Complex affine isoperimetric inequalities

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Abstract

Complex extensions of the Petty projection inequality and the Busemann-Petty centroid inequality are established.

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1 Introduction

One of the basic affine isoperimetric inequalities is Petty's projection inequality [35]: Among convex bodies of given volume, the ones whose polar projection bodies have maximal volume are precisely the ellipsoids.
What makes this inequality particularly interesting is the fact it strengthens and directly implies the classical isoperimetric inequality, see [27]. This insight was used in [43] to establish the Zhang-Sobolev inequality, an affine inequality far stronger than the classical sharp Sobolev inequality. Moreover, even decades after its discovery, Petty's projection inequality is still the focus of intense research. For example, it was recently shown to hold for sets of finite perimeter [39]. Additionally, it was extended to an Orlicz setting [18, 29, 32] as well as to rigid motion compatible Minkowski valuations [17].
Projection bodies, the objects under consideration in Petty's projection inequality, were introduced by Minkowski. Given a real convex body K (i.e. a non-empty compact convex subset of $\mathbb{R}^n$), its projection body is the convex body ΠK with support function

$$h_{\Pi K}(u) = \mathrm{vol}(K|u^\perp).$$

Here, $\mathrm{vol}(K|u^\perp)$ denotes the $(n-1)$-dimensional volume of the orthogonal projection of K onto the hyperplane orthogonal to u. Projection bodies have not only become a central notion in convex geometry, they also found applications in Minkowski geometry, stochastic geometry, geometric tomography, symbolic dynamics, and functional analysis (see, e.g., [10, 11, 14, 23, 37, 38]).

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The special role of projection bodies in the affine geometry of convex bodies was discovered by Ludwig in [25, 26], see also [16]. She proved that projection bodies are the only Minkowski valuations which are contravariant with respect to the *real* affine group. In [3], Abardia and Bernig proved a complex analog of Ludwig's characterization result. They classified all Minkowski valuations which are contravariant with respect to the *complex* affine group. As it turns out, there exists a whole family Π_C of such valuations. So from a valuation theoretic perspective, each member of this family serves as a complex analog of the projection body operator.

The operator Π_C is defined as follows. Let $\mathcal{K}(\mathbb{C}^n)$ denote convex bodies in $\mathbb{C}^n$. For $C \in \mathcal{K}(\mathbb{C})$ and $K \in \mathcal{K}(\mathbb{C}^n)$ the complex projection body $\Pi_C K$ is the convex body with support function

$$h_{\Pi_C K}(u) = nV(K, Cu),$$

where $Cu := \{cu : c \in C\}$ and $V(K, Cu)$ denotes the mixed volume of K and Cu. We refer the reader to Section 2 for the precise definitions.

In [3], Abardia and Bernig asked for an analogue of Petty's projection inequality for the operator Π_C. Our main result answers this question in the symmetric case. (Throughout we use the convention that $0 \cdot \infty = 0$.)

1.1 Theorem. *Let $C \in \mathcal{K}(\mathbb{C})$ and $K \in \mathcal{K}(\mathbb{C}^n)$. If K is centrally symmetric, then*

$$|K|^{2n-1}|\Pi_C^* K| \leq |B|^{2n-1}|\Pi_C^* B|. \tag{1}$$

If $\dim C = 1$, equality holds if and only if K is an ellipsoid. If $\dim C = 2$, equality holds if and only if K is an Hermitian ellipsoid.

Here, $|\cdot|$ stands for volume, $\Pi_C^* K$ denotes the polar of the complex projection body of K, and B is the complex unit ball in $\mathbb{C}^n$.

Cauchy's projection formula states that $\mathrm{vol}(K|u^\perp) = nV(K, [-1, 1]u)$. So if we compare the definitions of Π and Π_C, we see that Π equals $\Pi_{[-1,1]}$. Hence Theorem 1.1 contains the classical Petty projection inequality as a special case.

As was mentioned in the beginning, Petty's projection inequality is stronger than the isoperimetric inequality. We will show in Section 6 that each of the new inequalities (1) also strengthens and directly implies the isoperimetric inequality. Note that the inequalities (1) are invariant with respect to the complex linear group $\mathrm{GL}(n, \mathbb{C})$. In contrast, the complex isoperimetric inequality is invariant merely with respect to the unitary group. Consequently, the affine inequalities are stronger than their unitary counterparts. Similar phenomenons were also established in [13, 17, 19].

The theory of real convex bodies is classical and well established. But complex convex geometry gained momentum only recently, see e.g. [1, 2, 4–9, 21, 22, 41, 42]. In particular, complex projection bodies were studied in [24, 40]. Theorem 1.1 is part of this program.

A second fundamental affine isoperimetric inequality is the Busemann-Petty centroid inequality [34]: Among convex bodies of given volume, the ones whose centroid bodies have minimal volume are precisely the origin symmetric ellipsoids.

