# In Many Cases Optimal Configurations Are Almost Regular Hexagonal

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**Abstract.** This is a survey of results dealing with point sets which are optimal or almost optimal with respect to a given property and as a consequence are arranged in a regular or almost regular hexagonal pattern. The presented results and problems are from the geometry of numbers, from discrete geometry in the Euclidean plane, on the 2-sphere and on Riemannian 2-manifolds and from potential theory. They are related to problems of convexity, numerical integration and econometry and, presumably, to problems of biology, chemistry, physics, and engineering.

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# 1 Introduction

It is well known that in certain situations regular and almost regular hexagonal arrangements are optimal or, at least, close to optimal. Examples of this fact can be found in the geometry of numbers, in discrete geometry, in econometry and in other areas.

A related, more difficult problem is whether conversely, in such cases optimal or close to optimal arrangements are regular or almost regular hexagonal. Apart from some results in the geometry of numbers, the pertinent results are of rather recent origin. In this article such results will be discussed. In particular we will consider the following topics:

lattice packing and covering with circles and quantization of data,
the Epstein zeta function,
general packing and covering with circles in the plane and on Riemannian 2-manifolds; applications,
nets in the plane, on the 2-sphere and on Riemannian 2-manifolds,
the theorem on sums of moments and its stability counterpart in the plane and on Riemannian 2-manifolds; applications,
distribution of electric charges on the 2-sphere and on Riemannian 2-manifolds.

In architecture, the technical sciences, chemistry, crystallography, geology, botany, physiology, microbiology and other areas there are many cases where regular or almost regular hexagonal arrangements appear. Presumably this is due to the fact that engineers and nature have the tendency to optimize. Since some of the mathematical results that will be discussed seem to be related to certain situations in engineering and in the sciences where regular hexagons appear, these results may contribute to a better understanding of the latter. Unfortunately, clear-cut applications are frequently out-of-reach since in most cases it is difficult to specify precisely what is optimized. Most probably, in typical cases it is a whole bundle of properties.

# 2 Lattice packing and covering by circles and quantization of data

In this section we describe classical results on lattice packing and covering by circles from the geometry of numbers and a result on the best lattice quantizer in Euclidean 2-space $E^2$. For each result one may formulate a corresponding stability problem. This is done explicitly only in the case of Theorem 2.

## 2.1 Densest lattice packing by circles

A result which goes back to Lagrange [47] is the following.

**Theorem 1.** *Let $q(x,y) = ax^2 + 2bxy + cy^2$ be a positive definite binary quadratic form. Then*

$$\inf_{u,v \in Z, \neq 0,0} \{q(u,v)\} \leq \left(\frac{4D}{3}\right)^{1/2},$$

*where $Z$ is the ring of rational integers and $D = ac - b^2$. The equality sign is needed precisely if there is a linear transformation of the variables with integer coefficients and determinant $\pm 1$, i.e. an integer unimodular transformation, which transforms $q$ into a multiple of $x^2 + xy + y^2$.*

A *lattice* $L$ in $E^2$ is the system of all integer linear combinations of two linearly independent vectors $b_1, b_2 \in E^2$. The pair $\{b_1, b_2\}$ is then called a *basis* of $L$ and the area of the *fundamental parallelogram* $\{t_1 b_1 + t_2 b_2 : 0 \leq t_i < 1\}$ is the *determinant* $d(L)$ of $L$. The lattice $L$ is said to be *regular hexagonal* if it has a basis $\{b_1, b_2\}$ with $\|b_1\| = \|b_2\| = \|b_1 - b_2\|$. Here $\| \cdot \|$ is the ordinary Euclidean norm on $E^2$. Let $C$ be a compact convex set with non-empty interior in $E^2$, i.e. a *convex body* in $E^2$ and $L$ a lattice in $E^2$. The *density* of the *set lattice* $\{C + l : l \in L\}$ is defined as $|C|/d(L)$, where $| \cdot |$ stands for area in $E^2$. More intuitively, it is the 'total volume of the sets $C + l : l \in L$ divided by the total volume of $E^2$'. The set lattice $\{C + l : l \in L\}$ is a *lattice packing* by $C$ with *packing lattice* $L$ if the sets $C + l$ have pairwise disjoint interiors; it is a *lattice covering* by $C$ if the union of all its sets is $E^2$.

Regular hexagonal lattices *per se* and the related regular planar honeycombs (see 5.1) have been investigated in hundreds of articles. The pertinent results deal with crystallography, the Ising model in metallurgy, with mechanical, physical and chemical problems of plates with regular hexagonal cells, and other situations. In contrast, in this article we describe geometric and analytic properties which characterize the regular hexagonal lattices *among all* planar lattices.

It seems that Gauss [25] was the first to see that the above result of Lagrange is equivalent to the next statement.

**Theorem 2.** *Any lattice packing by the (solid Euclidean) unit circle in $E^2$ has density at most $\pi/2\sqrt{3} = 0.9068996\ldots$. Equality is attained precisely in the case when the packing lattice is regular hexagonal and has a basis $\{b_1, b_2\}$, where $\|b_1\| = \|b_2\| = 2$.*

Precise density bounds for lattice packings by balls are known up to dimension 8, for higher dimensions there exist only estimates. See Gruber and Lekkerkerker [36], Conway and Sloane [5] and Zong [75].

We formulate a stability counterpart of Theorem 2 as our first problem.

**Problem 1.** *Specify a ('simple' and 'small') function $f(\varepsilon)$ for $\varepsilon > 0$ such that the following hold: let $L$ be a lattice in $E^2$ which provides a packing by the unit circle of density greater than $(\pi/2\sqrt{3}) - \varepsilon$. Then $L$ has a basis $\{c_1, c_2\}$ such that $\|c_1 - b_1\|, \|c_2 - b_2\| \leq f(\varepsilon)$, where $b_1, b_2$ are suitable vectors in $E^2$ with $\|b_1\| = \|b_2\| = \|b_1 - b_2\| = 2$.*

Clearly, Problem 1 and the corresponding covering problem may be formulated also for higher dimensions. In 4.2 we state a related weak stability result concerning packing by geodesic circles on Riemannian 2-manifolds.

## 2.2 Thinnest lattice covering by circles

It is surprising that the covering result which corresponds to the packing result of Theorem 2 was not given until 1939 by Kershner [48] but then in a more general form, see 4.1. Here we state only the lattice case.

**Theorem 3.** *Any lattice covering by the unit circle in $E^2$ has density at least $2\pi/3\sqrt{3} = 1.2091996\ldots$. Equality is attained pecisely if the covering lattice is regular hexagonal and has a basis $\{b_1, b_2\}$, where $\|b_1\| = \|b_2\| = \|b_1 - b_2\| = \sqrt{3}$.*

The arithmetic version of this result is as follows.

