d-1}}, k_2 \ge k_1^{1+\varepsilon} \ge k^{(1+\varepsilon)^2},\\
&\le \Big(1+O\Big(\frac{1}{k^{\frac{1-\varepsilon}{2(d-1)}}}\Big)\Big)\Big(1+O\Big(\frac{1}{k^{\frac{1-\varepsilon}{2(d-1)}(1+\varepsilon)}}\Big)\Big)V_{k_2}(T, \|\cdot\|^2)k_2^{\frac{2}{d-1}}, k_2 \ge k^{(1+\varepsilon)^2},\\
&\dots\\
&\le \Big(1+O\Big(\frac{1}{t}\Big)\Big)\Big(1+O\Big(\frac{1}{t^{1+\varepsilon}}\Big)\Big)\Big(1+O\Big(\frac{1}{t^{(1+\varepsilon)^2}}\Big)\Big)\dots \operatorname{div}_{d-1}|T|^{\frac{d+1}{d-1}}, t = \frac{1}{k^{\frac{1-\varepsilon}{2(d-1)}}},\\
&\le \Big(1+O\Big(\frac{1}{t}\Big)\Big)\operatorname{div}_{d-1}|T|^{\frac{d+1}{d-1}},
\end{aligned}
$$

for each sufficiently large k, where the $O(\cdot)$ symbols up to the last one coincide and the constants in all $O(\cdot)$ symbols depend only on $d, \varepsilon, \vartheta$.

2.4 Estimates of $\delta^V(C, P_n)$

2.4.1 Lower Estimate of $\delta^V(C, P_n)$. We will prove the following:

(68) Let $0 < \varepsilon < \frac{1}{2(d-1)}$. Then

$$
\delta^V(C, P_n) \ge \frac{1}{2}\operatorname{div}_{d-1}A(C)^{\frac{d+1}{d-1}}\frac{1}{n^{\frac{2}{d-1}}} - O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{1}{2(d-1)}-\varepsilon}}\Big) \text{ as } n \to \infty,
$$

where the constant in $O(\cdot)$ depends only on C, ε.

The proof is split into four steps.

First, let n be so large that

(69) $\quad \dfrac{\mu}{n^{\frac{1}{d-1}}} \leq \alpha$ and $l = \lfloor n^{2(d-1)\varepsilon} \rfloor$ is sufficiently large in the sense of (12),

see (20), (8) and (12). Choose ι, κ, λ, C_i, p_i, H_i, "$'$", f_i, q_{iu}, $\kappa_C(\cdot)$, for $i = 1, \dots, l$, such that (12) – (16) hold. Let

(70) $$\delta = \frac{1 + 2(d-1)\varepsilon}{4(d-1)\varepsilon}.$$

Second, for each $i = 1, \dots, l$, consider an edge-to-edge tiling of H_i with squares of area $1/l^{\delta}$. Let $S'_{ij}, j = 1, \dots, j_i$, be the squares which meet the convex disc $(1 - \sqrt{d-1}\iota/l^{(\delta-1)/(d-1)})C'_i$. Let p'_{ij} be the center of S'_{ij} and $q_{ij} = q_{ip'_{ij}}$ the corresponding positive definite quadratic form, see 2.1.3. Let S_{ij} and p_{ij} be the inverse images of S'_{ij} and p'_{ij} on the lower side of C. Clearly,

(71) $$\operatorname{diam} S'_{ij} = \frac{\sqrt{d-1}}{l^{\frac{\delta}{d-1}}}.$$

The definition of S'_{ij} together with (12) and (71) shows that

(72) $$S'_{ij} \subset \Big(1 - \frac{\sqrt{d-1}\iota}{l^{\frac{\delta-1}{d-1}}}\Big)C'_i + \frac{\sqrt{d-1}}{l^{\frac{\delta}{d-1}}}B^{d-1} \subset C'_i \text{ for } j = 1, \dots, j_i.$$

It follows from (12) that the area of the strip between $(1 - \sqrt{d-1}\iota/l^{(\delta-1)/(d-1)})C'_i$ and $\operatorname{relbd} C'_i$ is at most $O(1/l^{\delta/(d-1)})O(1/l^{(d-2)/(d-1)}) = O(1/l^{(\delta+d-2)/(d-1)})$, where the constants in the $O(\cdot)$ symbols depend only on C. Thus, noting (12) – (16) and the definitions of S_{ij} and $A(\cdot)$ (compare (2) for the latter), we obtain that

