Optimal Configurations of Finite Sets in Riemannian 2-Manifolds

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Abstract. In this article we first extend Fejes Tth's basic result on sums of moments from the plane to Riemannian 2-manifolds. The extension as well as the original result show that for certain geometric or analytic problems regular hexagonal arrangements are optimal or almost optimal. Then a stability counterpart of the moment theorem for Riemannian 2-manifolds is given. It shows that, conversely, for several problems the optimal configurations are approximately regular hexagonal. Finally, the following applications are considered: (i) Description of the form of best approximating convex polytopes circumscribed to smooth convex bodies in $\mathbb{E}^3$ as the number of facets tends to infinity. (ii) Description of the convex polytopes with minimum isoperimetric quotient in 3-dimensional Minkowski spaces as the number of facets tends to infinity. (iii) Description of the arrangement of nodes in optimal numerical integration formulae for Hölder classes of functions of two variables as the number of nodes tends to infinity.

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1 Introduction

1.1 Amongst a small group of general results in discrete geometry is the following *theorem of Laszl Fejes Tóth on sums of moments*: let $f : [0, +\infty) \to [0, +\infty)$ be non-decreasing and let H be a convex 3, 4, 5, or 6-gon in the Euclidean plane $\mathbb{E}^2$. Then, for any finite set S in $\mathbb{E}^2$,

$$\int_H \min_{p \in S} \{f(\|p - x\|)\}\, dx \geq n \int_{H_n} f(\|x\|)\, dx, \tag{1.1}$$

where $n = \#S$ is the number of points of S and H_n is a regular hexagon in $\mathbb{E}^2$ of area $|H|/n$ and center at the origin o; $\|\cdot\|$ and $|\cdot|$ denote the Euclidean norm and the ordinary area measure in $\mathbb{E}^2$. The inequality (1.1), resp. a slightly different version of it, was first proved by Laszl Fejes Tóth [8] for the 2-sphere instead of $\mathbb{E}^2$, and only later for $\mathbb{E}^2$, see [9]. For alternative proofs, in some cases for surfaces of constant curvature and also for special non-monotone functions, see Laszl Fejes Tóth [10], Imre [24], Bollobas and Stern [2], Gabor Fejes Tóth [7], Florian [12], Papadimitriou [27], Haimovich and Magnanti [21] and, more recently, Böröczky and Ludwig [3] and the author [19].

Recent applications of the estimate (1.1) to asymptotic approximations of convex bodies in $\mathbb{E}^3$ by polytopes are due to Gruber [14] and Böröczky and Ludwig [3]. Other applications deal, for example, with planar packing and covering problems, problems of quantization of data, Gauss channels, optimal location in economics, the isoperimetric problem for special classes of convex polytopes in $\mathbb{E}^3$, the optimal choice of nodes in numerical integration formulae for functions of two variables and game theory. For references see [20]. Presumably, Fejes Tth's theorem can be applied also in biology.

Surprisingly, the inequality (1.1) may be arbitrarily weak: for suitable f the quotient of the left and right sides in (1.1) has no finite upper bound, even if S is chosen so as to minimize the left hand integral.

1.2 In view of the applications, two extensions of (1.1) suggest themselves, namely to higher dimensions and to Riemannian 2-manifolds. While the former at present seems to be out of reach, it is possible to extend (1.1) to Riemannian 2-manifolds. To enlarge the range of applications, we consider functions f and Riemannian 2-manifolds M with metric ϱ_M which satisfy rather mild conditions where, of course, functions as in the last paragraph have to be excluded, see 2.1.

Let J be a Jordan measurable set in M and for $n = 1, 2, \ldots$ let H_n denote a regular hexagon of $\mathbb{E}^2$ of area equal to the Riemannian area of J over n and with center o. Then

$$\inf_{\substack{S \subset M \\ \#S \leq n}} \int_J \min_{p \in S} \{f(\varrho_M(p,x))\}\, d\omega_M(x) \sim n \int_{H_n} f(\|x\|)\, dx \text{ as } n \to \infty.$$

As is the case for (1.1), also this result implies that for a series of geometric and analytic problems the regular hexagonal arrangements are close to optimal or even optimal. The more difficult question asks whether, in such situations, optimal and almost optimal configurations are almost regular hexagonal. As expected, the answer is affirmative and is a consequence of the following weak stability counterpart of the above result:

> Let
> $$\int_J \min_{p \in S_n} \{f(\varrho_M(p,x))\}\, d\omega_M(x) \sim n \int_{H_n} f(\|x\|)\, dx \,\text{as}\, n \to \infty,$$
> where (S_n) is a sequence of finite sets in M with $\#S_n \leq n$. Then S_n is uniformly distributed in J and asymptotically a regular hexagonal pattern;

(for definitions see 3.1). In case $f(t) = t^a$, where $0 < a \leq 2$, these results were announced by the author [19] and Gabor Fejes Tth has informed us of the proof of a similar stability result for the Euclidean plane.

1.3 We consider several applications of the above results:
(i) Approximation of convex bodies. A *convex body* in $\mathbb{E}^3$ is a compact convex set in $\mathbb{E}^3$ with non-empty interior. Let $\delta^V(.,.)$ denote the symmetric difference metric on the space of all convex bodies in $\mathbb{E}^3$. Given a convex body C, let $\mathcal{P}^c_{(n)}$ be the set of all convex polytopes circumscribed to C and having at most n facets. An intensively studied topic is the investigation the quantity $\delta^V(C, \mathcal{P}^c_{(n)}) = \inf\{\delta^V(C, P) : P \in \mathcal{P}^c_{(n)}\}$ and the description of the polytopes $P_n \in \mathcal{P}^c_{(n)}$ for which

the infimum is attained, that is, the *best approximating polytopes* of C in $\mathcal{P}^c_{(n)}$; see [16], [17]. Much is known about the asymptotics of $\delta^V(C, \mathcal{P}^c_{(n)})$ as $n \to \infty$, but a precise determination of $\delta^V(C, \mathcal{P}^c_{(n)})$ and description of P_n seems impossible at present. In this article it will be shown that

> for a sufficiently smooth convex body C in $\mathbb{E}^3$ the best approximating polytopes of C in $\mathcal{P}^c_{(n)}$ have asymptotically regular hexagonal facets (with respect to a suitable Riemannian metric) as n tends to infinity.

(ii) The isoperimetric problem in Minkowski space. Let $\mathbb{E}^d$ be endowed with a further norm. In this new normed space the volume $V(.)$ is the ordinary volume in $\mathbb{E}^d$, but there are different notions of surface area, e.g. those proposed by Busemann and Holmes - Thompson. These notions amount to the introduction of an o-symmetric convex body I, the *isoperimetrix*, such that the *surface area* of a convex body C is defined by $S_I(C) = \lim(V(C + \delta I) - V(C))/\delta$ as $\delta \to +0$. In the Euclidean case I is the Euclidean unit ball; see [31]. With the help of a result of Diskant [5], [6] we will prove that

> the convex polytopes in $\mathbb{E}^3$ with minimum isoperimetric quotient amongst those with n facets have asymptotically regular hexagonal facets (with respect to a suitable Riemannian metric) as n tends to infinity.

(iii) Numerical integration. Let M and J be as above and consider a class $\mathcal{F}$ of Riemann integrable functions $g : J \to \mathbb{R}$. For given sets of nodes $N_n = \{p_1, \dots, p_n\}$ in J and weights $W_n = \{w_1, \dots, w_n\}$ in $\mathbb{R}$ let the *error* and the *minimum error* be defined as follows

$$E(\mathcal{F}, N_n, W_n) = \sup_{g \in \mathcal{F}} \{ | \int_J g(x) d\omega_M(x) - \sum_{k=1}^n w_k \, g(p_k) | \},$$
$$E(\mathcal{F}, n) = \inf_{N_n, W_n} \{ E(\mathcal{F}, N_n, W_n) \}.$$

The problem arises to determine $E(\mathcal{F}, n)$ and to describe the optimal choices of nodes and weights. For general $\mathcal{F}$ (and for M of dimension greater than 2) one cannot expect much, but in the following case some information can be given: consider for a modulus of continuity $f : [0, \infty) \to [0, \infty)$ (see 6.1) which satisfies a weak growth condition (see 2.1), the Hölder class $\mathcal{H}^f$ of all functions $g : J \to \mathbb{R}$ such that $|g(x) - g(y)| \le f(\varrho_M(x, y))$ for all $x, y \in J$. Using the above results and differential geometric tools we will show that

$$E(\mathcal{H}^f, n) \sim n \int_{H_n} f(\|x\|) dx \text{ as } n \to \infty$$

> and, if (N_n) and (W_n) are sequences of nodes and weights, respectively, such that $E(\mathcal{H}^f, N_n, W_n) \sim E(\mathcal{H}^f, n)$ as $n \to \infty$, then N_n is uniformly distributed in J and asymptotically a regular hexagonal pattern.

The asymptotic formula for $M = \mathbb{E}^2$ and S^2 is, in essence, due to Babenko [1].

2 Extension of Fejes Tth's theorem to Riemannian 2-manifolds

2.1 The result

A function $f : [0, +\infty) \to [0, +\infty)$ satisfies the *growth condition* if

> $f(0) = 0$, f is continuous, strictly increasing and for each $\alpha > 1$ there are $\beta > \gamma > 1$ (and vice versa) such that $\gamma f(t) \leq f(\alpha t) \leq \beta f(t)$ for all sufficiently small $t \geq 0$. Here β and γ may be chosen arbitrarily close to 1 if α is sufficiently close to 1 (and vice versa).

Let M be a *Riemannian 2-manifold*. By this we mean that M is a 2-manifold M of differentiability class $\mathcal{C}^1$ with a metric tensor field having continuous coefficients. Let ϱ_M and ω_M denote the corresponding Riemannian metric and area measure. A set J in M is *Jordan measurable* if it has compact closure $\operatorname{cl} J$ and its boundary $\operatorname{bd} J$ has measure 0; see also 2.2.1.

Theorem 1. *Let $f : [0, +\infty) \to [0, +\infty)$ satisfy the growth condition and let J be a Jordan measurable set in M with $\omega_M(J) > 0$. Then*

$$\inf_{\substack{S \subset M \\ \#S \leq n}} \int_J \min_{p \in S} \{ f(\varrho_M(p, x)) \} d\omega_M(x) \sim n \int_{H_n} f(\|x\|) dx \text{ as } n \to \infty, \tag{2.1}$$

where H_n is a regular hexagon in $\mathbb{E}^2$ with area $\omega_M(J)/n$ and center o. The formula (2.1) continues to hold under the assumption that S is contained in J.

It follows from the proof that, in principle, it is possible to construct a sequence of sets $S = S_n$ which satisfy the asymptotic relation (2.1). In a more general version of the formula (2.1) there is a continuous weight factor under the left integral. In its proof the weight factor is used to define a new metric tensor field on M. Then Theorem 1 can be applied.

The proof of Theorem 1 is split into two parts.

2.2 Preparations for the proof

2.2.1 For a better understanding we first describe the definitions of ϱ_M, ω_M and of Jordan measurability. Let U be an open neighborhood of a point in M with corresponding homeomorphism h which maps U onto an open set U' in $\mathbb{E}^2$. For each $u \in U$ let $q_u(\,.\,)$ be the corresponding positive definite quadratic form on $\mathbb{E}^2$. A curve K in U is of class $\mathcal{C}^1$ if it has a parametrization $x : [a, b] \to U$ such that $u = h \circ x$ is a parametrization of a curve of class $\mathcal{C}^1$ in U'. The *length* of K then is defined as

$$\int_a^b q_{u(s)}^{1/2}(\dot{u}(s))\,ds.$$

By means of (an appropriate) dissection and addition, the notion of length can be defined for curves in M which are not contained in a single neighborhood. For $x, y \in M$ let their *Riemannian distance* $\varrho_M(x, y)$ be the infimum of the lengths of curves of class $\mathcal{C}^1$ in M connecting x and y. A set J in U is *Jordan measurable* if $h(J)$ is Jordan measurable as a subset of $\mathbb{E}^2$. Then its *area measure* $\omega_M(J)$ is defined as

$$\int_{h(J)} (\det q_u)^{1/2}\,du.$$

Again, by dissection and addition, we can define the concepts of Jordan measurability and area measure for sets in M which are not contained in a single neighborhood. This definition of Jordan measurability and the one in 1.2 are equivalent.

2.2.2 The moments $M(D,p)$ and $M(a,v)$. If D is a compact convex set in $\mathbb{E}^2$, its *moment* $M(D,p)$ with respect to a point p in $\mathbb{E}^2$ is defined as

$$\int_D f(\|p-x\|)\,dx. \tag{2.2}$$

Write $M(a,v)$ for the moment of a regular v-gon of area a with respect to its center.

A suitable linear transformation together with (1.1) yields the following.

(2.3) Let $q(\,.\,)$ be a positive definite quadratic form on $\mathbb{E}^2$ and $Q \subset \mathbb{E}^2$ a convex $3,4,5$, or 6-gon. Then for $m = 1, 2, \dots$, and any set S in $\mathbb{E}^2$ with $\#S \le m$ holds

$$\int_Q \min_{r\in S}\{f(q^{1/2}(r-s))\}\,ds\,(\det q)^{1/2} \ge mM\Big(\frac{|Q|(\det q)^{1/2}}{m}, 6\Big).$$

The growth condition (1.2) together with an argument of Babenko [1] and the application of a suitable linear transformation implies the next result.

(2.4) Let $\lambda > 1$, $q(\,.\,)$ a positive definite quadratic form on $\mathbb{E}^2$ and R a Jordan measurable set in $\mathbb{E}^2$. Then for each sufficiently large m there is a set $T \subset R$ with $\#T \le m$ with the following properties:

(i) The distance (with respect to the norm $q(\,.\,)^{1/2}$) of any point of R to the nearest point of T is $O(m^{-1/2})$.

(ii) $\displaystyle\int_R \min_{r\in T}\{f(q^{1/2}(r-s))\}\,ds\,(\det q)^{1/2} \le \lambda\, mM\Big(\frac{|R|(\det q)^{1/2}}{m}, 6\Big).$

2.2.3 Properties of $M(a,6)$. A special case of a result of the author [19] says that

(2.5) $M(a,6)$ is convex for $a \geq 0$.

Clearly,

(2.6) $M(0,6)=0$ and $M(a,6)$ is strictly increasing for $a \geq 0$.

As a consequence of (2.5) we will show that

(2.7) $mM\Big(\dfrac{a}{m},6\Big)$ is non-increasing for all $a>0$ and $m \geq 1$.

Let g be a convex non-decreasing function for $t \geq 0$ with $g(0)=0$. An inspection of the graph of g shows that $g(t)/t$ is non-decreasing for $t>0$. Hence $tg(1/t)$ is non-increasing for $t>0$. Applying this with $g(t)=M(at,6)$, where we take into account (2.5), (2.6), yields (2.7).

Since f satisfies the growth condition, the definition of $M(a,6)$ implies that

(2.8) $M(a,6)$ satisfies the growth condition.

2.2.4 The *Dirichlet-Voronoi cells* $D_M(S,p)$. For a finite set $S \subset M$ let

$$D_M(S,p)=\{x \in J : \varrho_M(p,x) \leq \varrho_M(q,x) \ \text{ for each}\, q \in S\},\ p \in S.$$

Then the following holds:

(2.9) Let (S_n) be a sequence of finite sets in M such that $\#S \leq n$ and
$\int\limits_J \min\limits_{p \in S_n}\{f(\varrho_M(p,x))\}d\omega_M(x) \to 0 \ \text{ as}\, n \to \infty.$
Then $\max\limits_{p \in S_n}\{\operatorname{diam}(D_M(S_n,p) \cap \operatorname{int} J)\} \to 0\,\text{as}\ n \to \infty.$

Here diam stands for diameter. For the proof of (2.9) it is sufficient to show the following statement: let $\varepsilon>0$, then for all sufficiently large n: given $p \in S_n$ and $x \in D_M(S_n,p) \cap \operatorname{int} J$, then $\varrho_M(p,x)<2\varepsilon$. The compactness of cl int J implies that there are finitely many points in int J such that the ε-neighborhoods of these points cover int J. If n is sufficiently large, each of these ε-neighborhoods contains at least one point of S_n; otherwise the integral in (2.9) cannot converge to 0. Consequently, for each $x \in$ int J the nearest point of S_n has distance less than 2ε from x. For each $x \in D_M(S_n,p) \cap$int J the point p is (one of) the nearest point(s) of S_n, we thus have that $\varrho_M(p,x)<2\varepsilon$. This concludes the proof of the above statement and thus of (2.9).

The next remark is obvious.

(2.10) Let $S \subset M$ be finite and $K \subset J$ Jordan measurable. Define
$S(K)=\{p \in S : \operatorname{int}(D_M(S,p) \cap K) \neq \emptyset\}$, $n(K)=\#S(K)$. Then
$\int\limits_K \min\limits_{p \in S}\{f(\varrho_M(p,x))\}d\omega_M(x)=\int\limits_K \min\limits_{p \in S(K)}\{f(\varrho_M(p,x))\}d\omega_M(x).$

2.2.5 In this subsection we use a system of (small) neighborhoods in M to transfer information from $\mathbb{E}^2$ to J and vice versa. Let $\lambda > 1$. For each $p \in M$ choose $U, h =$ "$'$" and $U' = h(U)$ as in 2.2.1, where the neighborhood U is so small that for the positive quadratic form $q(\,.\,)$ corresponding to p and h the following inequalities hold; here $q_u(\,.\,)$ is the positive quadratic form corresponding to $u \in U$ and h.

$$
\begin{aligned}
&\frac{1}{\lambda} q^{1/2}(x' - y') \le \varrho_M(x, y) \le \lambda q^{1/2}(x' - y') && \text{for } x, y \in U,\\
&\frac{1}{\lambda} (\det q)^{1/2} \le (\det q_u)^{1/2} \le \lambda (\det q)^{1/2} && \text{for } u \in U,\\
&\frac{1}{\lambda} |K'| (\det q)^{1/2} \le \omega_M(K) \le \lambda |K'| (\det q)^{1/2} && \text{for Jordan measurable } K \subset U.
\end{aligned}
$$

Let V be a Jordan measurable open neighborhood of p with $\operatorname{cl} V \subset U$. As p ranges over the compact set $\operatorname{cl} J$ the neighborhoods V form an open covering of $\operatorname{cl} J$. Thus there is a finite subcover. Therefore we may choose points $p_l \in J$, $l = 1, \dots, k$, say, and corresponding neighborhoods U_l, V_l, homeomorphisms $h_l =$ "$'$", and positive quadratic forms q_l such that

$$
(2.11)\quad \frac{1}{\lambda} q_l^{1/2}(x' - y') \le \varrho_M(x, y) \le \lambda q_l^{1/2}(x' - y') \text{ for } x, y \in U_l,
$$

$$
(2.12)\quad \frac{1}{\lambda} (\det q_l)^{1/2} \le (\det q_u)^{1/2} \le \lambda (\det q_l)^{1/2} \qquad \text{for } u \in U_l,
$$

$$
(2.13)\quad \frac{1}{\lambda} |K'| (\det q_l)^{1/2} \le \omega_M(K) \le \lambda |K'| (\det q_l)^{1/2}
$$
for Jordan measurable $K \subset U_l$.