**Theorem 4.** *Let $q(x, y)$ be a positive definite binary quadratic form. Then*

$$\sup_{\xi,\eta\in R} \inf_{u,v\in Z} \{q(\xi - u, \eta - v)\} \geq \left(\frac{2D}{3\sqrt{3}}\right)^{1/2}.$$

*The equality sign is needed precisely if there is an integer unimodular linear transformation of the variables which transforms $q$ into a multiple of $x^2 + xy + y^2$.*

The reason for the late appearence of the covering result seems to have been the fact that in classical Diophantine approximation and geometry of numbers the arithmetical version of the covering result has not attracted much interest. More generally, the 'inhomogeneous or covering problems' have attracted less interest than the 'homogeneous or packing problems'. In modern geometry of numbers and, in particular, in computational geometry of numbers this is not true any more.

Theorem 3 has been extended up to dimension 5 and for higher dimensions there exist estimates, see Gruber and Lekkerkerker [36] and Conway and Sloane [5].

### 2.3 Lattice quantizers

Let $L$ be a lattice in $E^2$. Its *Dirichlet-Voronoi cell* $D(L, o)$ is the set of all points in $E^2$ which are at least as close to the origin $o$ as to any other point of $L$:

$$D(L, o) = \{x \in E^2 : \|x\| \leq \|x - l\| \text{ for all } l \in L\}.$$

Now consider data where each datum is a point in $E^2$ and assume uniform distribution of data. It is impossible to store each datum, but given a datum one may store (one of) the point(s) in $L$ nearest to it and use the latter instead. This approximation process is called *quantization of data with respect to the lattice* $L$. It turns out that a good definition of the *error* of the approximation process is

$$E(L, \|\cdot\|^2) = \int\limits_{D(L,o)} \|x\|^2 dx.$$

See Conway and Sloane [5] and Section 6 below for more information.

An immediate consequence of the *moment lemma of Fejes Tóth* (see [22, 41]) and the fact that Dirichlet-Voronoi cells of planar lattices are $o$-symmetric hexagons or parallelograms is the following result which yields the best lattice quantizer in dimension 2:

**Theorem 5.** *Among all lattices in $E^2$ of the same determinant precisely the regular hexagonal lattices have minimum error of quantization.*

The problem to determine the best lattice quantizer was solved by Barnes and Sloane [2] in dimension three, but for higher dimensions only conjectures and estimates are available.

## 3 The Epstein zeta-function

We describe results on minima of the Epstein zeta-function and their connection with the optimal choice of nodes for numerical integration of functions in a Sobolev space.

### 3.1 Minima of the Epstein zeta-function

Given a lattice $L$ in $E^2$, the corresponding *Epstein zeta function* $\zeta(L, \cdot)$ is defined by

$$\zeta(L, s) = \sum_{l \in L, \neq o} \frac{1}{\|l\|^{2s}} \quad \text{for} \quad s > 1.$$

This zeta-function was first introduced by Epstein [10] in 1903. It is important for the determination of the potential of crystal lattices and for the lattice energy.

Moreover, it has applications in the dynamics of viscuous fluids. Sobolev [65] rediscovered it in 1961 in the context of numerical integration of functions in a Sobolev space. In the geometry of numbers the Epstein zeta-function was investigated in Great Britain in the 1950s and the 1960s and in Russia since the late 1960s. The motive for the investigations in Great Britain was purely number-theoretic, whereas the Russian school of geometry of numbers started its investigations with a problem of Sobolev about the optimal choice of nodes for numerical integration of functions in a Sobolev space. See Naas and Schmidt [55] for general information on the zeta function and Gruber and Lekkerkerker [36] for information in our context.

A major problem in this area is to determine for each $s > 1$ among all lattices $L$ of determinant 1 those for which $\zeta(L, s)$ is the absolute or a relative minimum. The following results were achieved by Rankin [59], Cassels [4], Ennola [8], Diananda [6] and Montgomery [52] (absolute minimum) and by Ryskov [60] (relative minima).

**Theorem 6.** *The following claims hold:*
*(i) Let $s > 1$. Then $\zeta(\cdot, s)$ attains its absolute minimum $m$ among all lattices of determinant 1 in $E^2$ precisely for the regular hexagonal lattices.*
*(ii)Let $s \geq 3$. Then the only relative minimum of $\zeta(\cdot, s)$ among all lattices of determinant 1 in $E^2$ is the absolute minimum.*

All proofs rely on heavy computations. So far this seems to prevent to find a related stability result.

**Problem 2.** *Let $s > 1$. Specify a ('simple' and 'small') function $f(\varepsilon)$ for $\varepsilon > 0$ with the following property: let $L$ be a lattice of determinant 1 in $E^2$ such that $\zeta(L, s) < m + \varepsilon$, where $m$ is the absolute minimum. Then $L$ has a basis $\{c_1, c_2\}$ such that $\|b_1 - c_1\|, \|b_2 - c_2\| \leq f(\varepsilon)$, where $b_1, b_2$ are suitable vectors such that $\|b_1\| = \|b_2\| = \|b_1 - b_2\| = (2/\sqrt{3})^{1/2}$.*

A solution of this problem might be helpful for the approximate description of the equilibrium distribution of $n$ unit charges on a sphere or a more general manifold for large $n$; a precise description seems to be out of reach. See Section 7.

One of the few higher-dimensional results about minima of the zeta-function says the following: any lattice (of determinant 1) which yields a densest packing of balls in $E^3$ is a relative minimum for the zeta-function for each $s > 3/2$. This was proved by Ennola [9] and for $s > 2$ by Sandakova [62].

### 3.2 Optimal choice of nodes in numerical integration formulae

In order to show the importance of the above minimum problem for the zeta-function we state the following special case of a result of Sobolev [65].

First, a definition is needed: Let $J$ be a Jordan measurable set in $E^2$ and consider a class $\mathcal{F}$ of Riemann integrable functions $g : J \to I\!R$. Given sets of nodes $N_n = \{p_1, \cdots, p_n\}$ in $J$ and weights $W_n = \{w_1, \cdots, w_n\}$ in $I\!R$, let the *error* be defined by

$$E(\mathcal{F}, N_n, W_n) = \sup_{g\in\mathcal{F}} \{ | \int_J g(x)\, dx - \sum_{k=1}^{n} w_k g(p_k) | \}.$$

Then the following result holds.

**Theorem 7.** *Let $L = BZ^2$ be a lattice of determinant 1 in $E^2$ where $B$ is a $(2,2)$ matrix. Let $F$ be a fundamental parallelogram of $L$ and let $\mathcal{F}$ be the family of all $L$-periodic functions $g : E^2 \to I\!R$ of class $\mathcal{C}^s$ for which the Sobolev norm*

$$\int_F \Big( \sum_{|\alpha|=s} (D^\alpha g)^2 \Big)^{1/2} dx$$

*is at most 1. For $n = 1, \cdots,$ consider the sets $N_{n^2} = ((1/n)L) \cap F$ and $W_{n^2} = \{1/n^2, \cdots, 1/n^2\}$ of $n^2$ nodes, resp. weights. Then*

$$E(\mathcal{F}, N_{n^2}, W_{n^2}) \leq \frac{1}{(2n\pi)^s} \zeta(B^{-T} Z^2, s)^{1/2}.$$

*The bound for the error is minimum if the lattice $B^{-T} Z^2$ is regular hexagonal.*

## 4 General packing and covering with circles in the plane and on Riemannian 2-manifolds; applications

In this section we consider density estimates for general packings and coverings of circles in $E^2$ from discrete geometry and their extensions to Riemannian 2-manifolds. In addition, corresponding stability results are presented. The density estimate and the corresponding stability result for coverings on Riemannian 2-manifolds yield results on the asymptotic approximation of smooth convex bodies in $E^3$ by convex polytopes with respect to the Hausdorff metric.