(73) $$A\Big(\bigcup_{i,j} S_{ij}\Big) \geq A(C) - lO\Big(\frac{1}{l^{\frac{\delta+d-2}{d-1}}}\Big) = A(C)\Big(1 - O\Big(\frac{1}{l^{\frac{\delta-1}{d-1}}}\Big)\Big),$$

where the constants in the $O(\cdot)$ symbols depend only on C.

Since by our choice of n proposition (69) holds, it follows from (20) and (8) that

(74) $$\|x - x^{\pi}\| \leq \frac{\beta\mu^2}{n^{\frac{2}{d-1}}} \text{ for } x \in P_n.$$

Next,

(75) let T'_{ij} be the square concentric with S'_{ij} and obtained from S'_{ij} by shrinking it with the factor

$$\Big(1 - 2\beta\mu^2\Big(\frac{l^{\delta}}{n^2}\Big)^{\frac{1}{d-1}} - 2\mu\Big(\frac{l^{\delta}}{n}\Big)^{\frac{1}{d-1}}\Big)\Big(\geq 1 - O\Big(\Big(\frac{l^{\delta}}{n}\Big)^{\frac{1}{d-1}}\Big)\Big).$$

Let T_{ij} be the inverse image of T'_{ij} on the lower side of C.

Third, consider for any pair of indices i, j the facets $F_{ijk}, k = 1, \dots, k_{ij}$, of P_n which touch C on its lower side and such that $F'_{ijk} \cap T'_{ij} \neq \phi$. By the definitions of S'_{ij}, T'_{ij}, (75), (20) and (72) we have the following:

(76) $F'_{ijk}, k = 1, \dots, k_{ij}$, cover T'_{ij},

(77) $F'_{ijk} \subset \operatorname{relint} S'_{ij} \subset C'_i$ and F'_{ijk} has distance at least $\beta\mu^2/n^{2/(d-1)}$ from $\operatorname{relbd} S'_{ij}$ and thus from $\operatorname{relbd} C'_i$.

(74), (77) and (12) imply that

(78) facets of P_n which correspond to different T_{ij} are distinct.

Hence

(79) $k_{11} + k_{12} + \ldots + k_{lj_l} \leq n.$

A further application of (74) and (77) shows that

(80) for each facet F_{ijk} of P_n the set above F_{ijk} and below C is contained in the set $P_{ni} = \{x \in P_n : x^\pi \in C_i\}$.

By (12),

(81) the sets $P_{ni}, i = 1, \dots, l$, form a dissection of $P_n \backslash \operatorname{int} C$.

Let p_{ijk} be the point where F_{ijk} touches C and let $q_{ijk} = q_{ip'_{ijk}}$ be the corresponding positive definite quadratic form, see 2.1.3.

Fourth, (81), (80), (77), (12), (13), the fact that F_{ijk} touches C at p_{ijk}, Taylor's formula applied to f_i at p'_{ijk}, (20), (12), (13), (77), Taylor's formula applied to the coefficients of q_{ij}, (71), (20), the fact that the sum of the areas of the facets of P_n is $O(1)$, (76), (78), (69), (70), (40), (46), (75), (45), (15), (13), (16), (71), the definition of surface integrals and of $A(\cdot)$ (compare (2)), (18), (79), (72), (12), (73), (70) and (69) together imply (68):