Consider the sets $I_l = J \cap (V_l \setminus (V_1 \cup \ldots \cup V_{l-1}))$ for $l = 1, \dots, k$. Since each V_l is Jordan measurable, $\operatorname{cl} V_l \subset U_l$ and $J \subset V_1 \cup \ldots \cup V_l$, we see that

(2.14) J is the disjoint union of the Jordan measurable sets $I_l \subset V_l \subset U_l, l = 1, \dots, k$.

2.3 Proof of Theorem 1

2.3.1 For the proof of Theorem 1 two estimates are needed. The first one is the following, the second one will be given in 2.3.2.

(2.15) *Let $0 < \alpha < 1$. Then for all sufficiently large n holds*

$$
\inf_{\substack{S \subset M\\ \#S \le n}} \int_J \min_{p \in S} \{ f(\varrho_M(p, x)\} d\omega_M(x) \ge \alpha\, n M\Big(\frac{\omega_M(J)}{n}, 6\Big).
$$

By the growth condition for f and (2.8) we may choose $\lambda > 1$ and a corresponding $\mu > 1$ so that

$$(2.16)\quad M\Big(\frac{a}{\lambda}, 6\Big) \geq \frac{1}{\mu} M(a, 6) \text{ for all sufficiently small } a \geq 0,$$
$$f\Big(\frac{t}{\lambda}\Big) \geq \frac{1}{\mu} f(t) \text{ for all sufficiently small } t \geq 0 \text{ and such that}$$
$$\frac{1}{\lambda^2 \mu^3} > \alpha.$$

For this λ choose $p_l, h_l = "\prime", U_l, V_l, I_l, q_l$, $l = 1, \dots, k$, as described in 2.2.5. Then, in particular, (2.11)-(2.14) hold. Next, choose for each l sets Q_{li}, $i = 1, \dots, i_l$, with the following properties:

(2.17) (i) The sets Q_{li}, $i = 1, \dots, i_l$, are compact and pairwise disjoint sets in $\mathrm{int} I_l \subset U_l$.

(ii) Q'_{li} is a square in $\mathbb{E}^2$ of edgelength $1/k_l$, say, (with respect to the norm $q_l^{1/2}(\,.\,)$), where k_l is a positive integer. Clearly, Q_{li} is Jordan measurable.

(iii) $\sum\limits_{l,i} \omega_M(Q_{li}) \geq \frac{1}{\lambda} \omega_M(J)$.

For $n = 1, 2 \dots$, choose sets S_n in M with $\# S_n \leq n$ such that

$$(2.18)\quad \inf_{\substack{S \subset M \\ \# S \leq n}} \int_J \min_{p \in S} \{ f(\varrho_M(p, x) \} \, d\omega_M(x) \geq \frac{1}{\lambda} \int_J \min_{p \in S_n} \{ f(\varrho_M(p, x)) \} \, d\omega_M(x).$$

Such a choice is possible since for any n the left side of (2.18) is positive. Define $S_{nli} = S_n(Q_{li})$ and $n_{li} = n(Q_{li})$ as in (2.10). Taking into account (2.14) and (2.17i), an application of (2.9) and (2.10) then shows that

(2.19) for all sufficiently large n hold:

(i) The sets $S_{nli}\,(\subset U_l)$, $l = 1, \dots, k$, $i = 1, \dots, i_l$, are pairwise disjoint.

(ii) $\sum\limits_{l,i} n_{li} \leq n$.

(iii) $\min\limits_{p \in S_n} \{ f(\varrho_M(p, x)) \} = \min\limits_{p \in S_{nli}} \{ f(\varrho_M(p, x)) \}$ for $x \in Q_{li}$.

Now, using (2.18), (2.17i), (2.19iii), (2.11), (2.12), the definition of the integral in M, (2.16), (2.3), (2.13), (2.6), (2.5), Jensen's inequality, (2.7),(2.19ii), (2.17iii), (2.6), and (2.16), we obtain the estimate (2.15):

$$
\begin{aligned}
&\inf_{\substack{S\subset M\\ \#S\le n}}\int\limits_J \min_{p\in S}\{f(\varrho_M(p,x)\}d\omega_M(x) \ge \frac{1}{\lambda}\int\limits_J \min_{p\in S_n}\{f(\varrho_M(p,x))\}d\omega_M(x)\\
&\quad\ge \frac{1}{\lambda}\sum_{l,i}\int\limits_{Q_{li}}\min_{p\in S_{nli}}\{f(\varrho_M(p,x))\}d\omega_M(x)\\
&\quad\ge \frac{1}{\lambda}\sum_{l,i}\int\limits_{Q'_{li}}\min_{r\in S'_{nli}}\{f\Big(\frac{1}{\lambda}\,q_l^{1/2}(r-s)\Big)\}ds\,\frac{1}{\lambda}\,(\det q_l)^{1/2}\\
&\quad\ge \frac{1}{\lambda^2\mu}\sum_{l,i}n_{li}M\Big(\frac{|Q'_{li}|(\det q_l)^{1/2}}{n_{li}},6\Big)\ge \frac{1}{\lambda^2\mu}\sum_{l,i}n_{li}M\Big(\frac{\omega_M(Q_{li})}{\lambda n_{li}},6\Big)\\
&\quad\ge \frac{1}{\lambda^2\mu}(n_{11}+\cdots+n_{ki_k})M\Big(\frac{\omega_M(Q_{11})+\cdots+\omega_M(Q_{ki_k})}{\lambda(n_{11}+\cdots+n_{ki_k})},6\Big)\ge \frac{1}{\lambda^2\mu}\,nM\Big(\frac{\omega_M(J)}{\lambda^2 n},6\Big)\\
&\quad\ge \frac{1}{\lambda^2\mu^3}\,nM(\frac{\omega_M(J)}{n},6)>\alpha\, nM\Big(\frac{\omega_M(J)}{n},6\Big)\text{ for all sufficiently large } n.
\end{aligned}
$$

2.3.2 The second estimate we require is as follows:

(2.20) *Let $\beta > 1$. Then for all sufficiently large n holds*

$$\inf_{\substack{S\subset J\\ \#S\le n}}\int\limits_J \min_{p\in S}\{f(\varrho_M(p,x)\}d\omega_M(x)\le \beta\, nM\Big(\frac{\omega_M(J)}{n},6\Big).$$

By the growth condition for f and (2.8) we may choose $\lambda > 1$ and a corresponding $\mu > 1$ so that

(2.21) $f(\lambda t)\le \mu f(t)$ for all sufficiently small $t\ge 0$,
$M(\lambda a,6)\le \mu M(a,6)$, for all sufficiently small $a\ge 0$, and such that
$\lambda^2\mu^5<\beta$.

For this λ choose $p_l,\dots,$ as described in 2.2.5. Next, consider for each l an edge-to-edge tiling of $\mathbb{E}^2$ with translates of a closed square of area

(2.22) $$\frac{\sum_{l=1}^k |I'_l|(\det q_l)^{1/2}}{(\det q_l)^{1/2}}\,\nu,$$

where $\nu > 0$ is so small that each of these k tilings has the following properties: let $R'_{lj}, j=1,\dots,j_l$, be the tiles of the lth tiling which meet I'_l, then

(2.23) $I'_l\subset R'_{l1}\cup\ldots\cup R'_{lj_l}\subset U'_l$, the R'_{lj}, $j=1,\dots,j_l$, have disjoint interiors,

(2.24) $|R'_{l1}|=\ldots=|R'_{lj_l}|$, $|I'_l|\le j_l|R'_{l1}|\le\lambda|I'_l|$.

Each $R_{lj}=h_l^{-1}(R'_{lj})\,(\subset U_l)$ is Jordan measurable. For the proof that

(2.25) $$\frac{j_0}{j_l}\le \lambda\,\frac{\sum_{l=1}^k|I'_l|(\det q_l)^{1/2}}{|I'_l|(\det q_l)^{1/2}},\text{ where } j_0=j_1+\ldots+j_k,$$

consider the second inequality in (2.24), multiply both sides by $(\det q_l)^{1/2}$, insert the value of $|R'_{l1}|$ and sum over all l. This gives $j_0\nu \le \lambda$ or $\nu \le \lambda/j_0$. Now consider the first inequality in (2.24), insert the value of $|R'_{l1}|$ and note that $\nu \le \lambda/j_0$. This implies (2.25).

Choose m_0 so large that for each of the positive definite quadratic forms $q_l(\,.\,)$ and each of the Jordan measurable sets $(R_{lj} \cap J)'$ the conclusion of (2.4) holds for all $m \ge m_0$. Then, using (2.6) we see that

(2.26) for each $m \ge m_0$ there are sets $T'_{mlj} \subset (R_{lj} \cap J)'$ with $\# T'_{mlj} \le m$ where

(i) The distance (with respect to the norm $q(\,.\,)^{1/2}$) of any point of $(R_{lj} \cap J)'$ from the nearest point of T_{mlj} is $O(m^{-1/2})$.

(ii) $\displaystyle\int_{(R_{lj}\cap J)'} \min_{r \in T'_{mlj}} \{f(q_l^{1/2}(r-s))\} ds (\det q_l)^{1/2} \le \lambda\, mM\Big(\frac{|R'_{lj}|(\det q_l)^{1/2}}{m}, 6\Big).$

After these preparations for the proof of (2.19), we assume first that n has the form $n = j_0 m$ with $m \ge m_0$. Let $T_n = \bigcup_{l,j} T_{mlj}$, where $T_{mlj} = h_l^{-1}(T'_{mlj})$ $(\subset R_{lj} \cap J)$. Clearly, $T_n \subset J$ and $\# T_n \le j_0 m = n$. This together with (2.14), (2.23), $T_n = \bigcup_{l,j} T_{mlj}$, (2.12) and the definition of the integral in M, (2.11), the growth condition for f, (2.26), (2.21), (2.26ii), (2.24), (2.6), (2.25), (2.6), (2.13), (2.14), and (2.21) implies that there is an $m_1 \ge m_0$ such that

$$
\begin{aligned}
(2.27)\quad \inf_{\substack{T\subset J\\ \#T \le n}} \int_J \min_{p\in T}\{f(\varrho_M(p,x)\}\, d\omega_M(x) &\le \int_J \min_{p\in T_n}\{f(\varrho_M(p,x)\}\, d\omega_M(x)\\
&\le \sum_{l,j} \int_{R_{lj}\cap I_l} \min_{p\in T_{mlj}} \{f(\varrho_M(p,x)\}\, d\omega_M(x)\\
&\le \lambda \sum_{l,j} \int_{(R_{lj}\cap I_l)'} \min_{r\in T'_{mlj}} \{f(\lambda q_l^{1/2}(r-s))\}\, ds\, (\det q_l)^{1/2}\\
&\le \lambda^2\mu \sum_{l,j} mM\Big(\frac{|R'_{lj}|(\det q_l)^{1/2}}{m}, 6\Big) \le \lambda^2\mu \sum_l j_l\, mM\Big(\frac{\lambda |I'_l|(\det q_l)^{1/2}}{j_l m}, 6\Big)\\
&\le \lambda^2\mu \sum_l j_l\, mM\Big(\frac{\lambda^2 \sum_l |I'_l|(\det q_l)^{1/2}}{j_0 m}, 6\Big) \le \lambda^2\mu\, j_0\, mM\Big(\frac{\lambda^3\omega_M(J)}{j_0 m}, 6\Big)\\
&\le \lambda^2\mu^4 nM\Big(\frac{\omega_M(J)}{n}, 6\Big) \text{ for all } n \text{ of the form } n = j_0 m, m \ge m_1
\end{aligned}
$$

For general n we argue as follows: let $m_2 \ge m_1$ be so large that $(m+1)/m \le \lambda$ for $m \ge m_2$. Then, given $n \ge j_0 m_2$, choose $m \ge m_2$ such that $j_0 m \le n < j_0(m+1)$. Combining this with (2.27), (2.6), and (2.21) finally yields the estimate (2.20):

$$
\begin{aligned}
\inf_{\substack{T\subset J\\ \#T \le n}} \int_J \min_{p\in T}\{f(\varrho_M(p,x)\} d\omega_M(x) &\le \inf_{\substack{T\subset J\\ \#T \le j_0 m}} \int_J \min_{p\in T}\{f(\varrho_M(p,x)\} d\omega_M(x)\\
&\le \lambda^2 \mu^4 j_0\, mM\Big(\frac{\omega_M(J)}{j_0 m}, 6\Big) \le \lambda^2\mu^4 nM\Big(\frac{\lambda\omega_M(J)}{n}, 6\Big)\\
&\le \lambda^2\mu^5 nM\big(\frac{\omega_M(J)}{n}, 6\big) \le \beta\, nM\Big(\frac{\omega_M(J)}{n}, 6\Big) \text{ for all } n \ge j_0 m_2.
\end{aligned}
$$

2.3.4 Theorem 1 is an immediate consequence of (2.15) and (2.20).□

3 Stability of hexagonal arrangements in Riemannian 2-manifolds

3.1 The stability result

Let M be a Riemannian 2-manifold as defined in 1.2 with Riemannian metric ϱ_M, area measure ω_M and the concept of Jordan measurability. Let (S_n) be a sequence of finite sets in M with $\#S_n \leq n$ and $\#S_n \to \infty$ as $n \to \infty$. S_n is said to be *uniformly distributed* in a Jordan measurable set J in M with $\omega_M(J) > 0$ if, for each Jordan measurable set K in J with $\omega_M(K) > 0$, we have

$$\frac{\#(K \cap S_n)}{n} \to \frac{\omega_M(K)}{\omega_M(J)} \text{ as } n \to \infty;$$

see Hlawka [23], p.58. S_n is *asymptotically a regular hexagonal pattern of edge-length* ϱ_n if the following holds: there exists a positive sequence (ϱ_n) and Landau symbols $o(n)$ and $o(1)$ such that for each point $p \in S_n$, with a set of at most $o(n)$ exceptions, the relation

$$\{x : \varrho_M(p,x) \leq 1.1\varrho_n\} \cap S_n = \{p, p_1, \dots, p_6\}, \text{ say},$$

holds, where

$$\varrho_M(p,p_j),\ \varrho_M(p_j,p_{j+1}) = (1 \pm o(1))\varrho_n \text{ for } j = 1, \dots, 6,\ p_7 = p_1.$$

$p_1, \dots, p_6$ are said to form up to $o(1)$ a *regular hexagon with center* p *of edge length* ϱ_n. ($(1 \pm o(1))\varrho_n$ denotes a quantity between $(1-o(1))\varrho_n$ and $(1+o(1))\varrho_n$.)

Theorem 2. *Let $f : [0,+\infty) \to [0,+\infty)$ satisfy the growth-condition, let J be a Jordan measurable set in M with $\omega_M(J) > 0$, and (S_n) a sequence of finite sets in M with $\#S_n \leq n$ for $n = 1, 2, \dots,$ such that*

$$(3.1)\quad \int_J \min_{p \in S_n}\{f(\varrho_M(p,x)\} d\omega_M(x) \sim \inf_{\substack{S \subset M \\ \#S \leq n}} \int_J \min_{p \in S}\{f(\varrho_M(p,x))\} d\omega_M(x) \text{ as } n \to \infty$$

Then S_n is uniformly distributed in J and is asymptotically a regular hexagonal pattern of edge length

$$\left(\frac{2\omega_M(J)}{\sqrt{3}n}\right)^{1/2}.$$

As was the case for Theorem 1, there is also a more general version of Theorem 2 with a weight factor in the left integral in (3.1).

After some preparations in 3.2 we first prove the result for the Euclidean plane in 3.3 and then extend it to Riemannian manifolds in 3.4. The former case is more difficult, but also the extension has to be done with more care than is usual in order not to loose the stability property.

3.2 Preparations for the proof

3.2.1 Simplifications. We note that f, or any continuous strictly increasing function on $[0,\infty)$ beween $(1-o(t))f(t)$ and $(1+o(t))f(t)$ as $t\to 0$, satisfies both the growth condition and (3.1) (note (2.9)). It therefore suffices to prove Theorem 2 under the additional assumption that f is continuously differentiable for $t>0$. Then

(3.2) $g:[0,+\infty)\to[0,+\infty)$, defined by $g(t)=f(t^{1/2})$ for $t\geq 0$, is continuously differentiable for $t>0$ and satisfies the growth condition.

Let G be a primitive of g with $G(0)=0$.

Positive constants independent of m, respectively n, will be denoted in many cases just by const. If const appears several times in the same or in consecutive expressions, this does not mean that it denotes necessarily the same constant. To simplify the presentation, constants are sometimes absorbed by Landau symbols and in some other cases constants are denoted by Greek letters. ϑ always denotes a suitable real with $0<\vartheta<1$.

A Landau symbol $o(1)$ (and similarly for the other Landau symbols used) is a positive function of m or n which converges to 0 as $m\to\infty$ or $n\to\infty$, respectively. If a Landau symbol appears several times in the same or in consecutive expressions, it may be different in each case. Some of our arguments are valid only if m or n is sufficiently large. In several cases this is assumed tacitly.

3.2.2 The moment lemma. The following result of Laszl Fejes Tth [9, 11] will be useful in the sequel; for the definition of the moments $M(D,p)$ and $M(a,p)$ see 2.2.2.

(3.3) Let D be a convex v-gon with area a and let $p\in\mathbb{E}^2$. Then $M(D,p)\geq M(a,v)$.