### 4.1 Circle packing and covering in the plane

Given a bounded set $J$ in $E^2$ or on a Riemannian 2-manifold, a system of (solid) circles of equal radii is called a *circle packing* of $J$ if the circles are contained in $J$ and have disjoint interiors. It is called a *circle covering* if $J$ is contained in the union of the circles. The sum of the areas of the circles divided by the area of $J$ - provided the latter exists - is the *density* of the packing, resp. covering. If $J$ is an unbounded set in $E^2$, these notions generalize in a natural way, but they may not be extended to unbounded sets on all manifolds. An example where such an extension is impossible is the hyperbolic plane.

An early result of Thue [69], proved by L.Fejes Tóth [17] and Segre and Mahler [64] in a more rigorous form, says that any circle packing of $E^2$ has density at most $\pi/2\sqrt{3}$. Kershner [48] proved that any circle covering of $E^2$ has density at least $2\pi/3\sqrt{3}$. These estimates are precise and equality is attained for the regular hexagonal lattice packings and coverings. Note that there are uncountably many other, essentially different circle packings and coverings for which equality also holds. Thue's result was extended to dimension 3 by Hales [42] who showed that the density of any packing of balls in $E^3$ is less or equal to the density of the densest lattice packing of balls. In higher dimensions only estimates are known. See Conway and Sloane [5] and Zong [75].

The results of Thue and Kershner are consequences of the following estimate of L.Fejes Tóth [18, 22].

**Theorem 8.** *Let $C$ be a compact convex set in $E^2$. Then the following statements hold:*

(i) *Any circle packing of $C$ consisting of two or more circles of equal radii has density less than $\pi/2\sqrt{3}$.*

(ii) *Any circle covering of $C$ consisting of two or more circles of equal radii has density greater than $2\pi/3\sqrt{3}$.*

It is not difficult to show the following asymptotic formulae.

**Theorem 9.** *Let $J$ be a compact set in $E^2$ bounded by a simply closed rectifiable curve. Then the following claims hold:*

(i) *For $n = 1, 2, \cdots$, let $\delta(n)$ be the maximum density of a packing of $J$ by $n$ circles of equal radii. Then*

$$\delta(n) = \frac{\pi}{2\sqrt{3}} + O\Big(\frac{1}{n^{1/2}}\Big) \ \text{as} \ n \to \infty.$$

(ii) *For $n = 1, 2, \cdots$, let $\vartheta(n)$ be the minimum density of a covering of $J$ by $n$ circles of equal radii. Then*

$$\vartheta(n) = \frac{2\pi}{3\sqrt{3}} + O\Big(\frac{1}{n^{1/2}}\Big) \ \text{as} \ n \to \infty.$$

For the next result we need a definition: let $S$ be a finite point set in $E^2$ and let $\delta > 0$. A point $p \in S$ is the *center of a regular hexagon in $S$ up to $\delta$* if there are $\sigma > 0$, the *edge length* of the hexagon, and points $p_1, \cdots, p_6 \in S$, such that

$$S \cap \{x : \|x - p\| \leq 1.1\,\sigma\} = \{p, p_1, \cdots, p_6\},$$

$$\|p - p_i\|, \|p_{i+1} - p_i\| = (1 \pm \delta)\sigma.$$

The choice of the number 1.1 is somewhat arbitrary and $(1 \pm \delta)\sigma$ means a quantity between $(1 - \delta)\sigma$ and $(1 + \delta)\sigma$.

The author's [32] stability counterpart of Theorem 8(ii) is as follows.

**Theorem 10.** *Let $H$ be a convex 3,4,5, or 6-gon and $\varepsilon > 0$ sufficiently small. Then for all coverings of $H$ by, say, $m$ congruent circles of sufficiently small radius and density less than $(2\pi/3\sqrt{3})(1 + \varepsilon)$ the following hold: in the set of centers of these circles, each center, with a set of less than $50\varepsilon^{1/3}m$ exceptions, is the center of a regular hexagon up to $500\varepsilon^{1/3}$. All these hexagons have the same edge length (in the above sense).*

In other words: if the radius of the circles of the covering is sufficiently small and the density very close to $2\pi/3\sqrt{3}$, then the covering is 'almost regular hexagonal'.

A corresponding result for the packing case can be proved in a similar way.

In view of applications two generalizations of Theorems 8 and 10 suggest themselves, to higher dimensions and to Riemannian manifolds. Except for the result of Hales, extensions to higher dimensions are out of reach. Extensions to Riemannian 2-manifolds are possible and will be described in subsection 4.3.

## 4.2 Circle packing and covering on the 2-sphere

Ever since the biologist Tammes [66] investigated the orifices on spherical pollen grains, problems of densest packing and thinnest covering of $S^2$ by $n$ circles of equal radii have attracted interest. We refer to the surveys of G.Fejes Tóth and Kuperberg [16], G.Fejes Tóth [14] and L. Fejes Tóth [23] and state only the following estimate of Habicht and van der Waerden [37]; see also van der Waerden [70].

**Theorem 11.** *For $n = 1, 2, \cdots$, let $\delta(n)$ be the maximum density of a packing of $S^2$ by $n$ circles of equal radii. Then*

$$\frac{\pi}{2\sqrt{3}} - \frac{\text{const}}{n^{1/6}} \leq \delta(n) \leq \frac{\pi}{2\sqrt{3}}.$$

In the next subsection we will see that packings of $S^2$ with $n$ circles of equal radii of maximum density necessarily are asymptotically regular hexagonal.

