A short analytic proof of Fejes Tóth's theorem on sums of moments

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Summary. This article contains a simple analytic proof of the theorem of L. Fejes Tóth on sums of moments using a bare minimum of geometric arguments.
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1. Introduction and Statement of the Result

To the more general results of discrete geometry belongs the following *theorem of L. Fejes Tóth on sums of moments*, where $\|\cdot\|$ and $|\cdot|$ denote the Euclidean norm and the ordinary area measure in Euclidean 2-space $\mathbb{E}^2$; see [**?**].

Theorem. *Let $f : [0, +\infty) \to \mathbb{R}$ be non-decreasing and let H be a convex $3, 4, 5,$ or 6-gon in $\mathbb{E}^2$. Then, for any set P of n points in $\mathbb{E}^2$,*

$$S = \int\limits_H \min\{f(\|x-p\|) : p \in P\} dx \geq n \int\limits_{H_n} f(\|x\|) dx, \tag{1}$$

where H_n is a regular hexagon in $\mathbb{E}^2$ of area $|H|/n$ and center at the origin o.

The great importance of this modest-looking result is due to its applications ranging from packing and covering problems for solid circles, problems of optimal location, errors of quantization of data and Gauss channels, the isoperimetric problem for convex polytopes in $\mathbb{E}^3$, and the optimal choice of nodes in numerical integration formulae to asymptotically best approximation of convex bodies in $\mathbb{E}^3$. See [**?**] for references. The theorem indicates that for certain geometric or other problems the regular hexagonal configurations are, at least, close to optimal. Considering this, the question arises whether in such situations optimal or close to optimal configurations are almost regular hexagonal. As expected, the — non-trivial — answer is yes and follows from a stability counterpart of Fejes Tóth's theorem for functions of the form $f(t) = t^a$ of Gruber [**?**]. This stability result can be extended to more general classes of functions but not to all functions f.

L. Fejes Tóth proved his estimate first for the 2-sphere and only then for $\mathbb{E}^2$, see [?, ?]. Alternative proofs, in some cases for surfaces of constant curvature, are due to L. Fejes Tóth [?], Imre [?], G. Fejes Tóth [?], and Florian [?]. An extension to Jordan measurable sets on 2-dimensional Riemannian manifolds can be obtained along the lines of [?]. All these proofs, in essence, are geometric.

It is the aim of this article to present an elementary analytic proof of the theorem on sums of moments which uses only a minimum of geometric tools. One motive for this proof is the fact that the analytic arguments can be refined to yield a corresponding stability result which will be published elsewhere. G. Fejes Tóth has announced a different proof.

2. Proof of the Theorem

It is sufficient to prove the Theorem for functions f with $f(0) = 0$ and positive continuous derivative on $(0, +\infty)$. Let $P = \{p_1, \ldots, p_n\}$ be a set of n points in $\mathbb{E}^2$. By replacing any point of P which is not in H by its closest point in H, the integral S on the left hand side of (1) is decreased. Thus we may suppose that P is contained in H. Since f is non-decreasing, S can be written as a sum of moments in the form

(2) $$S = \sum_{i=1}^{n} \int_{D_i} f(\|x - p_i\|)dx,$$

where $D_i = \{x \in H : \|x - p_i\| \leq \|x - p\| \text{ for each } p \in P\}$ for $i = 1, \ldots, n$.

D_i is the intersection of H with the Dirichlet–Voronoi cell of p_i with respect to P. It is a convex polygon of area a_i with v_i vertices, say. The so-called *moment lemma of L. Fejes Tóth* [?], p. 198, shows that

(3) $$\int_{D_i} f(\|x - p_i\|)dx \geq \int_{R_i} f(\|x\|)dx = M(a_i, v_i), \text{ say,}$$

where R_i is a regular v_i-gon with center o, area a_i, and v_i vertices.

Let g be defined by $g(r^2) = f(r)$ for $r \geq 0$. Then $g(0) = 0$ and g has positive continuous derivative on $(0, +\infty)$. Let G be such that $G(0) = 0$ and $G' = g$. Finally, let $h(a, v) = a/v \tan(\pi/v)$ for $a > 0, v \geq 3$. Clearly,

(4) if R is a regular polygon with center o, area a, and v vertices, then $h^{1/2}$ is its inradius, and

$$M(a, v) = \int_R f(\|x\|)dx = 2v \int_0^{\frac{\pi}{v}} \int_0^{\frac{h^{1/2}}{\cos\psi}} g(r^2) r\,dr\,d\psi = v \int_0^{\frac{\pi}{v}} G\left(\frac{h}{\cos^2\psi}\right)d\psi.$$

Define $M(a,v)$ for $a>0, v\geq 3$ by the latter integral.

After these preparations the main step of the proof of the Theorem is to show that the moment

(5) $\quad M(a,v)$ is convex for $a>0, v\geq 3$.