3.2.3 Properties of $M(a,v)$. Set $h(a,v)=a/(v\tan(\pi/v))$ for $a\geq 0, v\geq 3$. Then $h^{1/2}(a,v)$ is the inradius of a regular v-gon of area a. Using polar coordinates, we obtain

$$(3.4)\quad M(a,p)=2v\int_0^{\frac{\pi}{v}}\int_0^{\frac{h^{1/2}}{\cos\psi}} g(r^2)r\,dr\,d\psi = v\int_0^{\frac{\pi}{v}} G\Big(\frac{h}{\cos^2\psi}\Big)d\psi.$$

Define $M(a,v)$ for $a\geq 0$ and real $v\geq 3$ by (3.4).

The first needed property of $M(a,v)$ was proved in [19]. It says that

(3.5) $M(a,v)$ is convex for all $a\geq 0, v\geq 3$.

We draw two consequences of (3.5):

(3.6) $M(a,v) \geq M(b,v) + M_a(b,v)(a-b),$
$M(a,v) = M(b,v) + M_a(b,v)(a-b) + \frac{1}{2}M_{aa}(b+\vartheta(a-b),v)(a-b)^2$
for all $a,b \geq 0, v \geq 3$.

Since $g = G'$ is continuously differentiable for $t > 0$ (see (3.2)), the function $M(a,v)$ has continuous partial derivatives up to the second order by (3.4). Now, noting (3.5), we obtain (3.6).

(3.7) $M(a,v)$ is non-increasing in v for fixed $a > 0$ and, trivially, strictly increasing in a for fixed $v \geq 3$).

To see this, fix $a > 0$. Then $M(a,v)$ is convex in v by (3.5). The definition of $M(a,v)$ in (3.4) together with a simple continuity argument shows that $M(a,v) \to M(C,o)$ as $v \to +\infty$, where C is the (solid) circle of area a and center o. Thus $M(a,v)$ is a bounded convex function for $v \geq 3$, which implies (3.7).

Second,

(3.8) $\text{const}\, a\, g(a) \leq M(a,v) \leq \text{const}\, a\, g(a)$
for all sufficiently small $a \geq 0$ and all $v \geq 3$.

By (3.7) and its proof, $M(C,o) \leq M(a,v) \leq M(a,3)$. Since clearly $(a/2)g(a/2\pi) \leq M(C,o)$ and $M(a,3) \leq ag(4a/3\sqrt{3})$, an application of the growth condition for g (see (3.2)) yields (3.8).

Third,

(3.9) $\text{const}\, g(a) \leq M_a(a,v) \leq \text{const}\, g(a)$
for all sufficiently small $a \geq 0$ and all $v \geq 3$.

Propositions (3.4) and (3.2) imply that for all sufficiently small $a \geq 0$ and for $v \geq 3$,

$$M_a = v\, h_a \int_0^{\frac{\pi}{v}} g\Big(\frac{h}{\cos^2\psi}\Big)\frac{d\psi}{\cos^2\psi} \leq \frac{v}{v\tan\frac{\pi}{v}}\,\frac{\pi}{v}\, g\Big(\frac{a}{v\tan\frac{\pi}{v}\cos^2\frac{\pi}{v}}\Big)\frac{1}{\cos^2\frac{\pi}{v}} \leq \text{const}\, g(a),$$

$$M_a \geq v\, h_a \int_{\frac{\pi}{2v}}^{\frac{\pi}{v}} g\Big(\frac{h}{\cos^2\psi}\Big)\frac{d\psi}{\cos^2\psi} \geq \frac{v}{v\tan\frac{\pi}{v}}\,\frac{\pi}{2v}\, g\Big(\frac{a}{v\tan\frac{\pi}{v}\cos^2\frac{\pi}{2v}}\Big)\frac{1}{\cos^2\frac{\pi}{2v}} \geq \text{const}\, g(a).$$

Fourth,

(3.10) $M_{aa}(a,v) \geq \text{const}\,\dfrac{g(a)}{a}$
for all sufficiently small $a > 0$ and for $5 \leq v \leq 7$.

The inequality in (3.10) is a consequence of (3.4) and (3.2):

$$
\begin{aligned}
M_{aa} &= v\, h_a^2 \int_0^{\frac{\pi}{v}} g'\Big(\frac{h}{\cos^2\psi}\Big)\frac{d\psi}{\cos^4\psi} \geq \frac{v h_a^2}{2h}\int_0^{\frac{\pi}{v}} g'\Big(\frac{h}{\cos^2\psi}\Big)\frac{2h\sin\psi}{\cos^3\psi}d\psi \\
&= \frac{1}{2a\tan\frac{\pi}{v}}\,(g\Big(\frac{h}{\cos^2\frac{\pi}{v}}\Big) - g(h)) \geq \mathrm{const}\,\frac{g(a)}{a}
\end{aligned}
$$

for all sufficiently small $a \geq 0$ and for $5 \leq v \leq 7$.

As a preparation for the proof of (3.12) we show that

$$
\frac{M_{aa}(a,v)M_{vv}(a,v) - M_{av}^2(a,v)}{M_{aa}(a,v)} \geq \mathrm{const}\, a g(a) \tag{3.11}
$$

for all sufficiently small $a > 0$, and for $5 \leq v \leq 7$.

To see this we first specify two candidates for an upper bound of M_{aa}, one of which actually turns out to be an upper bound. We note that (3.2) implies $g' \geq 0$. Using this and (3.4), we obtain:

$$
M_{aa} = v h_a^2 \int_0^{\frac{\pi}{v}} g'(\frac{h}{\cos^2\psi})\frac{d\psi}{\cos^4\psi} \leq \left\{ \begin{array}{l} 2v h_a^2 \displaystyle\int_{\frac{\pi}{2v}}^{\frac{\pi}{v}} g'(\frac{h}{\cos^2\psi})\frac{d\psi}{\cos^4\psi}, \text{ or} \\ 2v h_a^2 \displaystyle\int_0^{\frac{\pi}{2v}} g'(\frac{h}{\cos^2\psi})\frac{d\psi}{\cos^4\psi} \end{array} \right\}
$$

for all $a > 0$ and $5 \leq v \leq 7$.

A representation of $M_{aa}M_{vv} - M_{av}^2$ from [19] and $g' \geq 0$ implies that

$$
\begin{aligned}
&M_{aa}M_{vv} - M_{av}^2 \\
&\quad = \frac{2\pi^2 a}{v^5\sin^2\frac{\pi}{v}}\int_0^{\frac{\pi}{v}} g'\Big(\frac{h}{\cos^2\psi}\Big)\Big(1 - \frac{\sin^2\psi}{\sin^2\frac{\pi}{v}}\Big)\frac{d\psi}{\cos^4\psi}\,\frac{2a\cos^2\frac{\pi}{v}}{v\sin^2\frac{\pi}{v}}\int_0^{\frac{\pi}{v}} g'\Big(\frac{h}{\cos^2\psi}\Big)\frac{\sin^2\psi}{\cos^4\psi}d\psi \\
&\quad \geq \mathrm{const}\, a^2 \int_0^{\frac{\pi}{2v}} g'\Big(\frac{h}{\cos^2\psi}\Big)\frac{d\psi}{\cos^4\psi}\int_{\frac{\pi}{2v}}^{\frac{\pi}{v}} g'\Big(\frac{h}{\cos^2\psi}\Big)\frac{d\psi}{\cos^4\psi}
\end{aligned}
$$

for all sufficiently small $a > 0$ and $5 \leq v \leq 7$,

This together with the above upper estimate for M_{aa} and (3.2) then yields (3.11):

$$
\frac{M_{aa}M_{vv} - M_{av}^2}{M_{aa}} \geq \left\{ \begin{array}{l} \mathrm{const}\, a^2 \displaystyle\int_0^{\frac{\pi}{2v}} g'\Big(\frac{h}{\cos^2\psi}\Big)\frac{d\psi}{\cos^4\psi} \geq \mathrm{const}\,\frac{a^2}{2h}\int_0^{\frac{\pi}{2v}} g'\Big(\frac{h}{\cos^2\psi}\Big)\frac{2h\sin\psi}{\cos^3\psi}\,d\psi, \text{ or} \\ \mathrm{const}\, a^2 \displaystyle\int_{\frac{\pi}{2v}}^{\frac{\pi}{v}} g'\Big(\frac{h}{\cos^2\psi}\Big)\frac{d\psi}{\cos^4\psi} \geq \mathrm{const}\,\frac{a^2}{2h}\int_{\frac{\pi}{2v}}^{\frac{\pi}{v}} g'\Big(\frac{h}{\cos^2\psi}\Big)\frac{2h\sin\psi}{\cos^3\psi}\,d\psi \end{array} \right\}
$$

$$\geq \left\{ \begin{array}{l} \text{const}\,(g(\frac{h}{\cos^2\frac{\pi}{v}}) - g(h)),\text{ or} \\ \text{const}\,a(g(\frac{h}{\cos^2\frac{\pi}{v}}) - g(\frac{k}{\cos^2\frac{\pi}{2v}})) \end{array} \right\} \geq \text{const}\,ag(a)$$

for all sufficiently small $a > 0$ and for $5 \leq v \leq 7$.

Fifth,

(3.12) $b^2 M_{aa}(a,v) + 2bM_{av}(a,v) + M_{vv} \geq \text{const}\,a\,g(a)$
for all sufficiently small $a > 0$ and for $b \in \mathbb{R}$, $5 \leq v \leq 7$.

The polynomial in b in (3.12) attains its minimum for $b = -M_{av}/M_{aa}$. This together with (3.11) implies (3.12):

$$b^2 M_{aa} + 2bM_{av} + M_{vv} \geq \frac{M_{av}^2}{M_{aa}} - \frac{2M_{av}^2}{M_{aa}} + M_{vv} = \frac{M_{aa}M_{vv} - M_{av}^2}{M_{aa}} \geq \text{const}\,a\,g(a)$$

for all sufficiently small $a > 0$ and for $5 \leq v \leq 7$.

Finally we show, sixth, the following estimate which refines (2.7):

(3.13) For each σ with $0 < \sigma < 1$ there is a $\tau > 1$ such that for each $a > 0$ and all sufficiently large $m = 1, 2, \ldots$, holds:

$$\sigma m M(\frac{a}{\sigma m}, 6) \geq \tau m M(\frac{a}{m}, 6).$$

Let $\lambda = 1/\sigma > 1$. Since $M(0,6) = 0$ and since $M(a,6)$ is convex for $a \geq 0$ by (3.5), an inspection of the graph of $M(a,6)$ shows that $M(a,6) + M_a(a,6)(\lambda - 1)a \geq \lambda M(a,6)$. Hence we have by (3.6), (3.7), (3.8) and (3.10) that

$$M(\lambda a, 6) \geq M(a,6) + M_a(a,6)(\lambda-1)a + \text{ const}\,M(a,6)(\lambda-1)^2$$

$$\geq \lambda M(a,6) + (\lambda-1)^2 \text{const}\,M(a,6),$$

or

$$\frac{1}{\lambda} M(\lambda a, 6) \geq (1 + \frac{(\lambda-1)^2}{\lambda}\text{const}\,) M(a,6) = \tau M(a,v),\text{ say,}$$

for all sufficiently small $a > 0$. Now replace a by a/m and $1/\lambda$ by σ and multiply both sides with m.

3.2.4 A lower estimate for $M(D,o)$. The following proposition is a refinement of the moment lemma (3.3) for 6-gons. It can be obtained along the lines of Hajs' [22] proof of (3.3) and so we omit its proof.

(3.14) Let D be a convex 6-gon with $|D| = a > 0$ and $o \in \operatorname{int} D$. Let $C(\delta) = C((1-\delta)(a/2\sqrt{3})^{1/2}, o)$, where $0 \leq \delta < 1$, be the circle of maximum radius with center o contained in D. Let $D(\delta)$ denote a convex 6-gon such that $|D(\delta)| = a$, $C(\delta) \subset D(\delta)$, the vertices of $D(\delta)$ are equidistant from o, the longest edge of $D(\delta)$ touches $C(\delta)$, and the 5 other edges all have the same length. Then $M(D, o) \geq M(D(\delta), o)$.

The proof of the next result is surprisingly tedious.

(3.15) Let D and $D(\delta)$ be as in (3.14). Then $M(D(\delta), o) \geq (1 + \operatorname{const} \delta^2) M(a, 6)$ for all sufficiently small $a > 0$ and $0 \leq \delta < 1$.

The first step is to show that

(3.16) $M(D(\delta), o)$ is non-decreasing for $0 \leq \delta < 1$.

Since $M(D(\delta), o)$ is clearly continuous in δ, it is sufficient to prove the following: let $0 < \delta < 1$, then for each $\varepsilon < \delta$ sufficiently close to δ we have $M(D(\varepsilon), o) \leq M(D(\delta), o)$. Let p, q, r be consecutive vertices of $D(\delta)$ such that the line segment $[q, r]$ is the longest edge of $D(\delta)$. It is the only edge of $D(\delta)$ touching $C(\delta)$ and $\|p - q\| < \|q - r\|$. Then, if $\varepsilon < \delta$ is sufficiently close to δ, the following hold: if we keep all vertices of $D(\delta)$ fixed, except q, and move q parallel to $[q, r]$ a suitable (small) distance in the direction $r - p$, we obtain a convex 6-gon E, say, such that $|E| = a, C(\varepsilon) \subset E$, precisely one edge of E touches $C(\varepsilon)$, and $M(E, o) < M(D(\delta), o)$. (To see the latter notice that E is obtained from $D(\delta)$ by keeping $D(\delta) \cap E$ pointwise fixed and shearing the triangle $D(\delta) \setminus E$ parallel to $[p, r]$ onto the triangle $E \setminus D(\delta)$. The shearing strictly decreases distances from o if ε is sufficiently close to δ.) By (3.14) $M(D(\varepsilon), o) \leq M(E, o)$. Since $M(E, o) < M(D(\delta), o)$, this concludes the proof of (3.16).

In the second step δ is replaced by a parameter φ as follows: for $0 \leq \delta < 1$ let $0 \leq \varphi < \pi/3$ be such that the angles under which the longest edge of $D(\delta)$ and the 5 other edges appear from o are $\pi/3 + 2\varphi$ and $\pi/3 - 2\varphi/5$, respectively. From now on we write $D(\varphi)$ rather than $D(\delta)$. Let $r = r(a, \varphi)$ be the circumradius of $D(\varphi)$. Since the longest edge of $D(\varphi)$ touches the circle $C(\delta) = C((1-\delta)(a/2\sqrt{3})^{1/2}, o)$ and $|D(\varphi)| = a$, it follows that

$$(1-\delta)(\frac{a}{2\sqrt{3}})^{1/2} = r\cos(\frac{\pi}{6} + \varphi),$$

$$\begin{aligned} a &= r^2(\sin(\frac{\pi}{6} + \varphi)\cos(\frac{\pi}{6} + \varphi) + 5\sin(\frac{\pi}{6} - \frac{\varphi}{5})\cos(\frac{\pi}{6} - \frac{\varphi}{5})) \\ &= r^2(\frac{1}{2}\sin(\frac{\pi}{3} + 2\varphi) + \frac{5}{2}\sin(\frac{\pi}{3} - \frac{2\varphi}{5})). \end{aligned}$$

Hence,

$$r^2 = r^2(a, \varphi) = 2a(\sin(\frac{\pi}{3} + 2\varphi) + 5\sin(\frac{\pi}{3} - \frac{2\varphi}{5}))^{-1},$$

$$\delta = \delta(\varphi) = 1 - (4\sqrt{3})^{1/2} \cos(\frac{\pi}{6} + \varphi)(\sin(\frac{\pi}{3} + 2\varphi) + 5\sin(\frac{\pi}{3} - \frac{2\varphi}{5}))^{-1/2}.$$

A numerical calculation readily implies that $1/4 \le \delta'(\varphi) \le 1$ for $0 \le \varphi < \pi/3$. Using the fact that $\delta(0) = 0$, we obtain

(3.17) $\frac{\varphi}{4} \le \delta(\varphi) \le \varphi$ for $0 \le \varphi < \frac{\pi}{3}$,

(3.18) $\delta(\varphi)$ is strictly increasing for $0 \le \varphi < \frac{\pi}{3}$.

In the third step of the proof of (3.15) it will be shown that

(3.19) $M(D(\varphi), o) \ge (1 + \mathrm{const}\, \varphi^2) M(a, 6)$ for $0 \le \varphi < \frac{\pi}{3}$.

Using (3.18), propositions (3.14) and (3.16) may be expressed as follows:

(3.20) $M(D, o) \ge M(D(\varphi), o)$ for $0 \le \varphi < \frac{\pi}{3}$.

(3.21) $M(D(\varphi), o)$ is non-decreasing for $0 \le \varphi < \frac{\pi}{3}$.