### 4.3 Circle packing and covering in Riemannian 2-manifolds

Let $M$ be a *Riemannian 2-manifold*. By this we mean that $M$ is a 2-manifold of differentiability class $\mathcal{C}^1$ with a metric tensorfield having continuous coefficients. Then, in particular, for any $p \in M$ there are a neighborhood $U$ of $p$ in $M$ and a homeomorphism $h : U \to U'$, where $U'$ is an open set in $E^2$, and for each $u \in U'$ there is a positive definite binary quadratic form $q_u$ on $E^2$, the coefficients of which depend continuously on $u$. A curve $K$ in $U$ is of class $\mathcal{C}^1$ if it has a parametrization $x : [a, b] \to U$ such that $u = h \circ x$ is a parametrization of class $\mathcal{C}^1$ of a curve in $U'$. The *length* of $K$ then is defined as

$$\int_a^b q_{u(s)}^{1/2}(\dot{u}(s))\, ds.$$

By dissecting suitably and adding, the notion of length can be defined for curves in $M$ not contained in a single neighborhood. For $x, y \in M$ let their *(Riemannian) distance* $\varrho_M(x, y)$ be the infimum of the lengths of the curves of class $\mathcal{C}^1$ in $M$ connecting $x$ and $y$. The *(solid Riemannian) circle* in $M$ with center $p \in M$ and radius $\varrho > 0$ is the set $\{x \in M : \varrho_M(p, x) \le \varrho\}$. A set $J$ in $U$ is *Jordan measurable* if $h(J)$ is Jordan measurable as a subset of $E^2$. Then its *(Riemannian) area measure* $\omega_M(J)$ is defined as

$$\int_{h(J)} (\det q_u)^{1/2}\, du.$$

Again, by dissecting suitably and adding, the notions of Jordan measurability and area measure can be defined for sets in $M$ which are not contained in a single neighborhood.

Theorems 8 and 9 have the following extension, see Gruber [29] for the covering result and its history; the packing result may be shown in a similar way.

**Theorem 12.** *Let $J$ be a Jordan measurable set in $M$ with $\omega_M(J) > 0$. Then the following hold:*

(i) *Let $\delta(n)$ denote the supremum of the densities of the packings of $J$ by $n$ circles of equal radii. Then*

$$\lim_{n\to\infty} \delta(n) = \frac{\pi}{2\sqrt{3}}.$$

(ii) *Let $\vartheta(n)$ denote the infimum of the densities of the coverings of $J$ by $n$ circles of equal radii. Then*

$$\lim_{n\to\infty} \vartheta(n) = \frac{2\pi}{3\sqrt{3}}.$$

In the planar case we were able to give information about the rate of convergence of the densities of the densest packing by $n$ circles and the thinnest covering by $n$ circles as $n \to \infty$, see Theorem 9. It is not known whether there are similar refinements available for general Riemannian 2-manifolds, but Habicht and van der Waerden [37] were able to show a refinement in the case of the 2-sphere, see Theorem 11 above.

Before stating a weak stability result corresponding to Theorem 12, some definitions are in place. Let $M$, $J$, $\varrho_M$ and $\omega_M$ be as above and let $(S_n)$ be a sequence of finite sets in $M$ such that $\# S_n = n$, where $\# S_n$ is the number of elements of $S_n$. We say that $S_n$ is *uniformly distributed in* $J$ (*with respect to* $\omega_M$) *as* $n \to \infty$, if for any Jordan measurable set $K$ in $J$ with $\omega_M(K) > 0$ we have that

$$\frac{\#(S_n \cap K)}{\# S_n} \to \frac{\omega_M(K)}{\omega_M(J)} \text{ as } n \to \infty.$$

We say that $S_n$ *is asymptotically a regular hexagonal pattern of edge length* $\sigma_n$ *in* $M$ (*with respect to* $\varrho_M$) *as* $n \to \infty$ if there are a positive sequence $(\sigma_n)$ and Landau symbols $o(n)$ and $o(1)$ such that the following hold: for all $p \in S_n$, with a set of at most $o(n)$ exceptions, we have that

$$\{x : \varrho_M(p, x) \leq 1.1\, \sigma_n\} \cap S_n = \{p, p_1, \cdots, p_6\}, \text{ say,}$$

where

$$\varrho_M(p, p_i), \varrho_M(p_i, p_{i+1}) = (1 \pm o(1))\sigma_n \text{ (as } n \to \infty).$$

Almost regular hexagonal pattern on surfaces, in many cases with faults, appear frequently in biology and microbiology. The following figure from Pum, Messner and Sleytr [57] shows a freeze-etched preparation of the surface of the archaebacterium *methanocorpusculum sinense.*

Early references exhibiting almost regular hexagonal pattern in biology are the classical books of Häckel [38, 39]. Among the numerous more recent references we mention Wehner and Gehring [74].

The stability counterpart of Theorem 12 can be formulated as follows.

**Theorem 13.** *Let $J$ be a Jordan measurable set in $M$ with $\omega_M(J) > 0$. For each $n = 1, 2, \cdots$, consider a packing, resp. covering of $J$ by $n$ circles of equal radii such that the densities of these packings, resp. coverings tend to $\pi/2\sqrt{3}$ and $2\pi/3\sqrt{3}$, respectively. Then, as $n \to \infty$, the set of centers of the nth packing, resp. covering is uniformly distributed in $J$ and asymptotically a regular hexagonal pattern in $J$ of edge length*

$$\Big(\frac{2\omega_M(J)}{\sqrt{3}\,n}\Big)^{1/2}.$$

The proof of the covering result is due to Gruber [32]. The packing result can be shown similarly.

Even for relatively simple Jordan measurable sets $J$ and Riemannian 2-manifolds $M$ - one example is the sphere $S^2$ - it is hopeless to find for sufficiently large $n$ *densest* circle packings, respectively *thinnest* circle coverings, see G.Fejes Tóth and Kuperberg [16]. In contrast to this, it is possible to construct packings and coverings with $n$ circles in $M$ such that the corresponding densities tend to $\pi/2\sqrt{3}$ and $2\pi/3\sqrt{3}$, respectively, as $n \to \infty$.

## 4.4 Approximation of convex bodies by polytopes with respect to the Hausdorff metric

A *convex body* in $E^d$ is a compact convex subset of $E^d$ with non-empty interior. The *(Pompeiu-)Hausdorff metric* $\delta^H$ on the space of all convex bodies in $E^d$ is defined as follows: given two convex bodies $C, D$, their Hausdorff distance $\delta^H(C, D)$ ist the maximum distance which a point of one of the two bodies can have from the other body. Given a convex body $C$, let $\mathcal{P}_n^i$ denote the family of all convex polytopes which are inscribed into $C$ and have $n$ vertices. A convex polytope $P_n \in \mathcal{P}_n^i$ is *best approximating* for $C$ with respect to $\delta^H$ if

$$\delta^H(C, P_n) = \inf\{\delta^H(C, P) : P \in \mathcal{P}_n^i\}.$$

Major problems in this area are to determine $\delta^H(C, P_n)$, at least asymptotically as $n \to \infty$, and to describe the best approximating polytopes $P_n$. See Gruber [31] for detailed information.

Assume now that the boundary $\mathrm{bd}\, C$ of a convex body $C$ is a surface of class $\mathcal{C}^2$ with positive Gauss curvature $\kappa_C$. Let $\mathrm{bd}\, C$ be endowed with the Riemannian metric $\varrho_{II}$ of the second fundamental form. Schneider [63] showed that the problem of approximating $C$ by inscribed convex polytopes with respect to the Hausdorff metric and the problem to cover $\mathrm{bd}\, C$ by circles of the same radius are closely related. Using the tools developed in the previous subsection, this leads to the following result.