As the Petty projection inequality, also the Busemann-Petty centroid inequality has been extended significantly [12, 18, 29, 31]. Among other applications, these extensions were used to prove affine Sobolev inequalities [20, 33] and information theoretic inequalities [30].
The definition of centroid bodies, the objects under consideration in the Busemann-Petty centroid inequality, dates back to Dupin. Given a real convex body $K \subset \mathbb{R}^n$ with non-empty interior, its centroid body is the convex body ΓK with support function

$$h_{\Gamma K}(u) = \frac{1}{|K|} \int_K |u \cdot x| \, dx.$$

Here, integration is with respect to Lebesgue measure and $\cdot$ denotes the standard Euclidean inner product. Let $K \in \mathcal{K}_o(\mathbb{C}^n)$ denote complex convex bodies with non-empty interior. With regard to the construction of complex projection bodies and the fact that $|u \cdot x| = h_{[-1,1]u}(x)$, we define the complex centroid body $\Gamma_C K$ of $K \in \mathcal{K}_o(\mathbb{C}^n)$ as the convex body with support function

$$h_{\Gamma_C K}(u) = \frac{1}{|K|} \int_K h_{Cu}(x) \, dx.$$

Integration in this definition is with respect to the push forward of Lebesgue measure under the canonical isomorphism between $\mathbb{R}^{2n}$ and $\mathbb{C}^n$. Our second result is the following complex Busemann-Petty centroid inequality.

1.2 Theorem. *Let $C \in \mathcal{K}(\mathbb{C})$ and $K \in \mathcal{K}_o(\mathbb{C}^n)$. If K is origin-symmetric, then*

$$|K|^{-1} |\Gamma_C K| \geq |B|^{-1} |\Gamma_C B|. \tag{2}$$

If $\dim C = 1$, equality holds if and only if K is an origin symmetric ellipsoid. If $\dim C = 2$, equality holds if and only if K is an origin symmetric Hermitian ellipsoid.

Note that, by construction, Γ equals $\Gamma_{[-1,1]}$. Hence Theorem 1.2 contains the classical Busemann-Petty centroid inequality as a special case.

2 Notation and preliminaries

For a complex number $c \in \mathbb{C}$ we write $\overline{c}$ for its complex conjugate and $|c|$ for its norm. If $\phi \in \mathbb{C}^{m \times n}$, then ϕ^* denotes the conjugate transpose of ϕ. We denote by $\cdot$ the standard Hermitian inner product on $\mathbb{C}^n$ being conjugate linear in the first argument, i.e. $x \cdot y = x^* y$ for all $x, y \in \mathbb{C}^n$. B stands for the complex unit ball $\{c \in \mathbb{C}^n : c \cdot c \leq 1\}$ and $\mathbb{S}^n$ for the complex unit sphere $\{c \in \mathbb{C}^n : c \cdot c = 1\}$. We write ι for the canonical isomorphism between $\mathbb{C}^n$ (viewed as a real vector space) and $\mathbb{R}^{2n}$, i.e.

$$\iota(c) = (\Re[c_1], \ldots, \Re[c_n], \Im[c_1], \ldots, \Im[c_n]), \quad c \in \mathbb{C}^n,$$

where $\Re$ and $\Im$ are the real and imaginary part, respectively. Note that

$$\Re[x \cdot y] = \iota x \cdot \iota y \tag{3}$$

for all $x, y \in \mathbb{C}^n$, where the inner product on the right hand side is the standard Euclidean inner product on $\mathbb{R}^{2n}$.
Let $\phi \in \mathrm{GL}(n, \mathbb{C})$ be decomposed in its real and imaginary part, i.e. $\phi = \Re[\phi] + i\Im[\phi]$. The real matrix representation $\mathbb{R}[\phi] \in \mathrm{GL}(2n, \mathbb{R})$ of ϕ is the block matrix

$$\mathbb{R}[\phi] = \begin{pmatrix} \Re[\phi] & -\Im[\phi] \\ \Im[\phi] & \Re[\phi] \end{pmatrix}.$$

It is not hard to show that

$$|\det \phi|^2 = |\det \mathbb{R}[\phi]| \tag{4}$$

as well as

$$\iota(\phi x) = \mathbb{R}[\phi]\iota x. \tag{5}$$

Next, we turn to basics from convex geometry. Throughout, let $K, L \in \mathcal{K}(\mathbb{C}^n)$ be convex bodies. K is an ellipsoid if

$$K = \{x \in \mathbb{C}^n : \iota x \cdot \phi \iota x \leq 1\} + t$$

for some positive definite symmetric matrix $\phi \in \mathrm{GL}(2n, \mathbb{R})$ and some $t \in \mathbb{C}^n$. A special class of ellipsoids are Hermitian ellipsoids. K is an Hermitian ellipsoid if

$$K = \{x \in \mathbb{C}^n : x \cdot \phi x \leq 1\} + t$$

for a positive definite Hermitian matrix $\phi \in \mathrm{GL}(n, \mathbb{C})$ and a $t \in \mathbb{C}^n$. Note that K is an Hermitian ellipsoid if and only if $K = \psi B + t$ for some $\psi \in \mathrm{GL}(n, \mathbb{C})$ and a $t \in \mathbb{C}^n$.
The volume $|K|$ of K is defined as the $2n$-dimensional Lebesgue measure of ιK, i.e. $|K| := |\iota K|$. Note that, by (5), $|\phi K| = |\iota(\phi K)| = |\mathbb{R}[\phi]\iota K|$. Thus relation (4) implies