Define

$$k = k(a, \varphi) = r^2(a, \varphi) \cos^2(\frac{\pi}{6} + \varphi) = 2a \cos^2(\frac{\pi}{6} + \varphi)(\sin(\frac{\pi}{3} + 2\varphi) + 5\sin(\frac{\pi}{3} - \frac{2\varphi}{5}))^{-1},$$

$$l = l(a, \varphi) = r^2(a, \varphi) \cos^2(\frac{\pi}{6} - \frac{\varphi}{5}) = 2a \cos^2(\frac{\pi}{6} - \frac{\varphi}{5})(\sin(\frac{\pi}{3} + 2\varphi) + 5\sin(\frac{\pi}{3} - \frac{2\varphi}{5}))^{-1}.$$

Then, considering the definition of $D(\varphi) = D(\delta)$ in (3.14), one may express $M(D(\varphi), o)$ as follows:

$$\text{(3.22)}\quad M(D(\varphi), o) = 2 \int_0^{\frac{\pi}{6}+\varphi} \int_0^{\frac{k^{1/2}}{\cos\psi}} g(r^2) r\, dr\, d\psi + 10 \int_0^{\frac{\pi}{6}-\frac{\varphi}{5}} \int_0^{\frac{l^{1/2}}{\cos\psi}} g(r^2) r\, dr\, d\psi$$
$$= \int_0^{\frac{\pi}{6}+\varphi} G\Big(\frac{k}{\cos^2\psi}\Big)\, d\psi + 5 \int_0^{\frac{\pi}{6}-\frac{\varphi}{5}} G\Big(\frac{l}{\cos^2\psi}\Big)\, d\psi.$$

Elementary calculations show that

$$\text{(3.23)}\quad k(a, 0) = \frac{a}{2\sqrt{3}},\ k_\varphi(a, 0) = -\frac{a}{3},\ k_{\varphi\varphi}(a, 0) = -\frac{4a}{15\sqrt{3}},\ k_\varphi(a, \varphi) < 0,$$
$$l(a, 0) = \frac{a}{2\sqrt{3}},\ l_\varphi(a, 0) = -\frac{a}{15},\ l_{\varphi\varphi}(a, 0) = \frac{28a}{15\sqrt{3}},\ l(a, \varphi),\ l_\varphi(a, \varphi) > 0,$$
$$r^2(a, \varphi) \ge r^2(a, 0) = \frac{2a}{3\sqrt{3}} \quad \text{for } 0 \le \varphi < 0.2.$$

Differentiating (3.22), integrating by parts, and taking into account that $-2k(a,\varphi)k_{\varphi\varphi}(a,\varphi)$, $l(a,\varphi)$, $l_{\varphi\varphi}(a,\varphi) > 0$ for $0 \le \varphi < 0.2$, that $g' \ge 0$ and $g(0) = 0$, we see that

$$
\begin{aligned}
(3.24)\ & M(D(0),o) = M(a,6),\\
& M_\varphi(D(\varphi),o) = G(r^2) + 5\frac{-1}{5}G(r^2)\\
& \quad + k_\varphi \int_0^{\frac{\pi}{6}+\varphi} g\Big(\frac{k}{\cos^2\psi}\Big)\frac{d\psi}{\cos^2\psi} + 5l_\varphi \int_0^{\frac{\pi}{6}-\frac{\varphi}{5}} g\Big(\frac{l}{\cos^2\psi}\Big)\frac{d\psi}{\cos^2\psi},\\
& M_\varphi(D(0),o) = 0,\\
& M_{\varphi\varphi}(D(\varphi),o) = k_\varphi \frac{g(r^2)}{\cos^2(\frac{\pi}{6}+\varphi)} + 5\frac{-1}{5}l_\varphi \frac{g(r^2)}{\cos^2(\frac{\pi}{6}-\frac{\varphi}{5})}\\
& \quad + k_{\varphi\varphi}\int_0^{\frac{\pi}{6}+\varphi} g\Big(\frac{k}{\cos^2\psi}\Big)\frac{d\psi}{\cos^2\psi} + k_\varphi^2 \int_0^{\frac{\pi}{6}+\varphi} g'\Big(\frac{k}{\cos^2\psi}\Big)\frac{d\psi}{\cos^4\psi}\\
& \quad + 5l_{\varphi\varphi}\int_0^{\frac{\pi}{6}-\frac{\varphi}{5}} g\Big(\frac{l}{\cos^2\psi}\Big)\frac{d\psi}{\cos^2\psi} + 5l_\varphi^2 \int_0^{\frac{\pi}{6}-\frac{\varphi}{5}} g'\Big(\frac{l}{\cos^2\psi}\Big)\frac{d\psi}{\cos^4\psi}\\
& \quad = \Big(\frac{k_\varphi}{\cos^2(\frac{\pi}{6}+\varphi)} - \frac{l_\varphi}{\cos^2(\frac{\pi}{6}+\frac{\varphi}{5})} + k_{\varphi\varphi}\tan(\frac{\pi}{6}+\varphi) + 5l_{\varphi\varphi}\tan(\frac{\pi}{6}-\frac{\varphi}{5})\Big)g(r^2)\\
& \quad - 2kk_{\varphi\varphi}\int_0^{\frac{\pi}{6}+\varphi} g'\Big(\frac{k}{\cos^2\psi}\Big)\frac{\sin^2\psi}{\cos^4\psi}\,d\psi + k_\varphi^2\int_0^{\frac{\pi}{6}+\varphi} g'\Big(\frac{k}{\cos^2\psi}\Big)\frac{d\psi}{\cos^4\psi}\\
& \quad - 2ll_{\varphi\varphi}\int_0^{\frac{\pi}{6}-\frac{\varphi}{5}} g'\Big(\frac{l}{\cos^2\psi}\Big)\frac{\sin^2\psi}{\cos^4\psi}\,d\psi + l_\varphi^2\int_0^{\frac{\pi}{6}-\frac{\varphi}{5}} g'\Big(\frac{l}{\cos^2\psi}\Big)\frac{d\psi}{\cos^4\psi}\\
& \quad = aJ(\varphi)g(r^2) - aK(\varphi)g(k) + aL(\varphi)g(l) \ \text{ for } 0 \le \varphi < 0.2,
\end{aligned}
$$

where all the expressions

$$
\begin{aligned}
(3.25)\ & I(\varphi) = \frac{1}{k}\Big(\frac{k_\varphi}{\cos^2(\frac{\pi}{6}+\varphi)} - \frac{l_\varphi}{\cos^2(\frac{\pi}{6}+\frac{\varphi}{5})} + k_{\varphi\varphi}\tan(\frac{\pi}{6}+\varphi) + 5l_{\varphi\varphi}\tan(\frac{\pi}{6}-\frac{\varphi}{5})\Big),\\
& J(\varphi) = I(\varphi) + \frac{1}{a}\Big(\frac{k_\varphi^2}{2k\sin(\frac{\pi}{6}+\varphi)} - l_{\varphi\varphi}\tan(\frac{\pi}{6}-\frac{\varphi}{5})\Big),\\
& K(\varphi) = \frac{1}{a}\frac{k_\varphi^2}{2k\sin(\frac{\pi}{6}+\varphi)},\ L(\varphi) = \frac{1}{a}\,l_{\varphi\varphi}\tan(\frac{\pi}{6}-\frac{\varphi}{5}) \text{ do not depend on } a.
\end{aligned}
$$

Propositions (3.23) and (3.25) imply that

$$
(3.26)\ J(0) = \frac{50\sqrt{3}-28}{225},\ K(0) = \frac{50\sqrt{3}}{225},\ L(0) = \frac{28}{225}.
$$

¿From $0 < k \le l \le r^2\cos^2(\pi/6 - \varphi/5) \le 0.286\, r^2$ for $0 \le \varphi < 0.2$, and (3.3) we obtain that $0 < g(k) \le g(l) \le \beta\, g(r^2)$ for $0 \le \varphi < 0.2$, where $0 < \beta < 1$. Hence (3.24) implies that

$$M_{\varphi\varphi}(D(\varphi),o) \geq a(J(\varphi) - (K(\varphi) - L(\varphi)\,\beta)\,g(r^2) \text{ for } 0 \leq \varphi < 0.2.$$

Since $J(0) - (K(0) - L(0)) = 0$, $K(0) - L(0) > 0$ by (3.26), $0 < \beta < 1$, and since $J(\varphi)$, $K(\varphi)$, $L(\varphi)$ are continuous in φ and do not depend on a by (3.25), we conclude further that

$$M_{\varphi\varphi}(D(\varphi),o) \geq \text{const}\, a\, g(r^2) \geq \text{const}\, a\, g(a)$$
for all sufficiently small $a > 0$ and for $0 \leq \varphi \leq \text{const}\,(< 0.2)$,

where we have used (3.23) and(3.2). Combining this with (3.24) and (3.8) then yields,

$$\begin{aligned} M(D(\varphi),o) &\geq M(D(0),o) + M_\varphi(D(0),o) + \frac{1}{2}M\varphi\varphi(D(\vartheta\varphi),o)\varphi^2 \\ &\geq M(a,6) + \text{const}\, a\, g(a)\,\varphi^2 \geq (1 + \text{const}\,\frac{a\,g(a)}{M(a,6)}\varphi^2)M(a,6) \\ &\geq (1 + \text{const}\,\varphi^2)M(a,6) \end{aligned}$$
for all sufficiently small $a > 0$ and for $0 \leq \varphi \leq$const.

Since by (3.21) $M(D(\varphi),o)$ is non-decreasing for $0 \leq \varphi < \pi/3$, we thus obtain (3.19), decreasing the constant in $(1 + \text{const}\,\varphi^2)$ if necessary.

Finally, noting that $D(\delta) = D(\varphi)$ for $\delta = \delta(\varphi)$, proposition (3.15) is an immediate consequence of (3.20) and (3.17).

3.2.5 The form of D.

(3.27) There is a Landau symbol $o(1)$ as $\delta \to 0$ for which the following hold: let D and $C(\delta)$ be as in (3.14) and let p_i, $i = 1.\dots,6$, be the mirror images of the center of $C(\delta)$ in the lines containing the edges of D. Then there is a regular 6-gon with center o, vertices q_i, and such that $\|q_i\| = (2a/\sqrt{3})^{1/2}$ and $\|p_i - q_i\| \leq o(1)\|q_i\|$ for $i = 1,\dots,6$.

This result is an immediate consequence of the next proposition.

(3.28) Let $\varepsilon > 0$. Then for each sufficiently small $\delta > 0$ one gets: for any convex 6-gon D such that $|D| = a$ and $C(\delta) = C((1-\delta)(a/2\sqrt{3})^{1/2},o) \subset D$ there is a regular convex 6-gon R with center o, $|R| = a$, and $\delta^H(D,R) < \varepsilon$.

Here δ^H is the usual Hausdorff metric on the space of compact convex sets in $\mathbb{E}^2$, see [?]. For the proof of (3.28) assume the contrary. Then there are $\delta_k > 0$ and convex 6-gons D_k such that $\delta_k \to 0$ as $k \to \infty$, $|D_k| = a$, $C(\delta_k) \subset D_k$, , and $\delta^H(D_k,R) \geq \varepsilon$. For each regular 6-gon R with center o and $|R| = a$. Since $C(\delta_k) \to C(0) = C((a/2\sqrt{3})^{1/2},o)$, $C(\delta_k) \subset D_k$, and $|D_k| = a$, the sequence (D_k) is bounded. Thus a version of Blaschke‘s selection theorem yields the following: by considering a suitable subsequence and renumbering, if necessary, we may assume that $D_k \to D$, say, where D is a convex 3, 4, 5, or 6-gon, $C(0) \subset D$, $|D| = a$, and $\delta^H(D,R) \geq \varepsilon$ for each regular 6-gon R with $|R| = a$ and center o. A simple isoperimetric inequality for polygons says that all convex 3, 4, 5, or 6-gons circumscribed to $C(0) = C((a/2\sqrt{3})^{1/2}, o)$ have area greater than a, except for the regular 6-gon with center o. Thus D must be a regular 6-gon with center o. Hence $\delta^H(D,R) = 0$ for the regular 6-gon $R = D$ with $|R| = a$ and center o. This contradiction concludes the proof of (3.28) and thus of (3.27).

3.3 Proof of Theorem 2 in the Euclidean case

3.3.1 In the first part we will prove the following proposition.

(3.29) *Let H be a convex 3, 4, 5, or 6-gon in $\mathbb{E}^2$ and (T_m) a sequence of finite sets in $\mathbb{E}^2$ with $\#T_m = m$ for $m = 1, 2, \dots,$ such that*

$$\int\limits_H \min_{p \in T_m} \{f(\|p - x\|)\}\, dx \sim mM(\frac{|H|}{m}, 6) \text{ as } m \to \infty.$$

Then T_m is asymptotically a regular hexagonal pattern of edge length

$$(\frac{2|H|}{\sqrt{3}m})^{1/2}.$$

3.3.2 Simplifications, notations, and remarks. Suitable small distortions of the sets T_m do not affect proposition (3.29). Thus we may assume that no vertex of H is equidistant from 2 points of T_m, no point on an edge of H is equidistant from 3 points of T_m, and no point of $\mathbb{E}^2$ is equidistant from 4 points of T_m for $m = 1, 2, \dots$ We clearly may assume that $|H| = 1$.

Define the *Dirichlet-Voronoi cells*

$$D(T_m, p) = \{x \in H : \|x - p\| \leq \|x - q\| \text{ for each } q \in T_m\},\ p \in T_m.$$

The sets $D(T_m, p)$ are convex polygons in H, possibly degenerate, that is, of dimension less than 2. For $i = 3, 4, \dots$, let D_{ij}, $j = 1, \dots, m_i$, be the proper, that is the non-degenerate i-gons among these polygons and for each D_{ij} let $p_{ij} \in T_m$ be such that $D_{ij} = D(T_m, p_{ij})$. Put $a_{ij} = |D_{ij}|$. Let $m_{prop} = m_3 + \dots + m_k$, where k is the maximum i with $m_i > 0$. The following properties are easy to check.

(3.30) The polygons D_{ij}, $i = 3, \dots, k$, $j = 1, \dots, m_i$, are proper convex polygons which tile H. A vertex of this tiling in $\mathrm{int} H$ is a vertex of precisely 3 such polygons, a vertex in the relative interior of an edge of H is a vertex of precisely 2 such polygons and a vertex of H is a vertex of precisely one of these polygons.

(3.31) $a_{31} + \dots + a_{3m_3} + \dots + a_{km_k} = 1\,(= |H|)$.

The assumptions of proposition (3.29), the fact that f is continuous and strictly increasing and the definition of the D_{ij} together with (3.3) imply that

$$(3.32)\ (m+o(m))M(\frac{1}{m},6) \geq \int_H \min_{p\in T_m}\{f(\|p-x\|)\}\,dx$$
$$= \sum_{i,j}\int_{D_{ij}} f(\|p_{ij}-x\|)\,dx = M(D_{31},p_{31})+\cdots+M(D_{km_k},p_{km_k})$$
$$\geq M(a_{31},3)+\cdots+M(a_{km_k},k).$$

3.3.3 Most polygons $D(T_m,p)$ are proper and contained in int$H$:

$$(3.33)\ m_{prop} = m - o(m).$$

If (3.33) did not hold, then $m_{prop} \leq \sigma m$ for infinitely many m, where $0<\sigma<1$. Considering only such m (in the proof of (3.33)), propositions (3.32), (2.10), (1.1), (2.7) and (3.13) together yield a contradiction and thus settle the proof of (3.33):

$$(m+o(m))M(\frac{1}{m},6) \geq \int_H \min_{p\in T_m}\{f(\|p-x\|)\}\,dx$$
$$= \int_H \min_{p\in T_m(H)}\{f(\|p-x\|)\}\,dx \geq m_{prop}M(\frac{1}{m_{prop}},6)$$
$$\geq \sigma\, m M(\frac{1}{\sigma\, m},6) > \tau m M(\frac{1}{m},6),\text{ where } \tau>1.$$

Next,

$$(3.34)\ m_{int} = \#\{D_{ij} : D_{ij} \subset \text{int}H\} = m_{prop} - o(m) = m - o(m).$$

For, if this were not the case, then $m_{int} \leq \sigma^2 m$ for infinitely many m, where $0<\sigma<1$. Consider in the following only such m. Let $\tau = \chi^2 > 1$ correspond to σ as in (3.13). By choosing $\chi\,(>1)$ closer to 1, if necessary, we may suppose that $\chi \leq 1/\sigma$. Since the first expression in (3.32) tends to 0 as $m\to\infty$, (2.9) implies that $\max\{\text{diam}\, D_{ij}\} \to 0$ as $m\to\infty$. Choose $K \subset \text{int}\, H$ homothetic to H with $|K| = |H|/\chi = 1/\chi$. Then for all sufficiently large m there are $m_K \leq m_{int}$ proper convex polygons among the sets $D_{ij}\cap K$. Thus (3.32), (2.10), (1.1) applied to K, (2.7), (3.13), the inequality $\chi m_{int} \leq \sigma m$ and the equality $\tau/\chi = \chi > 1$ lead to a contradiction, concluding the proof of (3.34):

$$(m+o(m))M(\frac{1}{m},6) \geq \int_H \min_{p\in T_m}\{f(\|p-x\|)\}\,dx \geq \int_K \min_{p\in T_m}\{f(\|p-x\|)\}\,dx$$
$$\geq \int_K \min_{p\in T_m(K)}\{f(\|p-x\|)\}\,dx \geq m_K M(\frac{|K|}{m_K},6) \geq m_{int}M(\frac{|K|}{m_{int}},6)$$
$$= \frac{1}{\chi}\chi\, m_{int}M(\frac{1}{\chi\, m_{int}},6) \geq \frac{1}{\chi}\sigma\, m M(\frac{1}{\sigma\, m},6) \geq \frac{\tau}{\chi} m M(\frac{1}{m},6).$$

3.3.4 Most D_{ij} are 6-gons. We need the auxiliary results (3.35)-(3.37).

(3.35) $3m_3+\cdots+km_k = 6(m_3+\cdots+m_k)-o(m) = 6m_{prop}-o(m) = 6m-o(m)$,
or $3m_3+\cdots+5m_5+7m_7+\cdots+km_k = 6m_{\neq 6}-o(m)\,(\leq 6m)$,

where $m_{\neq 6} = m_3+\cdots+m_5+m_7+\cdots+m_k$. Let $v, e, n = m_{prop}$ denote the numbers of vertices, edges, and facets of the tiling of H by the proper convex polygons D_{ij}. Proposition (3.30) shows that

$$3v = 3m_3+\cdots+km_k+v_{bd}+2v_H,\ 2e = 3m_3+\cdots+km_k+e_{bd},$$

where v_{bd}, v_H, e_{bd} are the numbers of vertices of the tiling in the relative interiors of the edges of H, of the vertices of H, and of the edges of the tiling on bd$H$, respectively. Euler's polytope formula then yields that

$$6 = 6v-6e+6n = -(3m_3+\cdots+km_k)+6n+2v_{bd}+4v_H-3e_{bd}.$$

Now, noticing that (3.33), (3.34) and $v_H \leq 6$ imply that $n = m_{prop} = m-o(m)$ and $2v_{bd}+4v_H-3e_{bd}-6 = o(m)$, proposition (3.35) follows.

(3.36) Let D, E be regular polygons with center o and such that $D \subset E$ and the circumcircle of D contains the incircle of E and vice versa. Then

$$\frac{1}{4} \leq \frac{|D|}{|E|} \leq 4,$$

as elementary calculations show. The fact that f satisfies the growth condition and thus is strictly increasing implies the next result.

(3.37) Let D, E be regular polygons with center o and such that $D \subset E$ and the circumcircle of D is contained in the interior of the incircle of E. Then there are $\lambda > 1 > \mu > 0$ such that

$$\lambda D \subset \mu E,\ \lambda D \not\subset \operatorname{int}\mu E,\ |D|+|E| = |\lambda D|+|\mu E|,$$

$$M(\lambda D, o)+M(\mu E, o) < M(D, o)+M(E, o).$$

After these preparations we prove that

(3.38) $m_{\neq 6} = m_3+\cdots+m_5+m_7+\cdots+m_k = o(m)$.