**Theorem 14.** *Let $C$ be a convex body in $E^3$ of class $\mathcal{C}^2$ with positive Gauss curvature $\kappa_C$ and let $\mathrm{bd}\, C$ be endowed with the Riemannian metric $\varrho_{II}$ and area measure $\omega_{II}$ of the second fundamental form. For $n = 4, 5, \cdots$, let $P_n \in \mathcal{P}_n^i$ be a best approximating convex polytope for $C$. Then the following statements hold:*

(i) $\delta^H(C, P_n) \sim \dfrac{1}{\sqrt{27}n}\omega_{II}(\mathrm{bd}\, C)$ *as* $n \to \infty$; $\omega_{II}(\mathrm{bd}\, C) = \displaystyle\int_{\mathrm{bd}\, C} \kappa_C^{1/2}(x)\, dx.$

(ii) *The set of vertices of $P_n$ is uniformly distributed in* $\mathrm{bd}\, C$ *and asymptotically a regular hexagonal pattern in* $\mathrm{bd}\, C$ *of edge length*

$$\Big(\frac{2\omega_{II}(\mathrm{bd}\, C)}{\sqrt{3}n}\Big)^{1/2}.$$

Confirming a conjecture of L.Fejes Tóth [22], (i) was proved by Schneider [63] for $\mathrm{bd}\, C$ of class $\mathcal{C}^3$ and by Gruber [29] in the present case. It extends to all dimensions. The constants in these results are related to the minimum density of coverings of $E^{d-1}$ by balls of equal radii. The statement about uniform distribution in (ii) was obtained by Glasauer and Schneider [27]; it holds in all dimensions. The assertion that the set of vertices is asymptotically a regular hexagonal pattern was proved by Gruber [32]. This seems to be difficult to generalize to higher dimensions.

While it is hopeless to construct best approximating polytopes, it is possible to find sequences of polytopes $(Q_n)$, where $Q_n \in \mathcal{P}_n^i$ such that $\delta^H(C, Q_n) \sim \omega_{II}(\mathrm{bd}\, C)/\sqrt{27}n$ as $n \to \infty$.

Similar results hold for circumscribed and general best approximating polytopes and also in the case where instead of vertices facets are considered. Results of this form are valid also for the Banach-Mazur distance and for Schneider's notion of distance, see [29].

# 5 Nets in the plane, on the 2-sphere and on Riemannian 2-manifolds

This section contains results from discrete geometry on nets in $E^2$, in particular the recently proved honeycomb conjecture, further some remarks on nets on the 2-sphere $S^2$ and, finally, a conjecture dealing with an extension of the honeycomb conjecture to Riemannian 2-manifolds and a corresponding stability result.

## 5.1 Nets in the Euclidean plane

Let $J$ be a *Jordan domain* in $E^2$. By this we mean a simply closed compact set bounded by a closed Jordan curve. Consider a dissection of $J$ into finitely many Jordan domains. The union of the boundaries of the Jordan domains of the dissection is a *net* in $J$, the domains are its *facets* and the sum of their perimeters is the *length* of the net. If $J$ is a convex polygon and all facets of a net in $J$ are convex, then the facets all are convex polygons.

**Theorem 15.** *Let $H$ be a convex 3, 4, 5, or 6-gon in $E^2$ and let a net in $H$ with convex facets, each of area $|H|/n$, be given. Then the length of the net is at least $12^{1/4}(|H|n)^{1/2}$.*

This result is a special case of an estimate of L. Fejes Tóth [22], p. 84. For a related result see G. Fejes Tóth [13]. For corresponding results dealing with more general sets $H$ and nets with non-convex facets we refer to Hales [43], Theorem 2. See also Theorem 16 below.

Theorem 15 or, more precisely, its proof implies that among the nets in $H$ with convex facets 'almost regular hexagonal nets' are close to optimal. This naturally leads to the converse question whether optimal nets with convex facets are 'almost regular hexagonal'. It is not too difficult to give a pertinent stability result. We refrain from doing so but in 5.3 a more general conjecture concerning weak stability of nets on Riemannian 2-manifolds will be presented.

The notion of net in a Jordan domain easily extends to (unbounded, locally finite) nets in the plane. By a *regular hexagonal net* or a *honeycomb* in the plane we then mean a net which is defined by a lattice tiling of $E^2$ by regular hexagons. The following result due to Hales [43] confirms the so-called *honeycomb conjecture*; for earlier work on this conjecture and for references see Morgan [54] and Phelan [56] and, in the 3-dimensional case, Wearie [73].

**Theorem 16.** *Let $N$ be a net in $E^2$, all facets of which have unit area and let $C$ be the solid unit circle in $E^2$. Then*

$$\limsup_{r\to+\infty} \frac{\operatorname{length}(N\cap rC)}{\sum\{\operatorname{area}(F\cap rC) : F \text{ facet of } N\}} \geq 2\sqrt{3},$$

*where equality holds if $N$ is a honeycomb.*

A major problem of the proof of Theorem 16 is the lacking convexity of the facets. While for $d = 2$ there are optimal nets all facets of which are convex, this does not seem to hold in case $d = 3$.

## 5.2 Nets on the 2-sphere

It is clear how to define *nets* on the 2-sphere $S^2$, but it is difficult to find nets with $n$ convex or, possibly, non-convex facets of given areas on $S^2$ of minimum length.

It follows from the theory of existence and structure of singularities (see Morgan [53] and Taylor [67]) that the nets of minimum length on $S^2$ with facets of given areas which may or may not be convex have the property that they consist of finitely many arcs of great circles which meet at trivalent equiangular vertices. Nets having this property but which are not necessarily of minimum length are called *Plateau nets*.

L.Fejes Tóth [21] has shown the following: if among the nets on $S^2$ with convex facets of the same area a net has minimum length and all its vertices are trivalent, then the net is regular, i.e. its facets are congruent regular spherical polygons. Thus, this can possibly happen only for $n = 2, 3, 4, 6, 12$.

Heppes [44, 45] has verified that there are 10 Plateau nets with convex facets on $S^2$ and that the five Plateau nets which are not regular are not solutions of the minimum length problem with convex facets. Whether the remaining five Plateau nets are solutions of the minimum length problem where the facets have equal areas and are convex or, more generally, not necessarily convex, remains open.

## 5.3 Nets on Riemannian 2-manifolds

Let $M$ be a Riemannian 2-manifold with Riemannian metric $\varrho_M$ and area measure $\omega_M$ as defined in 4.3. A *(geodesic) segment* in $M$ is a curve of class $\mathcal{C}^1$ in $M$ which connects two points in $M$ and has minimum length among all such curves. A continuous curve in $M$ consisting of finitely many segments is called a *(geodesic) polygon.* A set in $M$ is *(geodesically) convex* if with any two of its points it contains a segment connecting the points. A convex hexagon in $M$ is called *regular*, if its vertices have the same distance from a suitable point of the hexagon and any two consecutive vertices have the same distance. As for $E^2$ and $S^2$, we can define *nets* in $M$ or in a subset of it. Let $J$ be a Jordan measurable set in $M$, $(N_n)$ a sequence of nets in $J$, where $N_n$ has $n$ facets, and $(\sigma_n)$ a positive sequence. We say that $N_n$ is *asymptotically regular hexagonal of edge length* $\sigma_n$ if there are Landau symbols $o(n)$ and $o(1)$ such that up to $o(n)$ facets, each facet of $N_n$ contains a regular convex geodesic hexagon of edge length $\geq \sigma_n(1 - o(1))$ and is contained in such a hexagon of edge length $\leq \sigma_n(1 + o(1))$.