$$|\phi K| = |\det \phi|^2 |K| \tag{6}$$

for each $\phi \in \mathrm{GL}(n, \mathbb{C})$. In particular we have

$$|cK| = |c|^{2n} |K| \tag{7}$$

for all $c \in \mathbb{C}$.
In the sequel, we collect complex reformulations of well known results from convex geometry. These complex versions can be directly deduced from their real counterparts by an appropriate application of ι. The standard references for these real results are the books by Gardner [14], Gruber [15], and Schneider [36].
The convex body K is uniquely determined by its support function $h_K : \mathbb{C}^n \to \mathbb{R}$, where

$$h_K(x) = \max\{\Re[x \cdot y] : y \in K\}.$$

It is an easy consequence of (3) that

$$h_K = h_{\iota K} \circ \iota, \tag{8}$$

where $h_{\iota K}$ is the usual real support function, i.e. $h_L(x) = \max\{x \cdot y : y \in L\}$ for a convex body $L \subset \mathbb{R}^{2n}$ and $x \in \mathbb{R}^{2n}$. Moreover, the definition of support functions directly implies

$$h_{\lambda K} = \lambda h_K \tag{9}$$

for all $\lambda \geq 0$, as well as

$$h_{\phi K} = h_K \circ \phi^* \tag{10}$$

for all $\phi \in \mathrm{GL}(n, \mathbb{C})$.
Given two real numbers $c, d \geq 0$, the Minkowski sum $cK + dL$ is defined via

$$cK + dL = \{ck + dl : k \in K \text{ and } l \in L\},$$

or equivalently,

$$h_{cK+dL} = ch_K + dh_L.$$

In particular, the central symmetral ΔK of K is given by $\Delta K = \frac{1}{2}K + \frac{1}{2}(-K)$. Clearly,

$$\Delta(Cu) = (\Delta C)u \tag{11}$$

for all $C \in \mathcal{K}(\mathbb{C})$ and $u \in \mathbb{C}^n$.
The support function of an origin-symmetric body $C \in \mathcal{K}(\mathbb{C})$ can be written as an integral over the unit circle. In fact, every origin-symmetric planar convex body is a centered zonoid, and hence there exists a finite even Borel measure μ_C on $\mathbb{S}^1$ such that

$$h_C(u) = \int_{\mathbb{S}^1} h_{[-u,u]}\, d\mu_C \tag{12}$$

for every $u \in \mathbb{S}^1$. We therefore call μ_C the complex generating measure of C. The measure μ_C is actually unique. This follows from the following general result: If μ is a signed finite even Borel measure on $\mathbb{S}^n$, then

$$\int_{\mathbb{S}^n} h_{[-u,u]} d\mu = 0 \quad \text{for every } u \in \mathbb{S}^n \quad \Longleftrightarrow \quad \mu = 0. \tag{13}$$

The complex surface area measure S_K of K is the Borel measure on $\mathbb{S}^n$ defined for every Borel set $\omega \subset \mathbb{S}^n$ by

$$S_K(\omega) = \mathcal{H}^{2n-1}(\iota\{x \in K : \exists u \in \omega \text{ with } \Re[x \cdot u] = h_K(u)\}).$$

Here, $\mathcal{H}^{2n-1}$ stands for $(2n-1)$-dimensional Hausdorff measure on $\mathbb{R}^{2n}$. It is not difficult to show that surface area measures are translation invariant and

$$S_{cK}(\omega) = S_K(\bar{c}\,\omega) \tag{14}$$

for all $c \in \mathbb{S}^1$ and each Borel set $\omega \subset \mathbb{S}^n$. Up to translations, a body $K \in \mathcal{K}_o(\mathbb{C}^n)$ is uniquely determined by its surface area measure, i.e.

$$S_K = S_L \quad \iff \quad K = L + t \quad \text{for some } t \in \mathbb{C}^n. \tag{15}$$

Together with Minkowski's existence theorem for surface area measures, this allows us to define the complex Blaschke body ∇K as the unique origin-symmetric convex body with

$$S_{\nabla K} = \tfrac{1}{2} S_K + \tfrac{1}{2} S_{-K}.$$

From (14) and (15) it follows that for all $c \in \mathbb{S}^1$

$$\nabla(cK) = c\nabla K. \tag{16}$$

Moreover, by (15) we have

$$K = L + t \quad \text{for some } t \in \mathbb{C}^n \text{ and } L = -L \quad \Longrightarrow \quad \nabla K = L. \tag{17}$$

The complex mixed volume $V(K, L)$ of two bodies K and L is defined as

$$2nV(K, L) = \lim_{\varepsilon \to 0^+} \frac{|K + \varepsilon L| - |K|}{\varepsilon}.$$