Assume, on the contrary, that (3.38) does not hold. Then

(3.39) $m_{\neq 6} \geq \text{const}\, m$ for infinitely many m.

Only such m will be considered. (3.32), (3.31), (3.36), (3.37) and (3.33) imply that

$$(3.40)\quad (m+o(m))M(\frac{1}{m},6) \geq M(b_{31},3)+\cdots+M(b_{km_k},k), \text{ where}$$
$$b_{31}+\cdots+b_{km_k}=1 \text{ and } \frac{1}{5m} \leq b_{ij} \leq \frac{5}{m} \text{ for all sufficiently large } m.$$

It follows from (3.35) and (3.39) that the average number $i_{\neq 6}$ of vertices of the non-hexagonal polygons D_{ij} satisfies

$$(3.41)\quad i_{\neq 6} = \frac{1}{m_{\neq 6}}(3m_3+\cdots+5m_5+7m_7+\cdots+km_k)\begin{cases} \leq 6 \\ \to 6, \text{ as } m \to \infty. \end{cases}$$

Let $m_{<6} = m_3+\cdots+m_5$, $m_{>6} = m_7+\cdots+m_k$. The cases $m_{<6}=0$ and $m_{>6}=0$ for infinitely many m are excluded by (3.41). Thus we may assume that $m_{<6}, m_{>6}, m_{\neq 6} = m_{<6}+m_{>6} > 0$ for all sufficiently large m. Consider only such m. Let

$$(3.42)\quad b_6 = \frac{1}{m_6}(b_{61}+\cdots+b_{6m_6}) \text{ for } m_6>0 \text{ and } b_6 = \frac{1}{m} \text{ otherwise,}$$
$$b_{<6} = \frac{1}{m_{<6}}(b_{31}+\cdots+b_{5m_5}),\ i_{<6} = \frac{1}{m_{<6}}(3m_3+\cdots+5m_5),$$
$$b_{>6} = \frac{1}{m_{>6}}(b_{71}+\cdots+b_{km_k}),\ i_{>6} = \frac{1}{m_{>6}}(7m_7+\cdots+km_k),$$
$$b_{\neq 6} = \frac{1}{m_{\neq 6}}(m_{<6}b_{<6}+m_{>6}b_{>6}),\ i_{\neq 6} = \frac{1}{m_{\neq 6}}(m_{<6}i_{<6}+m_{>6}i_{>6}).$$

By (3.40),

$$(3.43)\quad \frac{1}{5m} \leq b_{<6},\ b_{\neq 6},\ b_{>6} \leq \frac{5}{m},\ i_{<6} \leq 5,\ i_{>6} \geq 7.$$

Consider the line segment $[(b_{<6}, i_{<6}), (b_{>6}, i_{>6})]$ and parametrize it in the form: $(a+bv, v) : i_{<6} \leq v \leq i_{>6}$. Clearly,

$$(3.44)\quad \frac{1}{5m} \leq a+bv \leq \frac{5}{m} \text{ for } 5 \leq v \leq 7 \text{ and } a+bi_{<6} = b_{<6}, a+bi_{>6} = b_{>6}.$$

This, combined with (3.12) and (3.2), shows that

$$(3.45)\quad \frac{d^2}{dv^2}M(a+bv,v) = M_{aa}(a+bv,v)b^2+2M_{av}(a+bv,v)b+M_{vv}(a+bv,v)$$
$$\geq \text{ const } \frac{1}{m}g(\frac{1}{m}) \text{ for all sufficiently large } m \text{ and for } 5 \leq v \leq 7.$$

Next it will be proved that

$$(3.46)\quad (1-\lambda)M(a+5b,5)+\lambda M(a+7b,7) \geq (1+\text{const})M(a+bv,v)$$
$$\text{for all sufficiently large } m \text{ and for } 5.5 \leq v = 5(1-\lambda)+7\lambda \leq 6.5.$$

To see this note first that (3.45) implies the following:

$$
\begin{aligned}
M(a+bi_\square, i_\square) &= M(a+bv,v) + \frac{d}{dv}M(a+bv,v)(i_\square - v)\\
&\quad + \frac{1}{2}\frac{d^2}{dv^2}M(a+b(v+\vartheta_\square(i_\square - v), (v+\vartheta_\square(i_\square - v))(i_\square - v)^2\\
&\geq M(a+bv,v) + \frac{d}{dv}M(a+bv,v)(i_\square - v) + \text{const}\frac{1}{m}g(\frac{1}{m})(i_\square - v)^2\\
&\quad \text{for all sufficiently large } m \text{ and for } 5 \leq v \leq 7 \text{ where } i_\square = 5 \text{ or } 7.
\end{aligned}
$$

This together with $5.5 \leq v = 5(1-\lambda) + 7\lambda \leq 6.5$, (3.44), (3.8) and (3.2) then yields (3.46):

$$
\begin{aligned}
&(1-\lambda)M(a+5b,5) + \lambda M(a+7b,7)\\
&\quad \geq M(a+bv,v) + 0 + \text{ const}\, mg(\frac{1}{m}) \geq (1+\text{const})M(a+bv,v)\\
&\quad \text{for all sufficiently large } m \text{ and for } 5.5 \leq v = 5(1-\lambda)+7\lambda \leq 6.5.
\end{aligned}
$$

Next we prove that

$$
\begin{aligned}
(3.47)\quad &(1-\mu)M(a+bi_{<6}, i_{<6}) + \mu M(a+bi_{>6}, i_{>6}) \geq (1+\text{const})M(a+bv,v)\\
&\text{for all sufficiently large } m \text{ and for } 5.5 \leq v = (1-\mu)i_{<6} + \mu i_{>6} \leq 6.5.
\end{aligned}
$$

Given μ, choose λ such that $v = (1-\mu)i_{<6} + \mu i_{>6} = 5(1-\lambda)+7\lambda$. Since $M(\, , \,)$ is convex by (3.5), propositions (3.46) and (3.44) then yield (3.47).

¿From (3.40), (3.5), (3.47), (3.43), (3.42), (3.41), (3.5), the fact that $m_{prop} = m_6 + m_{\neq 6}$, $m_6 b_6 + m_{\neq 6} b_{\neq 6} = 1$, (3.33), (3.35) and (3.41) and thus $m_6 6 + m_{\neq 6} i_{\neq 6} = 6m_{prop} - o(m)$, (3.7), (3.43) and (3.8), we obtain that

$$
\begin{aligned}
(m+o(m))M(\frac{1}{m},6) &\geq m_6 M(b_6,6) + m_{<6}M(b_{<6}, i_{<6}) + m_{>6}M(b_{>6}, i_{>6})\\
&\geq m_6 M(b_6,6) + m_{\neq 6}(\frac{m_{<6}}{m_{\neq 6}}M(b_{<6}, i_{<6}) + \frac{m_{>6}}{m_{\neq 6}}M(b_{>6}, i_{>6}))\\
&\geq m_6 M(b_6,6) + m_{\neq 6}M(b_{\neq 6}, i_{\neq 6}) + \text{ const } \; m_{\neq 6}M(b_{\neq 6}, i_{\neq 6})\\
&\geq (m_6 + m_{\neq 6})M(\frac{1}{m_6 + m_{\neq 6}}(m_6 b_6 + m_{\neq 6} b_{\neq 6}), \frac{1}{m_6 + m_{\neq 6}}(m_6 6 + m_{\neq 6} i_{\neq 6}))\\
&\qquad + \text{const}\, m_{\neq 6}M(b_{\neq 6}, i_{\neq 6})\\
&\geq (m-o(m))M(\frac{1}{m_{prop}}, 6) + \text{const}\, m_{\neq 6}M(b_{\neq 6}, i_{\neq 6})\\
&\geq (m-o(m))M(\frac{1}{m}, 6) + \text{const}\, m_{\neq 6}\frac{1}{m}g(\frac{1}{m}).
\end{aligned}
$$

By (3.8), this yields a contradiction to (3.39) and thus concludes the proof of (3.38):

$$
o(m)\frac{1}{m}g(\frac{1}{m}) \geq m_{\neq 6}\frac{1}{m}g(\frac{1}{m}).
$$

3.3.5 The total area of the non-hexagonal D_{ij} is small. If $m_{\neq 6} = 0$ for all m with a finite set of exceptions, we are finished. Assume now that $m_{\neq 6} > 0$ for infinitely many m. Only such m will be considered in the following. Let

$$a_6 = \frac{1}{m_6}(a_{61} + \cdots + a_{6m_6}),\ a_{\neq 6} = \frac{1}{m_{\neq 6}}(a_{31} + \cdots + a_{5m_5} + a_{71} + \cdots + a_{km_k}).$$

Then we have

$$(3.48)\ m_{\neq 6} a_{\neq 6} = o(1).$$

The average area of the polygons D_{ij} is $1/m_{prop}$ and thus by (3.33) is equal to $1/(m - o(m))$. Therefore, if $a_{\neq 6} \leq a_6$, then $a_{\neq 6} \leq 1/(m - o(m))$ and (3.46) is an immediate consequence of (3.38). Assume now that $a_{\neq 6} \geq 1/(m - o(m))$. If (3.48) does not hold, then

$$(3.49)\ m_{\neq 6} a_{\neq 6} \geq \text{const}(> 0) \text{ for infinitely many } m.$$

Only such m will be considered. Combining (3.32), (3.5), (3.41), (3.7), (3.6) with $a = a_6$ or $a_{\neq 6}$ and $b = 1/m$, (3.33), (3.31), (3.33), (3.9), (3.10), (3.2) and $a_{\neq 6} \geq 1/m_{prop} \geq 1/m$, we see that

$$\begin{aligned}
(m + o(m))M(\frac{1}{m}, 6) &\geq m_6 M(a_6, 6) + m_{\neq 6} M(a_{\neq 6}, i_{\neq 6}) \\
&\geq m_6 M(a_6, 6) + m_{\neq 6} M(a_{\neq 6}, 6) \\
&\geq (m_6 + m_{\neq 6}) M(\frac{1}{m}, 6) + (m_6(a_6 - \frac{1}{m}) + m_{\neq 6}(a_{\neq 6} - \frac{1}{m})) M_a(\frac{1}{m}, 6) \\
&\quad + \frac{m_{\neq 6}}{2} M_{aa}(\frac{1}{m} + \vartheta(a_{\neq 6} - \frac{1}{m}), 6)(a_{\neq 6} - \frac{1}{m})^2 \\
&\geq (m - o(m))M(\frac{1}{m}, 6) + o(1) g(\frac{1}{m}) \\
&\quad + \text{const}\, \frac{m_{\neq 6}\, g(\frac{1}{m} + \vartheta(a_{\neq 6} - \frac{1}{m}))}{\frac{1}{m} + \vartheta(a_{\neq 6} - \frac{1}{m})} (a_{\neq 6} - \frac{1}{m})^2 \\
&\geq (m - o(m))M(\frac{1}{m}, 6) + o(1) g(\frac{1}{m}) + \text{const}\, m_{\neq 6}\, g(\frac{1}{m}) \frac{(a_{\neq 6} - \frac{1}{m})^2}{a_{\neq 6}}.
\end{aligned}$$

Thus, taking into account (3.8),

$$(3.50)\ m_{\neq 6} \frac{(a_{\neq 6} - \frac{1}{m})^2}{a_{\neq 6}} \leq o(1).$$

Since $m_{\neq 6} = o(m)$ by (3.38), inequality (3.49) together with $a_{\neq 6}(\geq 1/m)$ shows that $a_{\neq 6}$ is an arbitrarily large multiple of $1/m$. Hence the left hand side in (3.50) is essentially equal to $m_{\neq 6} a_{\neq 6}$. Thus (3.50) and (3.49) are in contradiction and the proof of (3.48) is complete.

3.3.6 The area of most of the polygons D_{6j} is approximately $1/m$. To see this, we first give an upper estimate:

(3.51) $\dfrac{1}{m} \leq a_{6j} \leq \dfrac{1}{m} + o(\dfrac{1}{m})$ for all j with $a_{6j} \geq \dfrac{1}{m}$,
with a set of at most $o(m)$ exceptions.

$mM(1/m,6) \to 0$, $M(0,6)=0$ and $M(a,6)$ is strictly increasing and continuous for $a \geq 0$. Thus (3.32) implies that a_{6j} all are arbitrarily small for all sufficiently large m.From (3.32), (3.6) with $a = a_{6j}$, $b = 1/m$, (3.33), (3.38), (3.31), (3.48), (3.33), (3.38), (3.10),the remark just made, (3.9) and (3.2) it follows that

$$\begin{aligned}
&(m+o(m))M(\frac{1}{m},6) \geq M(a_{61},6) + \cdots + M(a_{6m_6},6)\\
&\quad \geq m_6 M(\frac{1}{m},6) + M_a(\frac{1}{m},6)(m_6 a_6 - \frac{m_6}{m})\\
&\quad\quad + \frac{1}{2}\sum_{j=1}^{m_6} M_{aa}(\frac{1}{m} + \vartheta_j(a_{6j} - \frac{1}{m}),6)(a_{6j} - \frac{1}{m})^2\\
&\quad \geq (m - o(m))M(\frac{1}{m},6) + M_a(\frac{1}{m},6)(1 - o(1) - (1 - o(1)))\\
&\quad\quad + \text{const} \sum\{\frac{g(\frac{1}{m} + \vartheta_j(a_{6j} - \frac{1}{m}))}{\frac{1}{m} + \vartheta_j(a_{6j} - \frac{1}{m})}(a_{6j} - \frac{1}{m})^2 : a_{6j} \geq \frac{1}{m}\}\\
&\quad \geq (m - o(m))M(\frac{1}{m},6) + o(1)g(\frac{1}{m})\\
&\quad\quad + \text{const}\, g(\frac{1}{m})\sum\{\frac{(a_{6j} - \frac{1}{m})^2}{a_{6j}} : a_{6j} \geq \frac{1}{m}\}.
\end{aligned}$$

Thus, using (3.8) and putting $a_{6j} = (1+\varepsilon_j)/m$, $\delta_j = \varepsilon^2/(1+\varepsilon_j)$ for $a_{6j} \geq 1/m$ and $\varepsilon_j = \delta_j = 0$ for $a_{6j} < 1/m$, we conclude that

$$(3.52)\ o(m)\frac{1}{m}g(\frac{1}{m}) \geq g(\frac{1}{m})\sum_{j=1}^{m_6}\frac{\varepsilon_j^2}{(1+\varepsilon_j)m}, \text{ or } \sum_{j=1}^{m_6}\delta_j \leq o(m).$$

Denote the Landau symbol in (3.52) by $\bar{o}(m)$. Let $\delta_j > (\bar{o}(m)/m)^{1/2}$ for precisely l indices. Then (3.52) implies that $l(\bar{o}(m)/m)^{1/2} \leq \bar{o}(m)$, or $l \leq o(m)$. Hence $\delta_j = \varepsilon_j^2/(1+\varepsilon_j) \leq (\bar{o}(m)/m)^{1/2} = o(1)$ for all j with a set of at most $(l \leq) o(m)$ exceptions. Thus, a fortiori, $\varepsilon_j \leq o(1)$ for all j with a set of at most $o(m)$ exceptions. Considering the definition of ε_j, this concludes the proof of (3.51).

The proof of the corresponding lower estimate

(3.53) $\dfrac{1}{m} - o(\dfrac{1}{m}) \leq a_{6j} \leq \dfrac{1}{m}$ for all j with $a_{6j} < \dfrac{1}{m}$,
with a set of at most $o(m)$ exceptions

is slightly longer. We first show that

(3.54) $\dfrac{1}{2m} \leq a_{6j} < \dfrac{1}{m}$ for all j with $a_{6j} < \dfrac{1}{m}$,
with a set of at most $o(m)$ exceptions.

Let m_{6+}, m_{60}, m_{6-}, m_{6--} be the numbers of indices j with $a_{6j} > 1/m, = 1/m$, $< 1/m$, and $< 1/2m$, respectively. Denote by $a_{6+}, \dots, a_{6--}$ the averages of the corresponding areas. (If $m_\square = 0$, let $a_\square = 1/m$, say.) Assume now that (3.54) does not hold. Then

(3.55) $(m_{6-} \geq) \, m_{6--} \geq \operatorname{const} m$ for infinitely many m.

¿From now on we consider only such m. Propositions (3.55), (3.33), (3.38), (3.31) and (3.48) imply that

(3.56) $\dfrac{1}{m} - a_{6-} \geq \dfrac{\text{const}}{m}, \; a_{6+} - \dfrac{1}{m} \geq \dfrac{\text{const}}{m},$

(3.57) $m_{6-} + m_{60} + m_{6+} (= m_6) = m - o(m)$, or $\dfrac{m_{6-}}{m} + \dfrac{m_{60}}{m} + \dfrac{m_{6+}}{m} = 1 - o(1),$
$m_{6-}a_{6-} + m_{60}a_{60} + m_{6+}a_{6+} = 1 - o(1),$

and thus

(3.58) $m_{6+}(a_{6+} - \dfrac{1}{m}) = -m_{6-}(a_{6-} - \dfrac{1}{m}) + o(1).$

The remark in the proof of (3.51) also shows that a_{6+} is arbitrarily small for all sufficiently large m. Combining (3.32), (3.5), (3.6) with $a = a_{6-}, a_{6+}$ and $b = 1/m$, (3.57), (3.58), (3.9), (3.10) the remark about a_{6+}, gives:

$$\begin{aligned}
(m + o(m))M(\frac{1}{m}, 6) &\geq (m_{6-} + m_{60} + m_{6+})M(\frac{1}{m}, 6) + (m_{6-}(a_{6-} - \frac{1}{m}) \\
&\quad + m_{6+}(a_{6+} - \frac{1}{m}))M_a(\frac{1}{m}, 6) + \frac{m_{6+}}{2} M_{aa}(\frac{1}{m} + \vartheta(a_{6+} - \frac{1}{m}), 6)(a_{6+} - \frac{1}{m})^2 \\
&\geq (m - o(m))M(\frac{1}{m}, 6) + o(1)g(\frac{1}{m}) + \operatorname{const} m_{6+} g(\frac{1}{m}) \frac{(a_{6+} - \frac{1}{m})^2}{a_{6+}}.
\end{aligned}$$

Hence, using (3.8), (3.58) and (3.55), resp. (3.56), it follows that

$$o(m)\frac{1}{m} g(\frac{1}{m}) \geq o(1)g(\frac{1}{m}) + m_{6+}(a_{6+} - \frac{1}{m}) \frac{a_{6+} - \frac{1}{m}}{a_{6+}} g(\frac{1}{m}),$$

or

$$o(1) \geq (m_{6-}(\frac{1}{m} - a_{6-}) + o(1)) \frac{a_{6+} - \frac{1}{m}}{a_{6+}}, \text{ or } o(1) \geq \frac{a_{6+} - \frac{1}{m}}{a_{6+}}.$$

If $a_{6+} \geq 2/m$ we would have $o(1) \geq 1/2$. If $a_{6+} \leq 2/m$, (3.56) would imply $o(1) \geq$ const follows. Both alternatives are impossible for sufficiently large m, and so the proof of (3.54) is complete.