$$
\begin{aligned}
\delta^V(C,P_n) &= V(P_n\backslash \mathrm{int} C) = \sum_i V(P_{ni}) \\
&\geq \sum_{i,j,k} V(\text{set above } F_{ijk} \text{ and below } C) \\
&\geq \sum_{i,j,k} \int_{F'_{ijk}} \{f_i(s) - f_i(p'_{ijk}) - \operatorname{grad} f_i(p'_{ijk})(s-p'_{ijk})\} ds \\
&\geq \frac{1}{2}\sum_{i,j,k}\{\int_{F'_{ijk}} q_{ijk}(s-p'_{ijk})ds - O\Big(\frac{1}{n^{\frac{3}{d-1}}}\Big)|F'_{ijk}|\} \\
&\geq \frac{1}{2}\sum_{i,j,k}\{\int_{F'_{ijk}} q_{ij}(s-p'_{ijk})ds - O\Big(\frac{1}{n^{\frac{\delta}{d-1}} n^{\frac{2}{d-1}}}\Big)|F'_{ijk}|\} - O(\frac{1}{n^{\frac{3}{d-1}}}), \\
&\geq \frac{1}{2}\sum_{i,j} \int_{T'_{ij}} \min_{k=1,\dots,k_{ij}} \{q_{ij}(s-p'_{ijk})\} ds - O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{1}{2(d-1)}+\varepsilon}}\Big) \\
&\geq \frac{1}{2}\operatorname{div}_{d-1}\sum_{i,j} (\det q_{ij})^{\frac{1}{d-1}} |T'_{ij}|^{\frac{d+1}{d-1}} \frac{1}{k_{ij}^{\frac{2}{d-1}}} - O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{1}{2(d-1)}+\varepsilon}}\Big) \\
&\geq \frac{1}{2}\operatorname{div}_{d-1}\Big(1 - O\Big(\Big(\frac{l^\delta}{n}\Big)^{\frac{1}{d-1}}\Big)\Big)^{\frac{d+1}{d-1}} \sum_{i,j} (\det q_{ij})^{\frac{1}{d-1}} |S'_{ij}|^{\frac{d+1}{d-1}} \frac{1}{k_{ij}^{\frac{2}{d-1}}} - O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{1}{2(d-1)}+\varepsilon}}\Big) \\
&\geq \frac{1}{2}\operatorname{div}_{d-1}\Big(1 - O\Big(\frac{1}{n^{\frac{1}{2(d-1)}-\varepsilon}}\Big)\Big) \sum_{i,j} \{\kappa_C(p'_{ij})^{\frac{1}{d+1}} (1 + (\operatorname{grad} f_i(p'_{ij})^2)^{\frac{1}{2}} |S'_{ij}|\}^{\frac{d+1}{d-1}} \frac{1}{k_{ij}^{\frac{2}{d-1}}} \\
&\quad - O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{1}{2(d-1)}+\varepsilon}}\Big) \\
&\geq \frac{1}{2}\operatorname{div}_{d-1}\Big(1 - O\Big(\frac{1}{n^{\frac{1}{2(d-1)}-\varepsilon}}\Big)\Big)\Big(1 - O(\frac{1}{l^{\frac{\delta}{d-1}}})\Big)^{\frac{d+1}{d-1}} \times \\
&\quad \times \sum_{i,j} \{\int_{S'_{ij}} \kappa_C(s)^{\frac{1}{d+1}} (1 + (\operatorname{grad} f_i(s))^2)^{\frac{1}{2}} ds\}^{\frac{d+1}{d-1}} \frac{1}{k_{ij}^{\frac{2}{d-1}}} - O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{1}{2(d-1)}+\varepsilon}}\Big) \\
&\geq \frac{1}{2}\operatorname{div}_{d-1}\Big(1 - O\Big(\frac{1}{n^{\frac{1}{2(d-1)}-\varepsilon}}\Big) - O\Big(\frac{1}{n^{\frac{1}{2(d-1)}+\varepsilon}}\Big)\Big) \sum_{i,j} A(S_{ij})^{\frac{d+1}{d-1}} \frac{1}{k_{ij}^{\frac{2}{d-1}}} - O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{1}{2(d-1)}+\varepsilon}}\Big) \\
&= \frac{1}{2}\operatorname{div}_{d-1}\Big(1 - O\Big(\frac{1}{n^{\frac{1}{2(d-1)}-\varepsilon}}\Big)\Big) \sum_{i,j} \{A(S_{ij})\frac{1}{k_{ij}}\}^{\frac{d+1}{d-1}} k_{ij} - O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{1}{2(d-1)}+\varepsilon}}\Big) \\
&\geq \frac{1}{2}\operatorname{div}_{d-1}\Big(1 - O\Big(\frac{1}{n^{\frac{1}{2(d-1)}-\varepsilon}}\Big)\Big) \{\sum_{i,j} A(S_{ij})\}^{\frac{d+1}{d-1}} \frac{1}{n^{\frac{2}{d-1}}} - O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{1}{2(d-1)}+\varepsilon}}\Big) \\
&\geq \frac{1}{2}\operatorname{div}_{d-1}\Big(1 - O\Big(\frac{1}{n^{\frac{1}{2(d-1)}-\varepsilon}}\Big)\Big)\Big(1 - O\Big(\frac{1}{l^{\frac{\delta-1}{d-1}}}\Big)\Big)^{\frac{d+1}{d-1}} A(C)^{\frac{d+1}{d-1}} \frac{1}{n^{\frac{2}{d-1}}} - O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{1}{2(d-1)}+\varepsilon}}\Big) \\
&\geq \frac{1}{2}\operatorname{div}_{d-1} A(C)^{\frac{d+1}{d-1}} \frac{1}{n^{\frac{2}{d-1}}} - \frac{1}{n^{\frac{2}{d-1}}}\Big(O\Big(\frac{1}{n^{\frac{1}{2(d-1)}-\varepsilon}}\Big) + O\Big(\frac{1}{n^{\frac{1}{2(d-1)}-\varepsilon}}\Big) + O\Big(\frac{1}{n^{\frac{1}{2(d-1)}+\varepsilon}}\Big)\Big) \\
&\geq \frac{1}{2}\operatorname{div}_{d-1} A(C)^{\frac{d+1}{d-1}} \frac{1}{n^{\frac{2}{d-1}}} - O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{1}{2(d-1)}-\varepsilon}}\Big),
\end{aligned}
$$