By our definition of volume we have

$$V(K, L) = V(\iota K, \iota L), \tag{18}$$

where the mixed volume on the right hand side is the usual real mixed volume in $\mathbb{R}^{2n}$. Obviously,

$$V(K, K) = |K|, \tag{19}$$

and if $\phi \in \mathrm{GL}(n, \mathbb{C})$, then (6) implies

$$V(\phi K, \phi L) = |\det \phi|^2 V(K, L). \tag{20}$$

Cauchy's projection formula states that the volume of an orthogonal projection can be expressed as a special mixed volume. More precisely,

$$\mathrm{vol}(\iota K | (\iota u)^\perp) = nV(\iota K, [-1, 1]\iota u), \tag{21}$$

for all $u \in \mathbb{S}^n$. The last fact for mixed volumes we want to mention is the representation

$$V(K, L) = \frac{1}{2n} \int_{\mathbb{S}^n} h_L dS_K. \tag{22}$$

We conclude this section with a few remarks on polar sets. Given $M \subset \mathbb{C}^n$, its polar set M^* is defined by

$$M^* = \{x \in \mathbb{C}^n : \Re[x \cdot y] \leq 1 \text{ for all } y \in M\}.$$

It is easy to see that

$$(\phi M)^* = \phi^{-*} M^*, \tag{23}$$

and in particular, for every $\lambda > 0$,

$$(\lambda M)^* = \lambda^{-1} M^*. \tag{24}$$

Let $K \in \mathcal{K}(\mathbb{C}^n)$ contain the origin in its interior. Then K^* belongs to $\mathcal{K}(\mathbb{C}^n)$ and also contains the origin in its interior. Moreover, K can be described not only by its support function, but also by its radial function. The latter is the function $\rho_K : \mathbb{C}^n \backslash \{0\} \to \mathbb{R}$ with

$$\rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in K\}.$$

On $\mathbb{C}^n \backslash \{0\}$ we have

$$\rho_{K^*} = h_K^{-1}. \tag{25}$$

Finally, the volume of K can be written by the polar formula for volume as

$$|K| = \frac{1}{2n} \int_{\mathbb{S}^n} \rho_K^{2n} d\sigma, \tag{26}$$

where σ stands for the the push forward with respect to ι^{-1} of $\mathcal{H}^{2n-1}$ on the $(2n-1)$-dimensional Euclidean unit sphere.

3 A characterization of Hermitian ellipsoids

We begin with the following symmetry property of Hermitian ellipsoids.

3.1 Lemma. *If $K \in \mathcal{K}(\mathbb{C}^n)$ is an Hermitian ellipsoid, then $c\nabla K = \nabla K$ for all $c \in \mathbb{S}^1$.*

Proof. By assumption, there exists a positive definite Hermitian matrix $\phi \in \mathrm{GL}(n, \mathbb{C})$ and a vector $t \in \mathbb{C}^n$ such that $K = E_\phi + t$, where

$$E_\phi := \{x \in \mathbb{C}^n : x \cdot \phi x \leq 1\}.$$

Since E_ϕ is origin-symmetric, it follows from (17) that $\nabla K = E_\phi$. Hence it suffices to prove for all $c \in \mathbb{S}^1$ that $cE_\phi = E_\phi$. So let $c \in \mathbb{S}^1$. Clearly,

$$cE_\phi = \{x \in \mathbb{C}^n : (c^{-1}x) \cdot \phi(c^{-1}x) \leq 1\}.$$

From the sesquilinearity of the Hermitian inner product we obtain

$$(c^{-1}x) \cdot \phi(c^{-1}x) = (\overline{c^{-1}} c^{-1})\, x \cdot \phi x = |c^{-1}|^2 x \cdot \phi x.$$

By assumption, c has norm one. Thus c^{-1} has also norm one which implies $cE_\phi = E_\phi$, as desired. $\square$

Our goal is to establish a converse of the last lemma. This will be done in Theorem 3.4. But in order to do so, we need some preparations.
Let $\psi \in \mathrm{GL}(n,\mathbb{C})$ be Hermitian. Then $x \cdot \psi x$ is real. Combining this with (3) yields $x \cdot \psi x = \Re[x \cdot \psi x] = \iota x \cdot \iota(\psi x)$. By (5) we obtain

$$x \cdot \psi x = \iota x \cdot \mathbb{R}[\psi] \iota x. \tag{27}$$

Now, let us establish a condition which ensures that a matrix $\phi \in \mathrm{GL}(2n,\mathbb{R})$ is the real matrix representation of some complex matrix $\psi \in \mathrm{GL}(n,\mathbb{C})$.

3.2 Lemma. *Let $\phi \in \mathrm{GL}(2n,\mathbb{R})$ be a positive definite symmetric matrix such that*

$$\phi \mathbb{R}[c\,\mathrm{Id}] = \mathbb{R}[c\,\mathrm{Id}]\phi$$

for some $c \in \mathbb{C}$ with $\Im[c] \neq 0$. Then there exists a positive definite Hermitian matrix $\psi \in \mathrm{GL}(n,\mathbb{C})$ with $\phi = \mathbb{R}[\psi]$.