**Conjecture on the size and the stability of nets.** *Let $J$ be a simply connected compact set in $M$ bounded by a closed Jordan curve of finite length. Then the following statements hold:*

(i) *Consider for each $n = 1, 2, \cdots$, a net in $J$ with $n$ facets, each a compact set of area $\omega_M(J)/n$ bounded by a closed Jordan curve, such that the net has minimum length among all such nets. Then the lengths of these nets are asymptotically equal to* $12^{1/4}(\omega_M(J)n)^{1/2}$.

(ii) *Consider for $n = 1, 2, \cdots$, a net in $J$ with $n$ facets, each a compact set of area $\omega_M(J)(1 \pm o(1))/n$ bounded by a closed Jordan curve, and such that the* *lengths of these nets are asymptotically equal to* $12^{1/4}(\omega_M(J)n)^{1/2}$. *Then* $N_n$ *is asymptotically regular hexagonal of edge length*

$$\Big(\frac{2\omega_M(J)}{3\sqrt{3}n}\Big)^{1/2}.$$

Part (i) of this conjecture may be considered as an extension of Theorems 15 and 16 to Riemannian 2-manifolds. For the proof of part (ii) a stability counterpart of Theorem 16 is needed first.

# 6 The theorem on sums of moments and its stability counterpart

In this section we consider Fejes Tóth's theorem on sums of moments from discrete geometry, extend it to Riemannian 2-manifolds and state a corresponding stability result. The extension and its stability counterpart are then applied to asymptotic approximation of smooth convex bodies in $E^3$ with respect to the symmetric difference metric, the isoperimetric problem for convex polytopes in $E^3$, and the optimal choice of nodes in numerical integration formulae for functions on a Riemannian 2-manifold.

## 6.1 Sums of moments in the Euclidean plane

Besides the Euler polytope formula and Koebe's representation theorem for planar graphs by circles the following *theorem of L.Fejes Tóth* [19, 22] *on sums of moments* is one of the few general results in planar discrete geometry.

**Theorem 17.** *Let* $f : [0, +\infty) \to [0, +\infty)$ *be non-decreasing and let* $H$ *be a convex 3, 4, 5, or 6-gon in* $E^2$. *Then*

$$\inf_{\substack{S\subset E^2\\ \#S=n}} \int_H \min_{p\in S}\{f(\|x-p\|)\}\,dx \geq n \int_{H_n} f(\|x\|)\,dx, \tag{1}$$

*where* $H_n$ *is a regular hexagon in* $E^2$ *of area* $|H|/n$ *and center at* $o$.

L.Fejes Tóth [19] first proved this result for $S^2$ and only then for $E^2$, see [22]. For alternative proofs, in some cases for surfaces of constant width and special non-decreasing $f$, see L. Fejes Tóth [20], Imre [46], G.Fejes Tóth [12], Florian [24], Haimovich and Magnanti [40], Böröczky and Ludwig [3] and Gruber [33].

The estimate (1) says that in certain situations regular hexagonal configurations of points are close to optimal. We warn the reader that stronger assertions do not hold: for suitable $f$ the two sides of the inequality (1) differ by arbitrarily large factors if $n$ is sufficiently large. Further, again for suitable $f$, optimal sets $S$ may be rather different from any regular hexagonal pattern.

Fejes Tóth's inequality has a large number of applications including problems of packing and covering of circles, of optimal location, of errors of quantization of data, and of Gauss channels, all in $E^2$, of the isoperimetric problem for 3-dimensional polytopes of given combinatorial type such as tetrahedra, hexahedra and dodecahedra, of the optimal choice of nodes for numerical integration formulae, game theory and of asymptotically best approximation of convex bodies by polytopes with respect to the symmetric difference metric. See [34, 35] for references.

## 6.2 Sums of moments on Riemannian 2-manifolds and its stability counterpart

In view of applications, two extensions of Theorem 17 come to mind: to higher dimensions and to Riemannian 2-manifolds. At present the former seems to be out of reach. What can be expected for Riemannian 2-manifolds? In general no inequality of the form (1) holds. Thus the best one can hope for is an estimate for the difference of the two sides. For general $f$ the two sides of (1) may differ by arbitrarily large factors. In order to get useful results we thus have to restrict to suitable classes of functions $f$. Such classes should contain sufficiently general functions and exclude pathological ones. We consider the class of functions $f : [0, +\infty) \to [0, +\infty)$ which satisfy the following *growth condition*:

> $f(0) = 0$, $f$ is continuous and strictly monotone and for any $\alpha > 1$ there are $\beta > \gamma > 1$ (and vice versa) such that $\gamma f(t) \le f(\alpha t) \le \beta f(t)$ for all sufficiently small $t \ge 0$. Here $\beta, \gamma$ may be chosen arbitrarily close to 1 if $\alpha$ is sufficiently close to 1 (and vice versa).

All positive powers of $t$, all strictly monotone analytic functions $f$ with $f(0) = 0$, their inverses and many other functions satisfy this condition. In [34] we considered the class of functions $f$ of the form $f(t) = t^\alpha$, where $0 < \alpha \le 2$. For most applications this is sufficient.

Let $M$ be a Riemannian 2-manifold with Riemannian metric $\varrho_M$ and area measure $\omega_M$.

The following result of the author [35] extends the sum theorem to $M$.

**Theorem 18.** *Let $f : [0, +\infty) \to [0, +\infty)$ satisfy the growth condition and let $J$ be a Jordan measurable set in $M$ with $\omega_M(J) > 0$. Then*

$$\inf_{\substack{S \subset M \\ \#S = n}} \int\limits_J \min_{p \in S} \{ f(\varrho_M(p, x)) \} \, d\omega_M(x) \sim n \int\limits_{H_n} f(\|x\|) \, dx \quad \text{as} \quad n \to \infty,$$

*where $H_n$ is a regular hexagon in $E^2$ of area $\omega_M(J)/n$ and center at $o$. This asymptotic formula continues to hold under the assumption that $S$ is contained in $J$.*

The proof of this result makes essential use of the growth condition and of the fact that $M$ is locally Euclidean.

It follows from the proof of Theorem 18 that it is possible to specify explicitly sequences $(S_n)$ of sets in $M$ or even in $J$ with $\#S_n = n$ for which the asymptotic relation of Theorem 18 holds.