for sufficiently large n, where the constants in the $O(\cdot)$ symbols depend on C and, possibly, ε.

2.4.2 Upper Estimate of $\delta^V(C, P_n)$. Finally the following result will be shown

(82) Let $\varepsilon > 0$. Then

$$\delta^V(C,P_n) \le \frac{1}{2}\operatorname{div}_{d-1} A(C)^{\frac{d+1}{d-1}} \frac{1}{n^{\frac{2}{d-1}}} + O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{1}{3(d-1)}-\varepsilon}}\Big) \text{ as } n \to \infty,$$

where the constant in $O(\cdot)$ depends only on C, ε.

Since it is sufficient to prove (82) for arbitrarily small $\varepsilon > 0$, we may assume that

(83) $\varepsilon > 0$ is so small that for

$$\delta = \frac{(9d-11)\varepsilon - 3(d-1)\varepsilon^2}{3(1-\varepsilon)}, \vartheta = 1 + \frac{1-3(d-1)\varepsilon}{\delta}, \omega = \frac{1-\varepsilon}{2(d-1)}$$

hold $0 < \delta < 1 < \vartheta, \delta\vartheta < 1$.

It is easy to see that

(84) $$(1-\delta\vartheta)\omega = \frac{\delta(\vartheta-1)}{d-1} = \frac{1}{3(d-1)} - \varepsilon.$$

We split the proof into several parts.

First,

(85) for all sufficiently large n, the number $l = \lfloor n^\delta \rfloor$ is sufficiently large in the sense of (12).

Choose ι, κ, λ, C_i, p_i, H_i, "$'$", f_i, q_{iu}, $\kappa_C(\cdot)$ for $i = 1, \dots, l$, such that (12) – (16) hold.

Second, for each i consider in H_i an edge-to-edge tiling with squares of area

$$\frac{|C_i'|A(C)}{l^\vartheta A(C_i)}.$$

Let $S_{ij}', j = 1, \dots, j_i$, be the squares which meet C_i' or are adjacent to such squares. By the definition the equi-affine surface area of $A(\cdot)$ (compare (2)), of surface integrals and by using (13) – (15), we see that $1/\lambda \le A(C_i)/|C_i'| \le \lambda$. Hence

(86) $$\frac{A(C)}{\lambda l^\vartheta} \le |S_{ij}'| = \frac{|C_i'|A(C)}{l^\vartheta A(C_i)} \le \frac{\lambda A(C)}{l^\vartheta},$$

$$\operatorname{diam} S_{ij}' \le \sqrt{d-1}\Big(\frac{\lambda A(C)}{l^\vartheta}\Big)^{\frac{1}{d-1}}.$$