Proof. Partition ϕ in four $n \times n$-matrices by

$$\phi = \begin{pmatrix} A & C \\ B & D \end{pmatrix}.$$

Moreover, the real matrix representation of $\mathbb{R}[c\,\mathrm{Id}]$ equals

$$\mathbb{R}[c\,\mathrm{Id}] = \begin{pmatrix} \Re[c]\mathrm{Id} & -\Im[c]\mathrm{Id} \\ \Im[c]\mathrm{Id} & \Re[c]\mathrm{Id} \end{pmatrix}.$$

Performing the matrix multiplications in our assumption $\phi \mathbb{R}[c\,\mathrm{Id}] = \mathbb{R}[c\,\mathrm{Id}]\phi$ with the above representations gives

$$\begin{pmatrix} \Re[c]A + \Im[c]C & \Re[c]C - \Im[c]A \\ \Re[c]B + \Im[c]D & \Re[c]D - \Im[c]B \end{pmatrix} = \begin{pmatrix} \Re[c]A - \Im[c]B & \Re[c]C - \Im[c]D \\ \Re[c]B + \Im[c]A & \Re[c]D + \Im[c]C \end{pmatrix}.$$

Comparing the upper left entries proves $C = -B$, whereas comparing the lower left entries proves $A = D$. Thus

$$\phi = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \tag{28}$$

which proves that $\phi = \mathbb{R}[\psi]$ where $\psi = A + iB$. Since $\phi \in \mathrm{GL}(2n,\mathbb{R})$, we infer from (4) that $\psi \in \mathrm{GL}(n,\mathbb{C})$.
Next, let us show that ψ is Hermitian. Since ϕ is supposed to be symmetric, i.e. $\phi = \phi^t$, representation (28) implies

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \begin{pmatrix} A^t & B^t \\ -B^t & A^t \end{pmatrix}.$$

Thus $A = A^t$ as well as $B = -B^t$. If we combine this with the definition of ψ we get $\psi = A + iB = A^t - iB^t = \psi^*$. Hence ψ is Hermitian.

It remains to show that ψ is positive definite. From (27) we know that $x \cdot \psi x = \iota x \cdot \phi \iota x$. Since, by assumption, ϕ is positive definite, ψ is therefore also positive definite. $\square$

Let $\phi \in \mathrm{GL}(2n, \mathbb{R})$ be a positive definite symmetric matrix. Then

$$E_\phi := \{x \in \mathbb{C}^n : \iota x \cdot \phi \iota x \leq 1\}$$

is an origin-symmetric ellipsoid. The next lemma proves that the defining matrix ϕ of E_ϕ is uniquely determined.

3.3 Lemma. *Let $\phi, \psi \in \mathrm{GL}(2n, \mathbb{R})$ be positive definite and symmetric. Then*

$$E_\phi = E_\psi \quad \Longleftrightarrow \quad \phi = \psi.$$

Proof. First, note that for an arbitrary symmetric $\theta \in \mathrm{GL}(2n, \mathbb{R})$ and $k, l = 1, \ldots, 2n$,

$$\frac{\partial}{\partial x_k \partial x_l} x \cdot \theta x = \frac{\partial}{\partial x_k \partial x_l} \sum_{s,t=1}^{2n} \theta_{st} x_s x_t = 2\theta_{kl}. \tag{29}$$

Now assume that $E_\phi = E_\psi$, since the other direction is trivial. Then $\rho_{E_\phi} = \rho_{E_\psi}$. It follows easily from the definition of radial functions that for all $x \in \mathbb{R}^{2n} \backslash \{0\}$

$$\rho_{E_\phi}(\iota^{-1} x) = (x \cdot \phi x)^{-1/2} \qquad \text{and} \qquad \rho_{E_\psi}(\iota^{-1} x) = (x \cdot \psi x)^{-1/2}.$$

Hence $x \cdot \phi x = x \cdot \psi x$ for all $x \in \mathbb{R}^{2n}$. Differentiating this equation with respect to x_k and x_l implies, by (29), that $\phi_{kl} = \psi_{kl}$ for all $k, l = 1, \ldots, 2n$. Thus $\phi = \psi$. $\square$

We are now in a position to establish the desired characterization of Hermitian ellipsoids.

3.4 Theorem. *Let $K \in \mathcal{K}(\mathbb{C}^n)$ be an ellipsoid. Then K is Hermitian if and only if $c \nabla K = \nabla K$ for some $c \in \mathbb{S}^1$ with $\Im[c] \neq 0$.*

Proof. First, assume that K is Hermitian. Then the assertion is an immediate consequence of Lemma 3.1.
Next, suppose that there exists a $c \in \mathbb{S}^1$ with $\Im[c] \neq 0$ and $c\nabla K = \nabla K$. Since K is an ellipsoid, there exists a positive definite symmetric $\phi \in \mathrm{GL}(2n, \mathbb{R})$ and a $t \in \mathbb{C}^n$ with $K = E_\phi + t$. By (17) we have

$$\nabla K = E_\phi. \tag{30}$$

It follows easily from the definition of E_ϕ and (5) that

$$cE_\phi = E_{\mathbb{R}[c\,\mathrm{Id}]\phi\mathbb{R}[c\,\mathrm{Id}]^{-1}}.$$

By our assumption and (30) we know that $E_\phi = cE_\phi$. So by the last equation

$$E_\phi = E_{\mathbb{R}[c\,\mathrm{Id}]\phi\mathbb{R}[c\,\mathrm{Id}]^{-1}}.$$

Now Lemma 3.3 proves

$$\phi = \mathbb{R}[c\,\mathrm{Id}]\phi\mathbb{R}[c\,\mathrm{Id}]^{-1}$$

and thus $\phi\mathbb{R}[c\,\mathrm{Id}] = \mathbb{R}[c\,\mathrm{Id}]\phi$. Lemma 3.2 reveals that there exists a positive definite Hermitian matrix $\psi \in \mathrm{GL}(n,\mathbb{C})$ with $\phi = \mathbb{R}[\psi]$. Thus $E_\phi = E_{\mathbb{R}[\psi]}$ and hence $K = E_{\mathbb{R}[\psi]} + t$. An application of (27) to the definition of $E_{\mathbb{R}[\psi]}$ implies that K is an Hermitian ellipsoid. □