In view of (3.54) it is sufficient for the proof of (3.53) to consider those j with $1/2m \leq a_{6j} < 1/m$. In this case the proof of (3.53) is similar to that of (3.51) and is therefore omitted.

3.3.7 Before proceeding to the final steps of the proof of proposition (3.29), we combine the results (3.33), (3.38), (3.34), (3.51), (3.53) and (3.48):

(3.59) Among the $m - o(m)$ proper convex polygons D_{ij} all up to $o(m)$ are 6-gons contained in $\operatorname{int} H$ of area beween $1/m - o(1/m)$ and $1/m + o(1/m)$. The total area of these 6-gons D_{6j} is $1 - o(1)$. Call these D_{6j} and also the corresponding points p_{6j} *nice*.

Next we show that

(3.60) each of the $m - o(m)$ nice points $p_{6j} \in T_m$, with a set of at most $o(m)$ exceptions, has the following property: the 6 points $p_1, \dots, p_6 \in T_m$ which together with p_{6j} determine D_{6j} (i.e. $D_{6j} = \{x : \|x - p_{6j}\| \leq \|x - p_1\|, \dots, \|x - p_6\|\}$) form up to $o(1)$ a regular 6-gon with center p_{6j} and edge length $(2/\sqrt{3}m)^{1/2}$. A nice point with this property will be called *good* and the corresponding points $p_1, \dots, p_6$ its *neighbors*. A point in T_m which is not good is *bad*.

For each nice D_{6j} let $C_{6j} = C((1 - \delta_j)(a_{6j}/2\sqrt{3})^{1/2}, p_{6j})$ be the circle with center p_{6j} and maximum radius contained in D_{6j}. Then (3.32), (3.15), (3.6) with $a = a_{6j}$ and $b = 1/m$, and (3.59) imply the following:

$$\begin{aligned}
(m + o(m))M(\frac{1}{m}, 6) &\geq \sum_{j=1}^{m_6} M(D_{6j}, p_{6j}) \geq \sum_{j=1}^{m_6} (1 + \text{ const } \delta_j^2) M(a_{6j}, 6) \\
&\geq \sum \{(1 + \text{const}\, \delta_j^2)(M(\frac{1}{m}, 6) + M_a(\frac{1}{m}, 6)(a_{6j} - \frac{1}{m}) : D_{6j} \text{ nice}\} \\
&\geq (m - o(m))M(\frac{1}{m}, 6) + \text{const}\, M(\frac{1}{m}, 6) \sum \{\delta_j^2 : D_{6j} \text{ nice}\} \\
&\quad - (m - o(m))M_a(\frac{1}{m}, 6) o(\frac{1}{m}),
\end{aligned}$$

or

$$o(m)M(\frac{1}{m}, 6) + o(1)M_a(\frac{1}{m}, 6) \geq M(\frac{1}{m}, 6) \sum \{\delta_j^2 : \ D_{6j} \text{ nice}\},$$

or

$$\sum \{\delta_j^2 : \ D_{6j} \text{ nice}\} \leq o(m)$$

by (3.8) and (3.9). Since by (3.59) there are $m - o(m)$ nice D_{6j}, we have to consider $m - o(m)$ numbers $\delta_j^2 \geq 0$ with sum at most $o(m)$. An argument similar to the one in the proof of (3.51) then shows that $\delta_j^2 \leq o(1)$ for all δ_j^2 with a set of at most $o(m)$ exceptions. This together with (3.28) finally yields (3.60).

(3.61) All points of T_m, with a set of at most $o(m)$ exceptions, are *very good*, that is, they are good and their neighbors are also good.

By (3.60) the distance of a good point to any of its neighbors is $(1 \pm \bar{o}(1))(2/\sqrt{3}m)^{1/2}$ where $\bar{o}(1)$ is the Landau symbol $o(1)$ in (3.60). For each bad point q consider all good points contained in the circle $C((1 + \bar{o}(1))((2/\sqrt{3}m)^{1/2}, q)$. For each of these good points p consider the circle $C((1 - \bar{o}(1))(1/2\sqrt{3}m)^{1/2}, p) \subset$

$D(T_m, p)$. These circles do not overlap and all are contained in the circle $C((1 + \bar{o}(1))(3/2)(2/\sqrt{3}m)^{1/2}, q)$. Comparing areas, we see that there are at most 9 such good circles for sufficiently small $\bar{o}(1)$, that is, for sufficiently large m. Now cancel all bad points and for each bad point q cancel the (at most 9) good points in $C((1 + \bar{o}(1))((2/\sqrt{3}m)^{1/2}, q)$. Since by (3.60) there are at most $o(m)$ bad points, this amounts to the cancellation of at most $o(m)$ points. Clearly, each of the remaining $m - o(m) - o(m) = m - o(m)$ good points has no bad neighbor and thus is very good, concluding the proof of (3.61).

Finally,

(3.62) if p is a very good point and $p_1, \dots, p_6$ are its neighbors, then $C(1.15(2/\sqrt{3}m)^{1/2}, p) \cap T_m = \{p, p_1, \dots, p_6\}$.

To see this note that by (3.61) $p, p_1, \dots, p_6$ are contained in the circle in (3.62) and this circle is contained in $D(T_m, p) \cup D(T_m, p_1) \cup \ldots \cup D(T_m, p_6)$ as an elementary argument shows. This yields (3.62).

3.3.8 Proposition (3.29) now follows from (3.60)-(3.62). □

3.4 Proof of Theorem 2 in the Riemannian case

3.4.1 The first part consists in showing that

(3.63) S_n *is uniformly distributed in* J.

The function f is continuous with $f(0) = 0$. We use the growth condition to see that the right side in (3.1) tends to 0 as $n \to \infty$. Hence

$$(3.64) \quad \int_J \min_{p \in S_n} \{f(\varrho_M(p, x))\}\, d\omega_M(x) \to 0.$$

In the first step of the proof of (3.63) we establish the following proposition, where for the definitions of $S_n(K)$ and $n(K)$ the reader is referred to (2.10):

(3.65) Let $K \subset J$ be Jordan measurable with $0 < \omega_M(K) < \omega_M(J)$. Then
$$\int_K \min_{p \in S_n(K)} \{f(\varrho_M(p, x))\}\, d\omega_M(x) \sim n(K) M(\frac{\omega_M(K)}{n(K)}, 6).$$

Since K is Jordan measurable and $\omega_M(K) > 0$, we have $\operatorname{int} K \neq \emptyset$. By (2.10) and (3.64) the integral in (3.65) tends to 0 as $n \to \infty$. An application of (2.9) to K instead of J then shows that $n(K) = \# S_n(K) \to \infty$ as $n \to \infty$. Thus if (3.65) did not hold, Theorem 1 applied to K implies that

$$(3.66) \quad \int_K \min_{p \in S_n(K)} \{f(\varrho_M(p, x))\}\, d\omega_M(x) \geq \lambda\, n(K) M(\frac{\omega_M(K)}{n(K)}, 6)$$

for infinitely many n, where $\lambda > 1$.

¿From now on only such n will be considered (in the proof of (3.65)). By (2.8) there is a $\mu > 1$ such that $\lambda M(a,6) \geq M(\mu\, a,6)$ for all sufficiently small $a \geq 0$. Choose a compact Jordan measurable set $L \subset \operatorname{int}(J \setminus K)$ with

(3.67) $\omega_M(L) > 0$ and $\mu\,\omega_M(K) + \omega_M(L) \geq \nu\,\omega_M(J)$, where $\nu > 1$.

Since our choice of L implies that the sets K and L have positive distance, we conclude from (2.9) and (2.10) that

(3.68) $K \cap L = \emptyset$, $S_n(K) \cap S_n(L) = \emptyset$, $n(K \cup L) = n(K) + n(L) \leq n$
for all sufficiently large n.

Using the argument that was applied to K we get $n(L) \to \infty$. Hence Theorem 1 applied to L shows that

$$(3.69) \quad \int\limits_L \min_{p \in S_n(L)} \{f(\varrho_M(p,x))\}\, d\omega_M(x) \gtrsim n(L) M(\frac{\omega_M(L)}{n(L)},6).$$

(The symbol $\gtrsim$ means the following: the limit inferior of the quotient of the left and the right side in (3.69) as $n \to \infty$ is at least 1.) By (2.8) we may choose $\xi > 1$ such that $M(\nu\, a,6) \geq \xi\, M(a,6)$ for all sufficiently small $a \geq 0$. Then, using obvious abbreviations, (2.1), (3.1), (3.68) (2.10), (3.66), (3.69), our choice of μ, (3.5) and Jensen's inequality, (3.68), (2.7), (2.6), (3.67) and our choice of ξ yield a contradiction and thus conclude the proof of (3.65):

$$\begin{aligned} nM(\frac{\omega_M(J)}{n},6) &\sim \int\limits_J \geq \int\limits_K + \int\limits_L \gtrsim \lambda\, n(K) M(\frac{\omega_M(K)}{n(K)},6) + n(L) M(\frac{\omega_M(L)}{n(L)},6) \\ &\geq n(K) M(\frac{\mu\,\omega_M(K)}{n(K)},6) + n(L) M(\frac{\omega_M(L)}{n(L)},6) \\ &\geq (n(K)+n(L)) M(\frac{\mu\,\omega_M(K) + \omega_M(L)}{n(K)+n(L)},6) \geq nM(\frac{\nu\,\omega_M(J)}{n},6) \\ &\geq \xi\, nM(\frac{\omega_M(J)}{n},6) \text{ for all sufficiently large } n. \end{aligned}$$

In the second step we show that

(3.70) $n(J) \sim n$.

Otherwise $n(J) < \sigma n$ for infinitely many n, where $0 < \sigma < 1$. Only such n will be considered. By (3.13)) there is $\tau > 1$ such that $\sigma\, nM(a/\sigma\, n,6) \geq \tau\, nM(a/n,6)$ for each $a \geq 0$ and all sufficiently large n. Then (2.1), (3.1), (2.10) with $K = J$, Theorem 1, and (2.7) yield a contradiction, concluding the proof of (3.70):

$$nM(\frac{\omega_M(J)}{n},6) \sim \int\limits_J \sim n(J)M(\frac{\omega_M(J)}{n(J)},6) \geq \sigma\, nM(\frac{\omega_M(J)}{\sigma\, n},6)$$
$$\geq \tau\, nM(\frac{\omega_M(J)}{n},6) \text{ for all sufficiently large } n.$$

We come to the third step:

(3.71) Let $K \subset J$ be Jordan measurable. Then,

(i) $n(K) = o(n)$ for $\omega_M(K) = 0$,

(ii) $n(K) \sim n$ for $\omega_M(K) = \omega_M(J)$,

(iii) $n(K) \sim \dfrac{\omega_M(K)}{\omega_M(J)} n$ for $0 < \omega_M(K) < \omega_M(J)$.

Assume first that $\omega_M(K) = 0$. If (3.71 i) did not hold, $n(K) \geq (\eta - 1)n$ for infinitely many n, where $\eta > 1$. Only such n will be considered. Choose a compact Jordan measurable set $L \subset \operatorname{int}(J \setminus K)$ with $\omega_M(L) > 0$. Since K and L have positive distance, (2.9) and (2.10) imply that

$S_n(K) \cap S_n(L) = \emptyset$, $n(K \cup L) = n(K) + n(L) \geq \eta\, n(L)$
for all sufficiently large n.

As before, $n(L) \to \infty$. By (3.13) we may choose $0 < \zeta < 1$ such that $\eta\, mM(a/\eta\, m, 6) \leq \zeta\, mM(a/m, 6)$ for each $a \geq 0$ and all sufficiently large m. These remarks together with (3.65) and (2.7) imply the following contradiction and thus finish the proof of (3.71 i):

$$n(L)M(\frac{\omega_M(L)}{n(L)},6) \sim \int\limits_L \sim \int\limits_{K\cup L} \sim n(K \cup L)M(\frac{\omega_M(K \cup L)}{n(K \cup L)},6)$$
$$\leq \eta\, n(L)M(\frac{\omega_M(L)}{\eta\, n(L)},6) \leq \zeta\, n(L)M(\frac{\omega_M(L)}{n(L)},6)$$
for all sufficiently large n.

Next assume that $\omega_M(K) = \omega_M(J)$. Then $J = (J \setminus K) \cup K$ where $\omega_M(J \setminus K) = 0$. Hence $n \sim n(J) \leq n(J \setminus K) + n(K) = o(n) + n(K) \leq o(n) + n$ by (3.70) and (3.71 i). Thus $n(K) \sim n$ and the proof of (3.71 ii) is complete. Finally assume that $0 < \omega_M(K) < \omega_M(J)$. Then $L = J \setminus K$ also satisfies $0 < \omega_M(L) < \omega_M(J)$. Clearly,

$$(3.72)\quad n \sim n(J) \leq n(K) + n(L) \begin{cases} \leq n(J) + n(\operatorname{bd} K) + n(\operatorname{bd} L) \leq n + o(n), \\ \geq n(J) - n(\operatorname{bd} K) - n(\operatorname{bd} L) \geq n - o(n). \end{cases}$$

by (3.70), (3.71 i), (2.10) and since the Dirichlet-Voronoi cells $\{x \in M : \varrho_M(p,x) \leq \varrho_M(q,x) \text{ for all } q \in S_n\}$, $p \in S_n$, all are connected. If (3.71 iii) were not true, then (3.70) shows that either

$$(3.73)\quad n(K) \leq (1-\kappa)\frac{\omega_M(K)}{\omega_M(J)} n \text{ and thus } \frac{\omega_M(K)}{n(K)} - \frac{\omega_M(J)}{n} \geq \kappa \frac{\omega_M(K)}{n(K)}$$
for infinitely many n, where $0 < \kappa < 1$,

or the corresponding inequality with K replaced by L would be true. In the latter case exchange K and L in the following argument. We consider only n satisfying (3.73). From (3.6), (3.9), (3.10), (3.2), and (3.73) we obtain that

$$
\begin{aligned}
(3.74)\quad M(\frac{\omega_M(K)}{n(K)},6) &= M(\frac{\omega_M(J)}{n},6) + M_a(\frac{\omega_M(J)}{n},6)(\frac{\omega_M(K)}{n(K)} - \frac{\omega_M(J)}{n}) \\
&\quad + \frac{1}{2}M_{aa}(\frac{\omega_M(J)}{n} + \vartheta\,(\frac{\omega_M(K)}{n(K)} - \frac{\omega_M(J)}{n}),6)(\frac{\omega_M(K)}{n(K)} - \frac{\omega_M(J)}{n})^2 \\
&\geq M(\frac{\omega_M(J)}{n},6) + M_a(\frac{\omega_M(J)}{n},6)(\frac{\omega_M(K)}{n(K)} - \frac{\omega_M(J)}{n}) \\
&\quad + \text{const}\,\frac{g(\frac{\omega_M(J)}{n})}{\frac{\omega_M(K)}{n(K)}}(\frac{\omega_M(K)}{n(K)} - \frac{\omega_M(J)}{n})^2, \\
M(\frac{\omega_M(L)}{n(L)},6) &\geq M(\frac{\omega_M(J)}{n},6) + M_a(\frac{\omega_M(J)}{n},6)(\frac{\omega_M(L)}{n(L)} - \frac{\omega_M(J)}{n})
\end{aligned}
$$

for all sufficiently large n,

Then (2.1), (3.1), $J = K \cup L$, $K \cap L = \emptyset$, (2.10), (3.65), (3.74), (3.73), (3.72), (3.8), (3.9) and (3.2) yield a contradiction:

$$
\begin{aligned}
M(\frac{\omega_M(J)}{n},6) &\sim \frac{1}{n}\int\limits_J = \frac{1}{n}\int\limits_K + \frac{1}{n}\int\limits_L \sim \frac{n(K)}{n}M(\frac{\omega_M(K)}{n(K)},6) + \frac{n(L)}{n}M(\frac{\omega_M(L)}{n(L)},6) \\
&\geq \frac{n(K)+n(L)}{n}M(\frac{\omega_M(J)}{n},6) \\
&\quad + M_a(\frac{\omega_M(J)}{n},6)(\frac{\omega_M(K)+\omega_M(L)}{n} - \frac{n(K)+n(L)}{n^2}\omega_M(J)) \\
&\quad + \text{const}\, g(\frac{\omega_M(K)}{n})\frac{n(K)}{\omega_M(K)}\frac{\omega_M(K)^2}{n(K)^2} \\
&\geq \frac{n-o(n)}{n}M(\frac{\omega_M(J)}{n},6) + M_a(\frac{\omega_M(J)}{n},6)(\frac{1}{n} - \frac{n+o(n)}{n^2})\omega_M(J) \\
&\quad + \text{const}\, g(\frac{\omega_M(K)}{n})\frac{\omega_M(K)}{n},
\end{aligned}
$$

or

$$
o(1)g(\frac{\omega_M(J)}{n})\frac{\omega_M(J)}{n} \geq g(\frac{\omega_M(J)}{n})o(\frac{1}{n}) + g(\frac{\omega_M(J)}{n})\frac{\omega_M(K)}{n},
$$

or $o(1) \geq \text{const} > 0$ for all sufficiently large n. This concludes the proof of (3.71 iii) and thus of (3.71).

Now, in order to prove (3.63), we note that for any Jordan measurable set $K \subset J$ with $\omega_M(K) > 0$ proposition (3.71) implies that

$$
\#(K \cap S_n) \left\{ \begin{array}{l} \leq \#S_n(K) = n(K) \\ \geq \#S_n(K) - \#S_n(\operatorname{bd} K) = n(K) - n(\operatorname{bd} K) \end{array} \right\} \sim \frac{\omega_M(K)}{\omega_M(J)}n.
$$

3.4.2 In the second part we show the following proposition:

(3.75) *Let* $0 < \varepsilon \le 0.1$. *Then for all sufficiently large* n *we have: each* $p \in S_n$, *with a set of at most* εn *exceptions, is, up to* ε, *the center of a regular 6-gon with vertices* $p_1, \dots, p_6 \in S_n$ *of edge length* $\varrho_n = (2\omega_M(J)/\sqrt{3}n)^{1/2}$ *and* $\{x : \varrho_M(p,x) \le 1.1\,\varrho_n\} \cap S_n = \{p, p_1, \dots, p_6\}$.