It follows from the definition of S_{ij}', (12), (85) and (86) that

(87) for all sufficiently large n holds

$$C_i' + \Big(\frac{A(C)}{\lambda l^\vartheta}\Big)^{\frac{1}{d-1}} B^{d-1} \subset \bigcup_{i,j} S_{ij}' \subset C_i' + 2\sqrt{d-1}\Big(\frac{\lambda A(C)}{l^\vartheta}\Big)^{\frac{1}{d-1}} B^{d-1}$$
$$\subset \Big(1 + 2\sqrt{d-1}\left(\frac{\lambda A(C)}{l^\vartheta}\right)^{\frac{1}{d-1}} \iota l^{\frac{1}{d-1}}\Big) C_i' = \Big(1 + \frac{\nu}{l^{\frac{\vartheta-1}{d-1}}}\Big) C_i' \subset \frac{1}{2} C',$$

where $\nu > 0$ is a constant.

Let p_{ij}' be the center of S_{ij}' and $q_{ij} = q_{ip_{ij}'}$ the corresponding positive definite quadratic form, see 2.1.3. Let S_{ij} and p_{ij} be the inverse images of S_{ij}' and p_{ij}' on the lower side of C.

Third, it will be shown that

(88) for all sufficiently large n, the following hold: let $p \in S_{ij}$ and $x \in H_p$ such that $x' \in S_{ij}'$. Then

$$\|x' - x^{\pi\prime}\| < \left(\frac{A(C)}{\lambda l^\vartheta}\right)^{\frac{1}{d-1}}.$$

By (87), (12), (13), (87) and (86) we have that

$$\|x - p\| \leq \operatorname{diam} S_{ij} \leq \kappa \operatorname{diam} S_{ij}' \leq \kappa\sqrt{d-1}\left(\frac{\lambda A(C)}{l^\vartheta}\right)^{\frac{1}{d-1}} \leq \alpha$$

for all sufficiently large n and thus sufficiently large l^ϑ, where α is from (8). An application of (8) then shows that for such n,

$$\|x' - x^{\pi\prime}\| \leq \|x - x^\pi\| \leq \beta\kappa^2(d-1)(\lambda A(C))^{\frac{2}{d-1}} \frac{1}{l^{\frac{2\vartheta}{d-1}}}$$

and therefore, by choosing n even larger, if necessary,

$$\|x' - x^{\pi\prime}\| < \left(\frac{A(C)}{\lambda l^\vartheta}\right)^{\frac{1}{d-1}}.$$

Fourth,

(89) Let $T_{ij}' = \Big(1 + \frac{\nu}{l^{\frac{\vartheta-1}{d-1}}}\Big)^{-1} S_{ij}'$ and $k_{ij} = \left\lfloor \frac{A(T_{ij})n}{A(C)} \right\rfloor$. Then $k_{ij} = \left\{ \begin{array}{l} \leq O(\frac{n}{l^\vartheta}) \\ \geq O(\frac{n}{l^\vartheta}) \end{array} \right\}$, where the constants in the $O(\cdot)$ symbols depend only on C.

The inequalities for k_{ij} follow from (86). Let T_{ij} be the inverse image of T_{ij}' on the lower side of $\operatorname{bd} C$. From the definition of the sets S_{ij}, (89), (87) and (12) it follows that

(90) for all sufficiently large n the sets $T_{ij}, i = 1, \dots, l,\ j = 1, \dots, j_i$, are non-overlapping subsets of $\mathrm{bd}\, C$.

Hence the definition of k_{ij} in (89) implies the following:

(91) for all sufficiently large n, holds $k_{11} + \ldots + k_{lj_l} \leq n$.

Fifth, in each square S'_{ij} choose points $p'_{ijk}, k = 1, \dots, k_{ij}$, such that

$$(92) \qquad V_{k_{ij}}(S'_{ij}, q_{ij}) = \int\limits_{S'_{ij}} \min_{k=1,\dots,k_{ij}} \{q_{ij}(s - p'_{ijk}\} ds,$$

see (40). Let $q_{ijk} = q_{ip'_{ijk}}$ be the corresponding positive definite quadratic forms, see 2.1.3. Finally, let Q_n be the intersection of the support halfspaces of C at the points p_{ijk}. Since the sets S_{ij} cover $\mathrm{bd}\, C$ by (87) and (12) and since $\max\{\mathrm{diam}\, S_{ij}, i = 1, \dots, l, j = 1, \dots, j_l\} \to 0$ as $n \to \infty$ by (86), (85) and (13), we see that

(93) for all sufficiently large n holds $Q_n \in \mathcal{P}^c_{(n)}$.