Let

$$I=\int_0^{\frac{\pi}{v}} g(\frac{h}{\cos^2\psi})\frac{d\psi}{\cos^2\psi},\ J=\int_0^{\frac{\pi}{v}} g'(\frac{h}{\cos^2\psi})\frac{d\psi}{\cos^4\psi},\ K=g(\frac{h}{\cos^2\frac{\pi}{v}}).$$

Elementary calculus yields the following:

$$\begin{aligned} M_{aa}&=vh_a^2J\quad (>0),\ M_{av}=(h_a+vh_{av})I+vh_ah_vJ-\frac{\pi h_a}{v\cos^2\frac{\pi}{v}}K,\\ M_{vv}&=(2h_v+vh_{vv})I+vh_v^2J+\frac{2\pi a}{v\cos^2\frac{\pi}{v}}(\frac{\pi}{v^3}-h_{av})K. \end{aligned}$$

Noting that

$$\begin{aligned} 2h_v^2-hh_{vv}&=\frac{2\pi^2a^2}{v^6\sin^2\frac{\pi}{v}},\ h+vh_v=\frac{\pi a}{v^2\sin^2\frac{\pi}{v}},\\ K-\frac{\cos\frac{\pi}{v}}{\sin\frac{\pi}{v}}I&=\frac{2a\cos^2\frac{\pi}{v}}{v\sin^2\frac{\pi}{v}}\int_0^{\frac{\pi}{v}} g'(\frac{h}{\cos^2\psi})\frac{\sin^2\psi}{\cos^4\psi}d\psi\quad (>0), \end{aligned}$$

a lengthy calculation then shows that

$$\begin{aligned} &M_{aa}M_{vv}-M_{av}^2\\ &=-vh_a(2h_vh_{av}-h_ah_{vv})IJ+\frac{2\pi^2ah_a^2}{v^3\cos^2\frac{\pi}{v}}JK-((h_a+vh_{av})I-\frac{\pi h_a}{v\cos^2\frac{\pi}{v}}K)^2\\ &=\frac{2\pi^2a\cos\frac{\pi}{v}}{v^5\sin^3\frac{\pi}{v}}IJ+\frac{2\pi^2a}{v^5\sin^2\frac{\pi}{v}}JK-\frac{\pi^2}{v^4\sin^2\frac{\pi}{v}\cos^2\frac{\pi}{v}}(K-\frac{\cos\frac{\pi}{v}}{\sin\frac{\pi}{v}}I)^2\\ &=\frac{2\pi^2a}{v^5\sin^2\frac{\pi}{v}}J(K-\frac{\cos\frac{\pi}{v}}{\sin\frac{\pi}{v}}I)-\frac{2\pi^2a}{v^5\sin^2\frac{\pi}{v}}\cdot\frac{v}{2a\cos^2\frac{\pi}{a}}(K-\frac{\cos\frac{\pi}{v}}{\sin\frac{\pi}{v}}I)^2\\ &=\frac{2\pi^2a}{v^5\sin^2\frac{\pi}{v}}(J-\frac{v}{2a\cos^2\frac{\pi}{a}}(K-\frac{\cos\frac{\pi}{v}}{\sin\frac{\pi}{v}}I))(K-\frac{\cos\frac{\pi}{v}}{\sin\frac{\pi}{v}}I)\\ &=\frac{2\pi^2a}{v^5\sin^2\frac{\pi}{v}}\cdot\int_0^{\frac{\pi}{v}} g'(\frac{h}{\cos^2\psi})(1-\frac{sin^2\psi}{sin^2\frac{\pi}{v}})\frac{d\psi}{\cos^4\psi}\cdot\frac{2a\cos^2\frac{\pi}{v}}{v\sin^2\frac{\pi}{v}}\cdot\int_0^{\frac{\pi}{v}} g'(\frac{h}{\cos^2\psi})\frac{\sin^2\psi}{\cos^4\psi}d\psi\\ &>0. \end{aligned}$$

Having proved that M_{aa} and $M_{aa}M_{vv}-M_{av}$ are positive for $a>0, v\geq 3$, it follows that the Hessian matrix of M is positive definite, which in turn implies (5).

Our next tool is the following simple consequence of *Euler's polytope formula*, see e.g. [**?**], p. 16:

(6) $\quad v_1 + \ldots + v_n \leq 6n.$

Since for fixed a the function $M(a, v)$ is convex in v by (5) and has a limit as $v \to +\infty$ (the moment of the circular disc with center o and area a), we see that

(7) $\quad M(a, v)$ is non-increasing in v for a fixed.

Now, combining (2), (3), (4), (5), applying Jensen's inequality for convex functions, and using (6) and (7),

$$S \geq \sum_{i=1}^{n} M(a_i, v_i) \geq nM(\frac{a_1 + \ldots + a_n}{n}, \frac{v_1 + \ldots + v_n}{n}) \geq nM(\frac{|H|}{n}, 6)$$

follows. This completes the proof of the Theorem.

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