In the following all distances in $\mathbb{E}^2$ will be with respect to the norm $q^{1/2}(\,.\,)$, respectively $q_l^{1/2}(\,.\,)$, see below.

First, some preparations are needed. In the proof of (3.75) the following consequence of proposition (3.29) (taking into account (3.62)) will be used.

(3.76) Let $0 < \varepsilon < 1$. Then for all sufficiently small $\delta > 0$ we have: let $q(\,.\,)$ be a positive definite quadratic form on $\mathbb{E}^2$ and $Q' \subset \mathbb{E}^2$ a unit square. Then each finite set $S' \subset \mathbb{E}^2$ for which $m = \#S'$ is sufficiently large and for which

$$\int\limits_{Q'} \min_{r \in S'}\{f(q^{1/2}(r-s))\}\,ds\,(\det q)^{1/2} \le (1+\delta)mM(\frac{|Q'|(\det q)^{1/2}}{m}, 6),$$

has the following property: each $r \in S'$, with a set of at most $(\varepsilon/3)m$ exceptions, is up to $\varepsilon/3$ the center of a regular 6-gon with vertices $r_1, \dots, r_6 \in S'$, edge length $\varrho = (2|Q'|(\det q)^{1/2}/\sqrt{3}m)^{1/2}$ and such that $\{s : q^{1/2}(r-s) \le 1.15\varrho\} \cap S' = \{r, r_1, \dots, r_6\}$.

Let $0 < \varepsilon \le 0.1$ and choose a corresponding $\delta > 0$ according to (3.76). Let $\lambda > 1$ and $\nu > 1$ correspond to it by the growth condition for f and M (see (2.8)), both so close to 1 that

(3.77) $f(\lambda t) \le \nu\, f(t)$ for all sufficiently small $t \ge 0$,
$M(\lambda\, a, 6) \le \nu\, M(a, 6)$ for all sufficiently small $a \ge 0$,
$\lambda^2\nu^2 \le 1+\delta$, $1.1\lambda \le \dfrac{1.15}{\lambda}$, $1+\varepsilon \le 1.1$,
$((1+\frac{\varepsilon}{3})\lambda - (1-\frac{\varepsilon}{3})\frac{1}{\lambda})(1+\frac{\varepsilon}{6})\lambda + (1+\frac{\varepsilon}{6})\lambda - \frac{1}{\lambda} \le \varepsilon.$

For this $\lambda > 1$ choose $p_l \in J$, U_l, V_l, $h_l =$ "$'$", $q_l(\,.\,)$, I_l, sets $Q_{li} \subset \operatorname{int} I_l \subset U_l$, $S_{nli} = S_n(Q_{li})$, $n_{li} = \#S_{nli}$, $l = 1, \dots, k$, $i = 1, \dots, i_l$, as in 2.4 such that in particular (2.11)-(2.14), (2.17 i,ii), (2.19 i,ii) hold, and, in addition, we have

(3.78) $\omega_M(J \setminus \bigcup\limits_{l,i} Q_{li}) < \dfrac{\varepsilon}{3}\omega_M(J)$,

(3.79) $n_{li} \sim \dfrac{\omega_M(Q_{li})}{\omega_M(J)} n$,

where (3.79) is an immediate consequence of (3.71).

Second,

(3.80) for all sufficiently large n hold

$$\int\limits_{Q'_{li}} \min_{r\in S'_{nli}} \{f(q_l^{1/2}(r-s))\}\, ds\, (\det q_l)^{1/2} \le (1+\delta) n_{li} M(\frac{|Q'_{li}|(\det q_l)1/2}{n_{li}}, 6).$$

We use (2.17 i) and (2.19 i) to see that $Q_{li}, S_{nli} \subset U_l$. Combining this with the definition of the integral in M, (2.12), (2.11), the fact that f is non-decreasing, (3.77), (3.65), (2.13), (3.7) and (3.77) shows that, for all sufficiently large n,

$$\begin{aligned}\int\limits_{Q'_{li}} \min_{r\in S'_{nli}} \{f(q_l^{1/2}(r-s))\}\, ds\, (\det q_l)^{1/2} &\le \lambda \int\limits_{Q_{li}} \min_{p\in S_{nli}} \{f(\lambda\, \varrho_M(p,x))\}\, d\omega_M(x)\\ &\le \lambda^2\nu\, n_{li} M(\frac{\omega_M(Q_{li})}{n_{li}}, 6) \le \lambda^2\nu\, n_{li} M(\frac{\lambda\, |Q'_{li}|(\det q_l)^{1/2}}{n_{li}}, 6)\\ &\le \lambda^2\nu^2\, n_{li} M(\frac{|Q'_{li}|(\det q_l)^{1/2}}{n_{li}}, 6) \le (1+\delta) n_{li} M(\frac{|Q'_{li}|(\det q_l)^{1/2}}{n_{li}}, 6).\end{aligned}$$

This completes the proof of (3.80).

Third,

(3.81) for all sufficiently large n hold: for all pairs l, i, each of the n_{li} points $r \in S'_{nli}$, with a set of at most $(2\varepsilon/3)n_{li}$ exceptions, is up to $\varepsilon/3$ the center of a regular 6-gon with vertices $r_1, \dots, r_6 \in S'_{nlii}$ of edge length ϱ'_{nli} where

$$(\frac{2|Q_{li}|(\det q_l)^{1/2}}{\sqrt{3} n_{li}})^{1/2} \le \varrho'_{nli} \le (1+\frac{\varepsilon}{6})(\frac{2|Q_{li}|(\det q_l)^{1/2}}{\sqrt{3} n_{li})^{1/2}})^{1/2},$$

$$\{s : q_l^{1/2}(r-s) \le 1.15\varrho'_{nli}\} \subset Q'_{li}\,,$$

$$\{s : q_l^{1/2}(r-s) \le 1.15\varrho'_{nli}\} \cap S'_{nli} = \{r, r_1, \dots, r_6\}.$$

The square Q'_{li} has edge length $1/k_l$, see (2.17 ii). Let $R'_{li} \subset Q'_{li}$ be a parallel strip of width 2ϑ along $\mathrm{bd}Q'_{li}$, where ϑ is so small that for $R_{li} = h_l^{-1}(R'_{li})$ we have $\omega_M(R_{li})/\omega_M(Q_{li}) < \varepsilon/6$. Then (3.71 iii) applied to Q_{li} and R_{li} implies that

(3.82) $n(R_{li}) < \dfrac{\varepsilon}{6} n_{li}$ for all sufficiently large n.

By (2.9) and (2.10),

(3.83) each point of S'_{nli} has distance $< \vartheta$ from Q'_{nli} for all sufficiently large n.

Let Q' be a square of edge length 1 which is obtained by fitting together edge-to-edge Q'_{nli} and suitable k^2-1 translates of it. Let S' be the union of S'_{nli} and its corresponding translates, and similarly for $S(R)'$ and R'. Since $S'_{nli} \setminus S_n(R_{li})' \subset \mathrm{int}Q'_{li}$, (3.82) yields,

(3.84) $(1-\dfrac{\varepsilon}{6})k_l^2 n_{li} \le m = \#S' \le k_l^2 n_{li}$ and $\#S(R)' \le \dfrac{\varepsilon}{6} k_l^2 n_{li}$
for all sufficiently large n.

The definition of Q' and S' and propositions (3.80), (3.84) and (2.7) together imply that

$$\int_{Q'} \min_{r\in S'}\{f(q_l^{1/2}(r-s))\}\,ds\,(\det q_l)^{1/2} \leq k_l^2 \int_{Q'_{li}} \min_{r\in S'_{nli}}\{f(q_l^{1/2}(r-s))\}\,ds\,(\det q_l)^{1/2}$$
$$\leq (1+\delta)k_l^2 n_{li} M(\frac{k_l^2|Q'_{li}|(\det q_l)^{1/2}}{k_l^2 n_{li}}, 6) \leq (1+\delta)mM(\frac{|Q'|(\det q_l)^{1/2}}{m}, 6)$$

for all sufficiently large n.

This permits us to apply (3.76). Thus, using (3.84) and the inequality $(1-\varepsilon/6)^{-1/2} \leq 1+\varepsilon/6$, we see that

(3.85) for all sufficiently large n: each of the m points $r \in S'$, with a set of at most $(\varepsilon/3)m$ exceptions, is up to $\varepsilon/3$ the center of a regular 6-gon with vertices $r_1,\dots,r_6 \in S'$, edge length $\varrho'_{nli} = (2|Q'|(\det q_l)^{1/2}/\sqrt{3}m)^{1/2}$ and such that $\{s : q_l^{1/2}(r-s) \leq 1.15\varrho'_{nli}\} \cap S' = \{r, r_1, \dots, r_6\}$. For ϱ'_{nli} the following inequalities hold:

$$(\frac{2|Q_{li}|(\det q_l)^{1/2}}{\sqrt{3}n_{li}})^{1/2} \leq \varrho'_{nli} \leq (1+\frac{\varepsilon}{6})(\frac{2|Q_{li}|(\det q_l)^{1/2}}{\sqrt{3}n_{li}})^{1/2},$$

$$1.15\varrho'_{nli}(1+\varepsilon/3) < \vartheta.$$

¿From the $m \geq (1-\varepsilon/6)k_l^2 n_{li}$ points of S' cancel first the at most $(2\varepsilon/6)m(\leq (2\varepsilon/6)k_l^2 n_{li})$ exceptional points and, second, the at most $(\varepsilon/6)k_l^2 n_{li}$ points of $S'(R')$, compare (3.84). There remain at least $(1-2\varepsilon/3)k_l^2 n_{li}$ points which consist of k_l^2 families. The first such family is contained in $Q'_{li} \setminus R'_{li}$. It has at least $(1-2\varepsilon/3)n_{li}$ points r, each of which has the properties described in (3.85) and since $1.15\varrho'_{nli}(1+\varepsilon/3) < \vartheta$ also the 6 neighbors of r are contained in Q'_{li} and by (3.83) also in S'_{nli}. The proof of (3.81) is complete.

Applying h_l^{-1} we will show , fourth, that (3.81) implies the following:

(3.86) for all sufficiently large n: for all pairs l, i each of the n_{li} points $p \in S_{nli}$, with a set of at most $(2\varepsilon/3)n_{li}$ exceptions, is up to ε the center of a regular 6-gon with vertices $p_1,\dots,p_6 \in S_{nli}$ of edge length $\varrho_n = (2\omega_M(J)/\sqrt{3}n)^{1/2}$ and $\{x : \varrho_M(p,x) \leq 1.1\varrho_n\} \cap S_n = \{p, p_1, \dots, p_6\}$.

Let $p \in S_{nli}$ be such that $r = p' \in S'_{nli}$ is one of the at least $(1-2\varepsilon/3)n_{li}$ points as described in (3.81). Let $r_1,\dots,r_6 \in S'_{nli}$ be its neighbors and let $p_j = h_l^{-1}(r_j)$. Then (3.81), (2.11), and (3.77) imply that

(3.87) for all sufficiently large n:

(i) $(1-\frac{\varepsilon}{3})\frac{1}{\lambda}\varrho'_{nli} \leq \frac{1}{\lambda}q_l^{1/2}(r-r_j)$
$\leq \varrho_M(p,p_j) \leq \lambda q_l^{1/2}(r-r_j) \leq (1+\frac{\varepsilon}{3})\lambda\varrho'_{nli} \leq 1.1\lambda\varrho'_{nli}$
and similarly for $\varrho_M(p_j,p_{j+1})$,

(ii) $\{x:\ \varrho_M(p,x)\le 1.1\lambda\varrho'_{nli}\}\subset\{x:\ \varrho_M(p,x)\le \dfrac{1.15}{\lambda}\varrho'_{nli}\}$
$\subset h^{-1}(\{s: q_l^{1/2}(r-s)\le 1.15\varrho'_{nli}\})\subset Q_{li}$,

(iii) $\{x:\ \varrho_M(p,x)\le 1.1\lambda\varrho'_{nli}\}\cap S_n=\{p,p_1,\dots,p_6\}$.

We estimate ϱ_n, defined in (3.86). From (3.85), (2.13) and (3.79) it follows that

$$\varrho'_{nli}\begin{cases}\le (1+\frac{\varepsilon}{6})(\frac{2|Q_{li}|(\det q_l)^{1/2}}{\sqrt{3}n_{li}})^{1/2} & \le (1+\frac{\varepsilon}{6})\lambda^{1/2}(\frac{2\omega_M(Q_{li})}{\sqrt{3}n_{li}})^{1/2}\\ \ge (\frac{2|Q_{li}|(\det q_l)^{1/2}}{\sqrt{3}n_{li}})^{1/2} & \ge \lambda^{-1/2}(\frac{2\omega_M(Q_{li})}{\sqrt{3}n_{li}})^{1/2}\end{cases}$$
$$\begin{cases}\le (1+\frac{\varepsilon}{6})\lambda\varrho_n,\\ \ge \lambda^{-1}\varrho_n,\end{cases}$$

or,

(3.88) $\dfrac{1}{\lambda}\varrho_n\le\varrho'_{nli}\le(1+\dfrac{\varepsilon}{6})\lambda\varrho_n$ for all sufficiently large n.

Thus we can conclude from (3.87 i) and (3.77) that

(3.89)
$$\begin{aligned}|\varrho_M(p,p_j)-\varrho_n| &\le |\varrho_M(p,p_j)-\varrho'_{nli}|+|\varrho'_{nli}-\varrho_n|\\ &\le ((1+\frac{\varepsilon}{3})\lambda-(1-\frac{\varepsilon}{3})\frac{1}{\lambda})\varrho'_{nli}+((1+\frac{\varepsilon}{6})\lambda-\frac{1}{\lambda})\varrho_n\\ &\le (((1+\frac{\varepsilon}{3})\lambda-(1-\frac{\varepsilon}{3})\frac{1}{\lambda})(1+\frac{\varepsilon}{6})\lambda+(1+\frac{\varepsilon}{6})\lambda-\frac{1}{\lambda})\varrho_n\le\varepsilon\varrho_n,\\ |\varrho_M(p_j,p_{j+1})-\varrho_n| &\le\dots\le\varepsilon\varrho_n\end{aligned}$$
for all sufficiently large n.

Hence, up to ε, p is the center of a regular 6-gon in S_n with vertices $p_1,\dots,p_6\in S_n$ of edge length ϱ_n. Finally, from (3.87 ii,iii), (3.88) and (3.89) we obtain $\{x:\varrho_M(p,x)\le 1.1\varrho_n\}\cap S_n=\{p,p_1,\dots,p_6\}$. The proof of (3.86) is complete.

Fifth, we show that

(3.90) for all sufficiently large n each point $p\in S_n$, with a set of at most εn exceptions, has the properties described in (3.86).

By (3.78)and (3.71) $n(J\setminus\bigcup Q_{li})<(\varepsilon/3)n$ and by (2.19 ii) $n_{11}+\dots+n_{ki_k}\le n$ for all sufficiently large n. Cancel from S_n all points of $S_n(J\setminus\bigcup Q_{li})$ and for each pair l,i the $(2\varepsilon/3)n_{li}$ points which do not have the properties described in (3.86). The remaining $(1-\varepsilon)n$ points of S_n clearly have the properties described in (3.86).

(3.86) und (3.90) together yield (3.75).

3.4.3 We finally show that (3.75) implies that

(3.93) *S_n is asymptotically a regular hexagonal pattern of edge length* $(\dfrac{2\omega_M(J)}{\sqrt{3}n})^{1/2}$.

Let $\varepsilon_k = 1/k$ for $k = 10, 11, \dots$, and choose $n_{10} < n_{11} < \dots$ such that (3.75) with $\varepsilon = \varepsilon_k$ holds for $n \geq n_k$. Define $g(n) = n$, $h(n) = 1$ for $1 \leq n < n_{10}$ and $g(n) = n/k$, $h(n) = 1/k$ for $n_k \leq n < n_{k+1}$. Then (3.75) says that each $p \in S_n$, with a set of at most $g(n)$ exceptions, is up to $h(n)$ the center of a regular 6-gon with vertices $p_1, \dots, p_6 \in S_n$ of edge length $(2\omega_M(J)/\sqrt{3}n)^{1/2}$ and $\{x : \varrho_M(p,x) \leq 1.1\varrho_n\} \cap S_n = \{p, p_1, \dots, p_6\}$. Since $g(n) = o(n)$ and $h(n) = o(1)$, the proof of (3.91) is complete.

3.4.4 Propositions (3.63) and (3.91) together yield Theorem 2.□

4 Volume approximation of convex bodies

4.1 The result

If the boundary of a convex body C of class $\mathcal{C}^2$ with Gauss curvature $\kappa_C > 0$ is endowed with a Riemannian metric, this metric induces, in any tangent plane of $\mathrm{bd}\, C$, an associated Euclidean metric. When considering tangent planes, we always mean this metric. For the Riemannian metric of the first fundamental form of equi-affine differential geometry see [26], [28]. The ordinary surface area measure on $\mathrm{bd}\, C$ is denoted by ω.

Theorem 3. *Let C be a convex body in $\mathbb{E}^3$ of class $\mathcal{C}^2$ with $\kappa_C > 0$. Let $\mathrm{bd}\, C$ be endowed with the Riemannian metric of the first fundamental form of equi-affine differential geometry. Then the following statements hold:*

(i) $\delta^V(C, \mathcal{P}^c_{(n)}) \sim \dfrac{5\,\omega^2_{EA}(\mathrm{bd}\, C)}{36\sqrt{3}\,n}$ *as* $n \to \infty$, *where* $\omega_{EA}(\mathrm{bd}\, C) = \int\limits_{\mathrm{bd}\, C} \kappa_C^{1/4}(x)d\omega(x)$.