The sets C_i form a dissection by (12) for sufficiently large n. Hence,

(94) for all sufficiently large n the sets $Q_{ni} = \{x \in Q_n : x^\pi \in C_i\}, i = 1, \dots, l$, form a dissection of $Q_n \backslash \mathrm{int} C$.

Clearly, the Dirichlet–Voronoi cells

(95) $D'_{ijk} = \{s \in S'_{ij} : q_{ij}(s - p'_{ijk}) \leq q_{ij}(s - p'_{ijm}) \text{ for } m = 1, \dots, k_{ij}\}$, $k = 1, \dots, k_{ij}$, form a dissection of the square S'_{ij}.

By (86), (89), (87), (14) and (41) we have that

$$(96) \qquad |D'_{ijk}| \left\{ \begin{array}{c} \leq O(\frac{1}{n}) \\ \geq O(\frac{1}{n}) \end{array} \right\} \text{ and } \mathrm{diam}\, D'_{ijk} \leq O\Big(\frac{1}{n^{\frac{1}{d-1}}}\Big),$$

where the constants in the $O(\cdot)$ symbols depend only on C.

Sixth, (93), (94), the definition of Q_n, (87), (88), (87), (95), Taylor's formula applied to f_i at p'_{ijk}, (13), (96), Taylor's formula applied to the coefficients of q_{ij}, (13), (86), (96), (91), (95), (85), (92), (87), (14), (51), (89), (85), (15), (84), Taylor's formula applied to $\kappa_C(1+\mathrm{grad}\, f_i)^2)^{1/2}$ at p'_{ijk}, (89), (87), (13), (16), (86), (85), (90) and (84) together imply the following: if n is so large that (85), (87), (88), (90), (91), (92) and (93) hold, then