Theorem 18, as well as the original theorem on sums of moments of L.Fejes Tóth, says that for a series of geometric and analytic problems regular or almost regular hexagonal configurations are, at least, close to optimal. Examples will be presented later. More difficult is the question whether optimal or almost optimal configurations are almost regular hexagonal. As expected, in many situations the answer is yes. This is a consequence of the following weak stability counterpart of Theorem 18 due to Gruber [35]:

**Theorem 19.** *Let $f : [0, +\infty) \to [0, +\infty)$ satisfy the growth condition, let $J$ be a Jordan measurable set in $M$ with $\omega_M(J) > 0$ and let $(S_n)$ be a sequence of finite sets in $M$ with $\#S_n = n$ and such that*

$$\int_J \min_{p \in S_n} \{f(\varrho_M(p, x))\}\, d\omega_M(x) \sim \inf_{\substack{S \subset M \\ \#S = n}} \int_J \min_{p \in S} \{f(\varrho_M(p, x))\}\, d\omega_M(x) \ \text{ as } \ n \to \infty.$$

*Then $S_n$ is uniformly distributed in $J$ and asymptotically a regular hexagonal pattern of edge length*

$$\Big(\frac{2\omega_M(J)}{\sqrt{3}n}\Big)^{1/2}.$$

The proof of this result is complicated. Roughly speaking, it consists of two parts: in the Euclidean case, in essence, it follows the proof of the sum theorem in Gruber [33], carefully estimating what is lost at the various steps. The extension to the Riemannian case requires more care than is common in such extensions in order not to loose the stability property. In a preliminary draft of the proof use was made of Vitali's covering theorem for manifolds to overcome this difficulty which then could be replaced by a more elementary argument.

G.Fejes Tóth [15] has announced an alternative, more geometric proof of a result similar to Theorem 19 in the Euclidean case.

### 6.3 Approximation of convex bodies by polytopes with respect to the symmetric difference metric

The ordinary surface area measure on bd$C$ is denoted by $\omega$. Define the *symmetric difference metric* $\delta^V$ by $\delta^V(C,D) = V(C \triangle D)$ for any two convex bodies $C, D$. If the boundary of a convex body $C$ of class $\mathcal{C}^2$ with Gauss curvature $\kappa_C > 0$ is endowed with a Riemannian metric, for example with the Riemannian metric of the fundamental form of equi-affine differential geometry (compare [49]), this metric induces in any tangent plane of $\operatorname{bd} C$ an associated Euclidean metric.

The theorem on sums of moments and the approximation of a convex body in $E^3$ by circumscribed convex polytopes with respect to $\delta^V$ are closely connected as noted by Gruber [28]. Thus an application of the last two theorems yields the following result, see Gruber [28, 35].

**Theorem 20.** *Let $C$ be a convex body in $E^3$ of class $\mathcal{C}^2$ with $\kappa_C > 0$ and let $\operatorname{bd} C$ be endowed with the Riemannian metric $\varrho_{EA}$ and area measure $\omega_{EA}$ of the fundamental form of equi-affine differential geometry. For $n = 4, 5, \cdots$, let $P_n$ denote a convex polytope with $n$ facets circumscribed to $C$ and best approximating with respect to $\delta^V$. Then the following statements hold:*

(i) $\delta^V(C, P_n) \sim \dfrac{5\omega_{EA}^2(\operatorname{bd} C)}{36\sqrt{3}n}$ *as* $n \to \infty$; $\omega_{EA}(\operatorname{bd} C) = \displaystyle\int\limits_{\operatorname{bd} C} \kappa_C^{1/4}(x)\, d\omega(x)$.

(ii) *As $n \to \infty$, the sets $C \cap \operatorname{bd} P_n$ are uniformly distributed in $\operatorname{bd} C$ and the facets $F$ of $P_n$ are asymptotically regular hexagons with center $C \cap F$ and edge length*
$$\Big(\frac{2\omega_{EA}(\operatorname{bd} C)}{3\sqrt{3}n}\Big)^{1/2}.$$

Except for very special cases, it would be a waste of time to look for best approximating polytopes, but it is possible to construct for each $n = 4, 5, \cdots$, a convex polytope $Q_n$ with $n$ facets circumscribed to $C$ and such that holds $\delta^V(C, Q_n) \sim 5\omega_{EA}^2(\operatorname{bd} C)/36\sqrt{3}n$ as $n \to \infty$.

A result analogous to Theorem 20(i) but for the mean width instead of $\delta^V$ is due to Glasauer and Gruber [26]. There are corresponding results dealing with inscribed or general polytopes, see Gruber [30] and Ludwig [51]. All these results extend to dimensions $d \geq 3$. Similarly, Theorem 20(ii) has a companion for inscribed polytopes with $n$ vertices with respect to the mean width distance. K. Böröczky, Jr., informs us about a result analogous to Theorem 20(ii) but for general polytopes instead of circumscribed ones. If instead of vertices facets are considered, similar results hold.

## 6.4 The isoperimetric problem for convex polytopes in a Minkowski space

Let $E^d$ be endowed with a further norm. In this new normed space the volume $V(\cdot)$ is the ordinary volume, but there are different natural notions of surface area, for example those proposed by Busemann and by Holmes-Thompson; Busemann's notion coincides with the $(d-1)-$dimensional Hausdorff measure defined by means of the new norm. These notions amount to the introduction of an $o-$symmetric convex body $I$, the *isoperimetrix*, such that the *surface area* $S_I(C)$ of a convex body $C$ is defined by

$$S_I(C) = \lim_{\delta \to +0} \frac{V(C + \delta I) - V(C)}{\delta}, \text{ where } C + \delta I = \{x + \delta y : x \in C, y \in I\}.$$

If $I$ is the solid Euclidean unit ball, we get the ordinary surface area. For more information see Thompson [68].

A result of Diskant [7] says that a polytope with minimum isoperimetric quotient (using $S_I$) amongst all convex polytopes with $n$ facets after a suitable homothety is circumscribed to the isoperimetrix $I$. Combining this with Theorem 20 yields the next result, see [35].

**Theorem 21.** *Let $I$ be an isoperimetrix in $E^3$ of class $\mathcal{C}^2$ with positive Gauss curvature. Assume that* $\mathrm{bd}\, I$ *is endowed with the Riemannian metric $\varrho_{EA}$ and area measure $\omega_{EA}$ of the fundamental form of equi-affine differential geometry. For $n = 4, 5, \cdots$, let $P_n$ be a convex polytope in $E^3$ with $n$ facets and minimum isoperimetric quotient $S_I(P_n)^3/V(P_n)^2$. Then the following results hold:*

(i) $\dfrac{S_I(P_n)^3}{V(P_n)^2} \sim 27V(I) + \dfrac{15\omega_{EA}^2(\mathrm{bd}\, I)}{4\sqrt{3}n}$ *as* $n \to \infty$.