4 Some properties of the complex projection body operator

In this section we collect some basic properties of complex projection bodies which will be used in the proof of the complex Petty projection inequality. Throughout this section let $C \in \mathcal{K}(\mathbb{C})$ and $K \in \mathcal{K}(\mathbb{C}^n)$.
Let us begin with rewriting the definition of Π_C. Indeed, by relation (22) we have

$$h_{\Pi_C K}(u) = \frac{1}{2}\int_{\mathbb{S}^n} h_{Cu}\, dS_K \tag{31}$$

for all $u \in \mathbb{S}^n$. Next, we note a simple homogeneity property of Π_C. From (31) as well as relation (9) we obtain for all $\lambda \geq 0$

$$\Pi_{\lambda C} K = \lambda\, \Pi_C K. \tag{32}$$

Moreover, it was shown in [3] that Π_C is translation invariant and $\mathrm{SL}(n,\mathbb{C})$-contravariant. The following theorem extends this contravariance to the whole general group $\mathrm{GL}(n,\mathbb{C})$. For $\phi \in \mathrm{GL}(n,\mathbb{C})$, we denote by ϕ^{-*} the inverse of the conjugate transpose of ϕ.

4.1 Theorem. *Let $\phi \in \mathrm{GL}(n,\mathbb{C})$ and $t \in \mathbb{C}^n$. Then $\Pi_C(\phi K + t) = |\det\phi|^2\phi^{-*}\Pi_C K$.*

Proof. The definition of mixed volumes immediately implies that they are translation invariant in their first argument. A glance at the definition of Π_C therefore reveals that Π_C is also translation invariant.
Now, let $\phi \in \mathrm{GL}(n,\mathbb{C})$. From the definition of Π_C, equality (20), and the fact that $\phi^{-1}Cu = C\phi^{-1}u$ we infer

$$h_{\Pi_C(\phi K)}(u) = nV(\phi K, Cu) = n|\det\phi|^2 V(K, \phi^{-1}Cu) = n|\det\phi|^2 V(K, C\phi^{-1}u).$$

The definition of Π_C again as well as (9) and (10) give

$$h_{\Pi_C(\phi K)}(u) = |\det\phi|^2 h_{\Pi_C K}(\phi^{-1}u) = h_{|\det\phi|^2\phi^{-*}\Pi_C K}(u).$$

Since convex bodies are uniquely determined by their support functions we are done. □

A special case of the last theorem is

$$\Pi_C(cK) = c\,\Pi_C K \tag{33}$$

for each $c \in \mathbb{S}^1$. Now, we determine the image of Π_C on balls.

4.2 Theorem. *If* $\dim C > 0$*, then* Π_C *maps balls to origin-symmetric balls.*

Proof. Since Π_C is translation invariant by Theorem 4.1, it suffices to prove the lemma for an origin-symmetric ball rB with radius $r > 0$. It is easy to see that $S_{rB} = r^{2n-1}\sigma$, and thus

$$h_{\Pi_C(rB)}(u) = \frac{r^{2n-1}}{2} \int_{\mathbb{S}^n} h_{Cu}\, d\sigma \tag{34}$$

for all $u \in \mathbb{S}^n$. Now fix some $u_0 \in \mathbb{S}^n$. Since $\mathrm{SU}(n)$ acts transitively on $\mathbb{S}^n$, we can find for every $u \in \mathbb{S}^n$ a $\vartheta_u \in \mathrm{SU}(n)$ with $\vartheta_u u_0 = u$. This implies $Cu = \vartheta_u C u_0$. If we plug this into (34) and use (10), we get

$$h_{\Pi_C(rB)}(u) = \frac{r^{2n-1}}{2} \int_{\mathbb{S}^n} h_{Cu_0} \circ \vartheta_u^*\, d\sigma.$$

But σ is $\mathrm{SU}(n)$-invariant, so the right hand side is independent from u and, since $\dim C > 0$, greater than zero. Hence $\Pi_C(rB)$ is an origin-symmetric ball. $\square$

4.3 Theorem. *If* $K \in \mathcal{K}_o(\mathbb{C}^n)$*, then* $\Pi_C(\nabla K) = \Pi_{\Delta C} K$*.*

Proof. A special case of (14) is the equality $S_{-K}(\omega) = S_K(-\omega)$ for every Borel set $\omega \subset \mathbb{S}^n$. If we plug the definition of ∇ into representation (31) and use the just mentioned fact, then

$$h_{\Pi_C(\nabla K)}(u) = \frac{1}{4} \int_{\mathbb{S}^n} h_{Cu}(v) + h_{Cu}(-v)\, dS_K(v).$$