$$\begin{aligned}
\delta^V(C,P_n) &= \delta^V(C,\mathcal{P}^c_{(n)}) \le \delta^V(C,Q_n) = \sum_i V(Q_{ni})\\
&\le \sum_i \int\limits_{C_i'+(\frac{A(C)}{\lambda l^\vartheta})^{\frac{1}{d-1}}B^{d-1}} \min_{j=1,\dots,j_i}\{\min_{k=1,\dots,k_{ij}}\{f_i(s)-f_i(p'_{ijk})-\operatorname{grad} f_i(p'_{ijk})(s-p'_{ijk})\}\}ds\\
&\le \sum_{i,j}\int\limits_{S'_{ij}} \min_{k=1,\dots,k_{ij}}\{f_i(s)-f_i(p'_{ijk})-\operatorname{grad} f_i(p'_{ijk})(s-p'_{ijk})\}ds\\
&\le \sum_{i,j,k}\int\limits_{D'_{ijk}} \{f_i(s)-f_i(p'_{ijk})-\operatorname{grad} f_i(p'_{ijk})(s-p'_{ijk})\}ds\\
&\le \frac12\sum_{i,j,k}\{\int\limits_{D'_{ijk}} q_{ijk}(s-p'_{ijk})ds + O\Big(\frac{1}{n^{\frac{3}{d-1}}}\Big)|D'_{ijk}|\}\\
&\le \frac12\sum_{i,j,k}\{\int\limits_{D'_{ijk}} q_{ij}(s-p'_{ijk})ds + O\Big(\frac{1}{n^{\frac{2}{d-1}}l^{\frac{\vartheta}{d-1}}}\Big)|D'_{ijk}|\} + O\Big(\frac{1}{n^{\frac{3}{d-1}}}\Big)\\
&\le \frac12\sum_{i,j}\{\int\limits_{Sij'} \min_{k=1,\dots,k_{ij}}\{q_{ij}(s-p'_{ijk})\}ds\} + O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{\delta\vartheta}{d-1}}}\Big)\\
&= \frac12\sum_{i,j} V_{k_{ij}}(S'_{ij},q_{ij}) + O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{\delta\vartheta}{d-1}}}\Big)\\
&\le \frac12\operatorname{div}_{d-1}\sum_{i,j}\Big(1+O\Big(\frac{1}{k_{ij}^\omega}\Big)\Big)(\det q_{ij})^{\frac{1}{d-1}}|S'_{ij}|^{\frac{d+1}{d-1}}\frac{1}{k_{ij}^{\frac{2}{d-1}}} + O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{\delta\vartheta}{d-1}}}\Big)\\
&\le \frac12\operatorname{div}_{d-1}\Big(1+O\Big(\frac{1}{n^{(1-\delta\vartheta)\omega}}\Big)\Big)\Big(1+\frac{\nu}{n^{\frac{\delta(\vartheta-1)}{d-1}}}\Big)\times\\
&\quad\times\sum_{i,j}(\det q_{ij})^{\frac{1}{d-1}}|T'_{ij}|^{\frac{d+1}{d-1}}\frac{A(C)^{\frac{2}{d-1}}}{A(T_{ij})^{\frac{2}{d-1}}}\frac{1}{n^{\frac{2}{d-1}}}\Big(1+O\Big(\frac1n\Big)\Big)^{\frac{2}{d-1}} + O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{\delta\vartheta}{d-1}}}\Big)\\
&\le \frac12\operatorname{div}_{d-1}\Big(1+O\Big(\frac{1}{n^{(1-\delta\vartheta)\omega}}\Big)+O\Big(\frac{1}{n^{\frac{\delta(\vartheta-1)}{d-1}}}\Big)+O\Big(\frac1n\Big)\Big)\times\\
&\quad\times\sum_{i,j}\{\kappa_C(p'_{ij})^{\frac{1}{d+1}}(1+(\operatorname{grad} f_i(p'_{ij}))^2)^{\frac12}|T'_{ij}|\}^{\frac{d+1}{d-1}}\frac{A(C)^{\frac{2}{d-1}}}{A(T_{ij})^{\frac{2}{d-1}}}\frac{1}{n^{\frac{2}{d-1}}} + O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{\delta\vartheta}{d-1}}}\Big)\\
&\le \frac12\operatorname{div}_{d-1}\Big(1+O\Big(\frac{1}{n^{\frac{1}{3(d-1)}-\varepsilon}}\Big)\Big)\Big(1+O\Big(\frac{1}{l^{\frac{\vartheta}{d-1}}}\Big)\Big)^{\frac{d+1}{d-1}}\times\\
&\quad\times\sum_{i,j}\{\int\limits_{T'_{ij}}\kappa_C(s)^{\frac{1}{d+1}}(1+(\operatorname{grad} f_i(s))^2)^{\frac12}ds\}^{\frac{d+1}{d-1}}\frac{A(C)^{\frac{2}{d-1}}}{A(T_{ij})^{\frac{2}{d-1}}}\frac{1}{n^{\frac{2}{d-1}}} + O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{\delta\vartheta}{d-1}}}\Big)\\
&\le \frac12\operatorname{div}_{d-1}\Big(1+O\Big(\frac{1}{n^{\frac{1}{3(d-1)}-\varepsilon}}\Big)+O\Big(\frac{1}{n^{\frac{\delta\vartheta}{d-1}}}\Big)\Big)A(C)^{\frac{2}{d-1}}\sum_{ij}A(T_{ij})\frac{1}{n^{\frac{2}{d-1}}} + O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{\delta\vartheta}{d-1}}}\Big)\\
&\le \frac12\operatorname{div}_{d-1}A(C)^{\frac{d+1}{d-1}}\frac{1}{n^{\frac{2}{d-1}}} + O\Big(\frac{1}{n^{\frac{2}{d-1}+\frac{1}{3(d-1)}-\varepsilon}}\Big),
\end{aligned}$$

where the constants in the $O(\cdot)$ symbols depend only on C and, possibly, ε. This concludes the proof of (82).

2.4.3 Conclusion. The Theorem finally follows from (68) and (82).

3 Acknowledgement

For their helpful hints I am obliged to Professors Goodey and Schnitzer.

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