(ii) *Let the convex polytope $P_n \in \mathcal{P}^c_{(n)}$ be best approximating of C with respect to the symmetric difference metric δ^V for $n = 4, 5, \dots$. Then, as $n \to \infty$, the sets $C \cap \mathrm{bd}\, P_n$ are uniformly distributed in $\mathrm{bd}\, C$ and the facets F of P_n are asymptotically regular hexagons with centers at the points of $C \cap F$ and edge length*

$$\left(\frac{2\,\omega_{EA}(\mathrm{bd}\, C)}{3\sqrt{3}\,n}\right)^{1/2}.$$

(i) is simply a restatement of the result in Gruber [14]. (i), as well as the remark about the uniform distribution in (ii), extends to all dimensions, see Gruber [15] and Glasauer and Gruber [13]. The result (ii) complements analogous results of Gruber [18] for the Hausdorff metric, the Banach-Mazur distance and a measure of distance due to Schneider. A result corresponding to Theorem 3 for the mean width approximation of C by inscribed convex polytopes can be proved along the lines of the proof of Theorem 3. Karoly Brczky, Jr. has informed us of a result similar to Theorem 3 for general polytopes instead of circumscribed ones.

4.2 The proof

4.2.1 We consider two different ways in which $\mathrm{bd}\, C$ can be thought of as a Riemannian 2-manifold. First we can make use of the ordinary second differential form and, for the second, we use the first differential form of equi-affine differential geometry. For the latter see [26],[28]. Let $\varrho_{II}, \varrho_{EA}, \omega_{II}, \omega_{EA}$ denote the corresponding Riemannian metrics and area measures on $\mathrm{bd}\, C$. If $J \subset \mathrm{bd}\, C$ is Jordan measurable with respect to ω, then it is also Jordan measurable with respect to ω_{EA} and vice versa.

The following propositions are either well known or easy to show using the definitions; $\|.\|$ is the ordinary Euclidean norm.

(4.1) $d\omega_{II}(x) = \kappa_C^{1/4}(x) d\omega_{EA}(x)$ for $x \in \mathrm{bd}\, C$.

(4.2) For each $\lambda > 1$ there is $\delta > 0$ such that
$$\frac{1}{\lambda} \le \frac{\varrho_{II}(x,y)}{\varrho_{EA}(x,y)\kappa_C^{1/8}(x)} \le \lambda \text{ for } x, y \in \mathrm{bd}\, C, \|x-y\| \le \delta.$$

(4.3) For each $\lambda > 1$ there is $\delta > 0$ such that
$$\frac{1}{2\lambda}\varrho_{II}^2(x,y) \le \mathrm{dist}(x, H_C(y)) \le \frac{\lambda}{2}\varrho_{II}^2(x,y) \text{ for } x, y \in \mathrm{bd}\, C, \|x-y\| \le \delta,$$
where $\mathrm{dist}(x, H_C(Y))$ is the distance between x and the support plane $H_C(y)$ of C at y, measured along the normal of $\mathrm{bd}\, C$ through x.

(4.4) For each $\lambda > 1$ there is $\delta > 0$ such that
$$\frac{1}{\lambda}\int_U \mathrm{dist}(x, H_C(x))d\omega(x) \le V(U, H_C(y)) \le \lambda \int_U \mathrm{dist}(x, H_C(y))d\omega(x)$$
for all $y \in \mathrm{bd}\, C$ and all Jordan measurable neighborhoods U of y of diameter $\le \delta$, where by $V(U, H_C(y))$ we mean the volume of the set of all points on a normal through a point of U between U and $H_C(y)$.

4.2.2 For $n = 4, 5, \ldots$, choose $S_n \subset \mathrm{bd}\, C$ with $\# S_n = n$ such that

(4.5)
$$\frac{1}{2}\int_{\mathrm{bd}\, C} \min_{p \in S_n}\{\varrho_{EA}^2(p,x)\} d\omega_{EA}(x) \sim \frac{n}{2}\int_{H_n} \|x\|^2 dx = \frac{5\,\omega_{EA}^2(\mathrm{bd}\, C)}{36\sqrt{3}\, n} \text{ as } n \to \infty.$$

Since the right side in (4.5) tends to 0, the sets S_n become increasingly dense in $\mathrm{bd}\, C$ if n is sufficiently large. Thus the intersection of the supporting halfspaces of C at the points of S_n is a convex polytope Q_n, say, for all sufficiently lare n. The maximum diameter of the facets of Q_n tends to 0. Thus (4.5), (4.1)-(4.4) and the fact that P_n is best approximating imply that

(4.6)
$$\frac{5\,\omega_{EA}^2(\mathrm{bd}\, C)}{36\sqrt{3}\, n} \sim \frac{1}{2}\int_{\mathrm{bd}\, C} \min_{p \in S_n}\{\varrho_{EA}^2(p,x)\} d\omega_{EA}(x)$$
$$\sim \frac{1}{2}\int_{\mathrm{bd}\, C} \min_{p \in S_n}\{\varrho_{EA}^2(p,x)\}\kappa_C^{1/4} d\omega(x) \sim \frac{1}{2}\int_{\mathrm{bd}\, C} \min_{p \in S_n}\{\varrho_{II}^2(p,x)\} d\omega(x)$$
$$\sim \delta^V(C, Q_n) \ge \delta^V(C, P_n)) \text{ as } n \to \infty.$$

Since $\delta^V(C, P_n) \to 0$, the maximum diameter of the facets of P_n tends to 0. Thus (4.4)-(4.1) and Theorem 1 imply that

$$
\begin{aligned}
(4.7)\quad \delta^V(C, P_n)) &\sim \frac{1}{2} \int\limits_{\mathrm{bd}\, C} \min_{p \in C \cap \mathrm{bd}\, P_n} \{\varrho_{II}^2(p, x)\} d\omega(x) \\
&\sim \frac{1}{2} \int\limits_{\mathrm{bd}\, C} \min_{p \in C \cap \mathrm{bd}\, P_n} \{\varrho_{EA}^2(p, x)\} \kappa_C^{1/4} d\omega(x) \\
&= \frac{1}{2} \int\limits_{\mathrm{bd}\, C} \min_{p \in C \cap \mathrm{bd}\, P_n} \{\varrho_{EA}^2(p, x)\} d\omega_{EA}(x) \gtrsim \frac{5\, \omega_{EA}^2(\mathrm{bd}\, C)}{36\sqrt{3}\, n} \text{ as } n \to \infty.
\end{aligned}
$$

Propositions (4.6) and (4.7) together imply Theorem 3(i) and, in addition,

$$
\frac{1}{2} \int\limits_{\mathrm{bd}\, C} \min_{p \in C \cap \mathrm{bd}\, P_n} \{\varrho_{EA}^2(p, x)\} d\omega_{EA}(x) \sim \frac{5\, \omega_{EA}^2(\mathrm{bd}\, C)}{36\sqrt{3}\, n} \text{ as } n \to \infty.
$$

Theorem 2 then shows that $C \cap \mathrm{bd}\, P_n$ is uniformly distributed in $\mathrm{bd}\, C$ and asymptotically forms a regular hexagonal pattern in $\mathrm{bd}\, C$ of edge length equal to $(2\, \omega_{EA}(\mathrm{bd}\, C)/3\sqrt{3}\, n)^{1/2}$ with respect to the Riemannian metric of the first fundamental form of equi-affine differential geometry. Elementary arguments now imply that each $p \in C \cap \mathrm{bd}\, P_n$, with a set of at most $o(n)$ exceptions, is up to $o(1)$ the center of a regular hexagonal facet of P_n of edge length $(2\, \omega_{EA}(\mathrm{bd}\, C)/\sqrt{3}\, n)^{1/2}$ with respect to the associated Euclidean metric in the plane containing the facet. This concludes the proof of Theorem 3(ii).□

5 The isoperimetric problem for polytopes in Minkowski spaces

5.1 The result

Theorem 4. *Let I be an isoperimetrix in $\mathbb{E}^3$ of class $\mathcal{C}^2$ with positive Gauss curvature. Let $\mathrm{bd}\, I$ be endowed with the Riemannian metric of the first fundamental form of equi-affine differential geometry. For $n = 4, 5, \ldots$, let P_n be a convex polytope in $\mathbb{E}^3$ with n facets and minimum isoperimetric quotient $S_I(P_n)^3/V(P_n)^2$. Then the following are true:*

(i) $\dfrac{S_I(P_n)^3}{V(P_n)^2} \sim 27\, V(I) + \dfrac{15\, \omega_{EA}^2(\mathrm{bd}\, I)}{4\sqrt{3}\, n}$ *as* $n \to \infty$.

(ii) *By replacing P_n by a homothetic copy, if necessary, we may assume that P_n is circumscribed to I for $n = 4, 5, \ldots$ Then, as $n \to \infty$, the set $I \cap \mathrm{bd}\, P_n$ is uniformly distributed in $\mathrm{bd}\, I$ and the facets F of P_n are asymptotically regular hexagons with center $I \cap F$ and edge length*

$$
\left(\frac{2\, \omega_{EA}(\mathrm{bd}\, I)}{3\sqrt{3}\, n}\right)^{1/2}.
$$

(i) and the statement in (ii) about the uniform distribution of $I \cap P_n$ in $\mathrm{bd}\, I$ extend to all dimensions. This can be proved using results of Gruber [15] and Glasauer and Gruber [13].

5.2 The proof

Diskant's result [5] allows us to assume that P_n is circumscribed to I. Using the definitions of $S_I(\,.\,)$ and $V(\,.\,)$, we have, as in the Euclidean case, $S_I(P_n) = 3\,V(P_n)$. Thus

$$\frac{S_I(P_n)^3}{V(P_n)^2} = 27\,V(P_n). \tag{5.1}$$

Hence, since P_n minimizes the isoperimetric quotient, it must have minimum volume among all convex polytopes with at most n facets and which are circumscribed to I. Equality (5.1) together with Theorem 3 thus yields Theorem 4. □

6 Optimal numerical integration formulas

In this section we assume that M is is of class $\mathcal{C}^3$ with metric tensorfield of class $\mathcal{C}^2$, compare section 2.

6.1 The result

A *modulus of continuity* is a function $f : [0,\infty) \to [0,\infty)$ such that $f(s+t) \leq f(s) + f(t)$ for all $s, t \geq 0$. Recall the definitions of nodes, weights, errors and minimum errors from section 1.

Theorem 5. *Let $f : [0,\infty) \to [0,\infty)$ be a modulus of continuity which satisfies the growth condition, let J be a Jordan measurable set in M with $\omega_M(J) > 0$ and let $\mathcal{H}^f = \{g : J \to I\!R : |g(x) - g(y)| \leq f(\varrho_M(x,y))$ for all $x, y \in J\}$. Then the following statements hold:*

(i) $$E(\mathcal{H}^f, n) = \inf_{N_n, W_n} \{E(\mathcal{H}^f, N_n, W_n)\} \sim n \int_{H_n} f(\|x\|)dx \text{ as } n \to \infty.$$

(ii) *Let (N_n) and (W_n) be sequences of nodes and weights such that*

$$E(\mathcal{H}^f, N_n, W_n) \sim E(\mathcal{H}^f, n) \text{ as } n \to \infty.$$

Then N_n is uniformly distributed and forms asymptotically a regular hexagonal pattern in J of edge length

$$\Big(\frac{2\,\omega_M(J)}{\sqrt{3}\,n}\Big)^{1/2}.$$

Note that from the proof of Theorem 1 it follows that, in principle, it is possible to construct sequences (N_n) and (W_n) of nodes and weights such that $E(\mathcal{H}^f, N_n, W_n) \sim E(\mathcal{H}^f, n)$. The statement about the uniform distribution in (ii) can be extended to all dimensions.

6.2 The proof

In essence, our proof of part (i) follows Babenko [1] but with additional difficulties.

6.2.1 First some preparations:

(6.1) For each point in M there is a (geodesically) convex open neighborhood N such that the following hold: for any $p, q \in N, p \neq q$, the *bisector* $B_N(p, q) = \{x \in N : \varrho_M(p, x) = \varrho_M(q, x)\}$ is Jordan measurable of measure 0.

Since M is of class $\mathcal{C}^3$ with metric tensorfield of class $\mathcal{C}^2$, each point of M has an open neighborhood N in M with the following properties: (i) N is contained in a Jordan measurable neighborhood. (ii) Any two points $x, y \in N$ are connected by a unique geodesic segment of length $\varrho_M(x, y)$ in N; thus, in particular, N is convex. (iii) For each $p \in N$ the geodesics starting at p cover $N \setminus \{p\}$ schlicht. (iv) For each $p \in N$ there is a diffeomorphism $D_p : N \setminus \{p\} \to [0, 2\pi) \times (0, +\infty)$ such that $D_p(N \setminus \{p\}) \supset [0, 2\pi) \times (0, a)$ for a suitable $a > 0$ and for each geodesic G starting at p, the image $D_p(G \cap (N \setminus \{p\}))$ is a vertical line segment of the form $\{b\} \times (0, c)$. See Brauner [4] and Kobayashi and Nomizu [25].

Now, let $p, q \in N, p \neq q$. By (ii), each geodesic starting at p contains at most one point of the bisector $B_N(p, q)$ and the bisector obviously is closed in $N \setminus \{p\}$. Thus $D_p(B_N(p, q))$ is closed in $D_p(N \setminus \{p\})$ and of Lebesgue measure 0 by Fubini's theorem. Thus $D_p(B_N(p, q))$ is Jordan measurable and of measure 0. The latter implies that $B_N(p, q)$ is Jordan measurable in N and of measure 0, concluding the proof of (6.1). For an alternative proof of (6.1) use our assumption about M to show that M is an Alexandrov space and the fact that in Alexandrov spaces bisectors have measure 0. The latter is a consequence of a result of Shiohama and Tanaka [30].

(6.2) Let $N_n = \{p_1, \ldots, p_n\} \subset J$. If N_n is sufficiently dense in the interior of J, $\mathrm{int}J$, then the Dirichlet-Voronoi cells

$$D_i = D_M(N_n, p_i) = \{x \in J : \varrho_M(p_i, x) \leq \varrho_M(p_j, x) \, \text{for each} \, p_j \in N_n\}$$

are Jordan measurable sets which tile J.

Consider a finite cover of $\mathrm{cl}\, J$ by neighborhoods as described in (6.1). Then, if N_n is sufficiently dense in $\mathrm{int}J$, each D_i and all points $p_j \in N_n$ which are needed to define it, are contained in one of these neighborhoods, say N. The boundary $\mathrm{bd}\, D_i$ then consists of subsets of bisectors $B_N(p_i, p_j)$ and of subsets of $\mathrm{bd}\, J$. By (6.1) and since J is Jordan measurable, it follows that each D_i has boundary of measure 0 and thus is Jordan measurable. In addition, the overlap with any other D_j (is empty or) has measure 0. The proof of (6.2) is complete.

(6.3) Let $N_n = \{p_1, \dots, p_n\} \subset J$ be sufficiently dense in int$J$ and let $W_n = \{w_1, \dots, w_n\} \subset I\!R$. Then

$$E(\mathcal{H}^f, N_n, W_n) \geq E(\mathcal{H}^f, N_n, \overline{W}_n) = \int_J \min_{p \in N_n} \{f(\varrho_M(p, x))\} \, d\omega_M(x),$$

where $\overline{W}_n = (\overline{w}_1, \dots, \overline{w}_n)$ with $\overline{w}_i = \omega_M(D_i)$.

We assume that N_n is so dense in int$J$, that (6.2) holds. Then the definition of $E(\mathcal{H}^f, N_n, \overline{W}_n)$, proposition (6.2), the fact that the function $\min\{f(\varrho_M(p_i, x)) : p_i \in N_n\}$ is in the class $\mathcal{H}^f$, is 0 at the points p_i, and the definition of $E(\mathcal{H}^f, N_n, W_n)$ together imply that

(6.4)
$$\begin{aligned} E(\mathcal{H}^f, N_n, \overline{W}_n) &= \sup_{g \in \mathcal{H}^f} \{ | \int_J g(x) \, d\omega_M(x) - \sum_{i=1}^n \overline{w}_i g(p_i) | \} \\ &\leq \sup_{g \in \mathcal{H}^f} \{ \sum_{i=1}^n \int_{D_i} |g(x) - g(p_i)| \, d\omega_M(x) \} \leq \sum_{i=1}^n \int_{D_i} f(\varrho_M(p_i, x)) \, d\omega_M(x) \\ &= \int_J \min_{p_i \in N_n} \{f(\varrho_M(p_i, x))\} \, d\omega_M(x) \leq E(\mathcal{H}^f, N_n, W_n). \end{aligned}$$

This confirms the inequality in (6.3). Putting $W_n = \overline{W}_n$ in (6.4), equality holds throughout, which yields the equality in (6.3).

6.2.2 For the proof of Theorem 5(i) note that $E(\mathcal{H}^f, n) \to 0$ as $n \to \infty$. Thus for all sufficiently large n it is sufficient to consider in the definition of $E(\mathcal{H}^f, n)$ only such sets of nodes N_n which are sufficiently dense in int$J$ in the sense of proposition (6.3). Then

$$\begin{aligned} E(\mathcal{H}^f, n) &= \inf_{N_n, W_n} E(\mathcal{H}^f, N_n, W_n)) = \inf_{N_n} E(\mathcal{H}^f, N_n, \overline{W}_n) \\ &= \inf_{N_n} \{ \int_J \min_{p \in N_n} \{f(\varrho_M(p, x))\} \, d\omega_M(x) \} \text{ for all sufficiently large } n. \end{aligned}$$

Now apply Theorem 1.

6.2.3 For the proof of Theorem 5(ii) the fact that $E(\mathcal{H}^f, n) \to 0$ as $n \to \infty$ and the assumption of part (ii) imply that N_n is sufficiently dense in int$J$ in the sense of (6.3) for all sufficiently large n. Proposition (6.3) and the definition of $E(\mathcal{H}^f, n)$ then show that

$$\begin{aligned} E(\mathcal{H}^f, n) &\sim E(\mathcal{H}^f, N_n, W_n) \geq E(\mathcal{H}^f, N_n, \overline{W}_n) \\ &= \int_J \min_{p \in N_n} \{f(\varrho_M(p, x))\} \, d\omega_M(x) = E(\mathcal{H}^f, N_n, \overline{W}_n) \geq E(\mathcal{H}^f, n) \end{aligned}$$

for all sufficiently large n. This, in turn, implies that,

$$\int_J \min_{p \in N_n} \{f(\varrho_M(p, x))\} \, d\omega_M(x) \sim E(\mathcal{H}^f, n).$$

Combining this with (i) and Theorem 2 yields (ii) and thus concludes the proof of Theorem 5.□

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