From (10) we know that $h_{Cu}(-v) = h_{-(Cu)}(v)$. Thus the definition of Δ implies

$$h_{\Pi_C(\nabla K)}(u) = \frac{1}{2} \int_{\mathbb{S}^n} h_{\Delta(Cu)}(v)\, dS_K(v).$$

Since, by (11), $\Delta(Cu) = (\Delta C)u$, a glance at (31) concludes the proof. $\square$

4.4 Corollary. *If* $K \in \mathcal{K}_o(\mathbb{C}^n)$ *is centrally symmetric, then* $\Pi_C K = \Pi_{\Delta C} K$*.*

Proof. By assumption, K is the translate of an origin-symmetric body. From (17) we infer that ∇K is a translate of K. The translation invariance of Π_C therefore gives $\Pi_C K = \Pi_C \nabla K$. Now Theorem 4.3 prove the assertion. $\square$

We conclude this section by relating Π_C to the classical projection body operator. Define $\Pi K := \Pi_{[-1,1]} K$. By the definition of $\Pi_{[-1,1]}$, relation (18), and equality (21), we have

$$h_{\Pi K}(u) = nV(K, [-1,1]u) = nV(\iota K, [-1,1]\iota u) = \mathrm{vol}(\iota K | (\iota u)^{\perp})$$

for all $u \in \mathbb{S}^n$. Keeping (8) in mind, we arrive at

$$\iota \Pi K = \Pi \iota K, \tag{35}$$

which justifies our notation Π for $\Pi_{[-1,1]}$.

Let us also give a more explicit definition of Π. Indeed, the equality $h_{[-1,1]u}(v) = |\Re[u \cdot v]|$ together with (31) show, for all $u \in \mathbb{S}^n$,

$$h_{\Pi K}(u) = \frac{1}{2} \int_{\mathbb{S}^n} |\Re[u \cdot v]| dS_K(v). \tag{36}$$

Finally, we reformulate a well known injectivity property of projection bodies. Since $\Delta[-1,1] = [-1,1]$, Theorem 4.3 yields $\Pi\nabla K = \Pi K$. So for $K, L \in \mathcal{K}_o(\mathbb{C}^n)$, the equality $\Pi K = \Pi L$ holds if and only if $\Pi\nabla K = \Pi\nabla L$. It follows from (13) that this happens precisely if $S_{\nabla K} = S_{\nabla L}$. By (15) this holds if and only if $\nabla K = \nabla L + t$ for some $t \in \mathbb{C}^n$. Due the origin-symmetry of Blaschke bodies, $t = 0$ and we arrive at

$$\Pi K = \Pi L \quad \iff \quad \nabla K = \nabla L \quad \text{for all } K, L \in \mathcal{K}_o(\mathbb{C}^n). \tag{37}$$

5 Proof of the complex Petty projection inequality

We begin with rewriting the support function of Cu for $C \in \mathcal{K}(\mathbb{C})$ and $u \in \mathbb{C}$.

5.1 Lemma. *Let $C \in \mathcal{K}(\mathbb{C})$. Then $h_{Cu}(v) = h_C(u \cdot v)$ for all $u, v \in \mathbb{S}^n$.*

Proof. If follows from the definition of support functions and the linearity of the Hermitian inner product in the second argument that

$$h_{Cu}(v) = \max_{c \in C}\{\Re[v \cdot (cu)]\} = \max_{c \in C}\{\Re[(v \cdot u)c]\}.$$

Furthermore, by the conjugate symmetry of the Hermitian inner product, the definition of the Hermitian inner product in $\mathbb{C}$, and the definition of support functions

$$h_{Cu}(v) = \max_{c \in C}\{\Re[\overline{u \cdot v} c]\} = \max_{c \in C}\{\Re[(u \cdot v) \cdot c]\} = h_C(u \cdot v). \qquad \square$$

5.2 Lemma. *Let $C \in \mathcal{K}(\mathbb{C})$ be origin-symmetric. Then, for all $u, v \in \mathbb{S}^n$,*

$$h_{Cu}(v) = \int_{\mathbb{S}^1} |\Re[cu \cdot v]| d\mu_C(c),$$

where μ_C is the complex generating measure of C.

Proof. Let $u, v \in \mathbb{S}^n$. From (12) we know that

$$h_C(u \cdot v) = \int_{\mathbb{S}^1} h_{[u \cdot v, u \cdot v]}(c) d\mu_C(c).$$

It is easy to see that $h_{[-w,w]}(c) = |\Re[c \cdot w]|$ for all $c, w \in \mathbb{C}$. Thus

$$h_C(u \cdot v) = \int_{\mathbb{S}^1} |\Re[c \cdot (u \cdot v)]| d\mu_C(c). \tag{38}$$

By the definition of the Hermitian inner product in $\mathbb{C}$ and the sesquilinearity of the Hermitian inner product in $\mathbb{C}^n$

$$c \cdot (u \cdot v) = \overline{c}(u \cdot v) = (cu) \cdot v.$$

Plugging this into the integrand of (38) gives

$$h_C(u \cdot v) = \int_{\mathbb{S}^1} |\Re[cu \cdot v]| d\mu_C(c).$$

Combining this with Lemma 5.1 yields the desired representation of $h_{Cu}(v)$. $\square$

Based on the representation of the last lemma, we will now see that Π_C is an average over multiples of Π.