(ii) *By replacing $P_n$ by a suitable homothetic copy, if necessary, we may assume that $P_n$ is circumscribed to $I$ for all $n$. Then, if $n \to \infty$, the set $I \cap \mathrm{bd}\, P_n$ is uniformly distributed in* $\mathrm{bd}\, I$ *and the facets $F$ of $P_n$ are asymptotically regular hexagons with center $I \cap F$ and edge length*

$$\Big(\frac{2\omega_{EA}(\mathrm{bd}\, I)}{3\sqrt{3}n}\Big)^{1/2}.$$

(i) and the statement in (ii) about the uniform distribution of $I \cap P_n$ in $\mathrm{bd}\, I$ hold in all dimensions. This can be proved using results of Gruber [30] and Gruber and Glasauer [26]. Diskant's isoperimetric result cited above generalizes a classical theorem of Lindelöf [50] for the Euclidean case.

## 6.5 Optimal numerical integration formulae on Riemannian 2-manifolds

In this subsection we assume that $M$ is a 2-dimensional manifold of class $\mathcal{C}^3$ with metric tensorfield of class $\mathcal{C}^2$. Let $\varrho_M$ and $\omega_M$ denote the Riemannian metric and area measure on $M$, respectively. Let $J$ be a Jordan measurable set in $M$ with

$\omega_M(J) > 0$ and consider a class $\mathcal{F}$ of Riemann integrable functions $g : J \to I\!R$. Given sets of nodes $N_n = \{p_1, \cdots, p_n\}$ in $J$ and weights $W_n = \{w_1, \cdots, w_n\}$ in $I\!R$, let the *error* and the *minimum error* be defined by

$$E(\mathcal{F}, N_n, W_n) = \sup_{g \in \mathcal{F}} \{ | \int_J g(x)\, d\omega_M(x) - \sum_{k=1}^{n} w_k g(p_k) | \},$$

$$E(\mathcal{F}, n) = \inf_{N_n, W_n} \{ E(\mathcal{F}, N_n, W_n) \}.$$

The problems arise to determine $E(\mathcal{F}, n)$ and to describe the optimal choices of nodes and weights. For general $\mathcal{F}$ (and also for $M$ of dimension greater than 2) precise answers are out of reach, but we are able to give information in the following case. We say that $f : [0, +\infty) \to [0, +\infty)$ is a *modulus of continuity* if it satisfies the inequality

$$f(s+t) \le f(s) + f(t) \;\text{ for all }\; s, t \ge 0.$$

Given such a function $f$, define the *Hölder class* $\mathcal{H}^f$ *with modulus of continuity* $f$ by

$$\mathcal{H}^f = \{ g : J \to I\!R : |g(x) - g(y)| \le f(\varrho_M(x, y)) \;\text{ for all }\; x, y \in J \}.$$

Now our result can be formulated as follows.

**Theorem 22.** *Let $J$ be a Jordan measurable set in $M$ with $\omega_M(J) > 0$, let $f : [0, +\infty) \to [0, +\infty)$ be a modulus of continuity which satisfies the growth condition (see 6.2 above), and let $\mathcal{H}^f$ be the corresponding Hölder class. Then hold the following claims:*

(i) $$E(\mathcal{H}^f, n) = \inf_{N_n, W_n} \{ E(\mathcal{H}^f, N_n, W_n) \} \sim n \int_{H_n} f(\|x\|)\, dx \;\text{ as }\; n \to \infty.$$

(ii) *Let $(N_n)$ and $(W_n)$ be sequences of nodes and weights such that*

$$E(\mathcal{H}^f, N_n, W_n) \sim E(\mathcal{H}^f, n) \;\text{ as }\; n \to \infty.$$

*Then $N_n$ is uniformly distributed in $J$ and asymptotically a regular hexagonal pattern in $M$ of edge length*

$$\Big( \frac{2\omega_M(J)}{\sqrt{3} n} \Big)^{1/2}.$$

This result is contained in [35]. For $M = E^2, S^2$ the asymptotic formula (i), in essence, is due to Babenko [1].

It is possible to construct explicitly sequences $(N_n)$ and $(W_n)$ of nodes and weights such that $E(\mathcal{H}^f, N_n, W_n) \sim E(\mathcal{H}^f, n)$ as $n \to \infty$.

The statement about the uniform distribution in (ii) holds for all dimensions.

# 7 Distribution of electric charges

This section deals with equilibrium distributions of electric charges which move without friction on $S^2$ and, more generally, on bounded sets in Riemannian 2-manifolds.

## 7.1 Electric charges on the 2-sphere

A problem which has been investigated ever since Thompson published his plum pudding model of the atom and to which botanists, chemists, physicists, computer scientists and many mathematicians contributed is the following: let $n$ unit charges move without friction on the unit sphere $S^2$. What is their minimum potential energy and what are their *equilibrium configurations*, i.e. the configurations with minimum potential energy? For Coulomb forces the potential energy of $n$ unit charges at the points $p_1, \cdots, p_n \in S^2$ is (up to a constant) given by

$$\sum_{i<j} \frac{1}{\|p_i - p_j\|}.$$

Besides Coulomb force also forces which are determined by other powers of the Euclidean distance have been investigated. For surveys we refer to Saff and Kuijlaars [61] and Erber and Hockney [11].

Using tools from potential theory, Wagner [71, 72] and Rakhmanov, Saff and Zhou [58] have shown that the minimum potential energy of $n$ unit charges on $S^2$ equals

$$\frac{n^2}{2} - r_n, \text{ where } \operatorname{const} n^{3/2} \le r_n \le \operatorname{const} n^{3/2},$$

with suitable constants. Presumably the remainder term $r_n$ is asymptotically of the form $\operatorname{const} n^{3/2}$.

It has been conjectured (see Saff and Kuijlaars) that for $n \to \infty$ the equilibrium configurations are - using our notation - asymptotically regular hexagonal pattern on $S^2$ with respect to the ordinary metric on $S^2$. Computer experiments support this conjecture, but it seems to bedifficult to confirm it. One reason for this is the following: the above expression for the potential energy of the equilibrium configuration is based on an averaging argument. Hence there are configurations which differ substantially from regular hexagonal pattern, but have essentially the same potential energy. Thus the first step in the proof of the conjecture is to refine the above estimate for the potential energy. A second step then would be to prove a corresponding weak stability result.

## 7.2 Electric charges on Riemannian 2-manifolds

We formulate just the following conjecture, where $M$ is a Riemannian 2-manifold with Riemannian matric $\varrho_M$ and area measure $\omega_M$.

**Conjecture on equilibrium configurations.** *Let $J$ be a Jordan domain on $M$ bounded by a piecewise smooth curve. Assume that electric charges move without friction on $J$ and its boundary and that unit charges at points $x, y \in J$ repel each other by a force of the form $f(\varrho_M(x,y))$, where $f : (0,+\infty) \to (0,+\infty)$ is, say, continuous and strictly decreasing. For $n = 1, 2, \cdots$, let $S_n$ be an equilibrium configuration of $n$ unit charges on $J$. Then $S_n$ is asymptotically a regular hexagonal pattern of edge length*

$$\Big(\frac{2\omega_M(J)}{\sqrt{3}n}\Big)^{1/2}.$$

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