5.3 Lemma. *Let $C \in \mathcal{K}(\mathbb{C})$ be origin-symmetric. Then, for $u \in \mathbb{S}^n$ and $K \in \mathcal{K}(\mathbb{C}^n)$,*

$$h_{\Pi_C K}(u) = \int_{\mathbb{S}^1} h_{\overline{c}\Pi K}(u) d\mu_C(c). \tag{39}$$

Moreover, the total mass $|\mu_C|$ satisfies

$$|\mu_C| = \left(\frac{|\Pi^* B|}{|\Pi_C^* B|} \right)^{\frac{1}{2n}}.$$

Proof. By the integral representation (31) and Lemma 5.2 we have

$$h_{\Pi_C K}(u) = \frac{1}{2} \int_{\mathbb{S}^n} h_{Cu}(v) dS_K(v) = \frac{1}{2} \int_{\mathbb{S}^n} \int_{\mathbb{S}^1} |\Re[cu \cdot v]| d\mu_C(c) S_K(v).$$

Now Fubini's theorem and (36) yield

$$h_{\Pi_C K}(u) = \frac{1}{2} \int_{\mathbb{S}^1} \int_{\mathbb{S}^n} |\Re[(cu) \cdot v]| S_K(v) d\mu_C(c) = \int_{\mathbb{S}^1} h_{\Pi K}(cu) d\mu_C(c).$$

Since, by equality (10), $h_{\Pi K}(cu) = h_{\overline{c}\Pi K}(u)$, we arrive at the desired representation of $h_{\Pi_C K}(u)$.
It remains to calculate the total mass of μ_C. By Lemma 4.2, $h_{\Pi B}$ is constant on $\mathbb{S}^n$. From (10) we deduce that $h_{\overline{c}\Pi B}$ is also constant and $h_{\overline{c}\Pi B} = h_{\Pi B}$. Consequently, relation (39) and the homogeneity property (9) yield

$$h_{\Pi_C B} = |\mu_C| h_{\Pi B} = h_{|\mu_C| \Pi B}.$$

In other words, $\Pi_C B = |\mu_C| \Pi B$. Keeping (24) in mind, polarizing gives

$$\Pi_C^* B = |\mu_C|^{-1} \Pi^* B.$$

Taking the volume on both sides and using (7) we obtain the claimed value for $|\mu_C|$. $\square$

We now relate the volume of a general complex projection body $\Pi_C^* K$ to that of $\Pi^* K$.

5.4 Lemma. *Let $C \in \mathcal{K}(\mathbb{C})$ be origin-symmetric with $\dim C > 0$ and $K \in \mathcal{K}_o(\mathbb{C}^n)$. Then*

$$|\Pi_C^* K| \leq |\mu_C|^{-2n} |\Pi^* K|, \tag{40}$$

with equality if and only if there exists a point $d \in \mathbb{S}^1$ with $\bar{c}\nabla K = d\nabla K$ for μ_C-almost every $c \in \mathbb{S}^1$.

Proof. From (25) and the polar formula for volume (26) we get

$$|\Pi_C^* K| = \frac{1}{2n} \int_{\mathbb{S}^n} h_{\Pi_C K}(u)^{-2n} d\sigma(u).$$

Since C is assumed to be at least one-dimensional, $|\mu_C| > 0$, and by (39) we obtain

$$|\Pi_C^* K| = \frac{|\mu_C|^{-2n}}{2n} \int_{\mathbb{S}^n} \left[\frac{1}{|\mu_C|} \int_{\mathbb{S}^1} h_{\bar{c}\Pi K}(u) d\mu_C(c) \right]^{-2n} d\sigma(u).$$

Now an application of Jensen's inequality, Fubini's theorem, and the polar formula for volume (26) give

$$\begin{aligned}
|\Pi_C^* K| &\leq \frac{|\mu_C|^{-2n-1}}{2n} \int_{\mathbb{S}^n} \int_{\mathbb{S}^1} h_{\bar{c}\Pi K}(u)^{-2n} d\mu_C(c) d\sigma(u) \\
&= \frac{|\mu_C|^{-2n-1}}{2n} \int_{\mathbb{S}^1} \int_{\mathbb{S}^n} h_{\bar{c}\Pi K}(u)^{-2n} d\sigma(u) d\mu_C(c) \\
&= |\mu_C|^{-2n-1} \int_{\mathbb{S}^1} |\bar{c}\Pi^* K| d\mu_C(c).
\end{aligned}$$

From (7) we know that $|\bar{c}\Pi^* K| = |\Pi^* K|$ for all $c \in \mathbb{S}^1$. This proves (40).
It remains to establish the equality condition. To do so, let us first prove the following equivalence for $K \in \mathcal{K}_o(\mathbb{C}^n)$:

$$\begin{gathered}
\forall u \in \mathbb{S}^n : c \mapsto h_{\bar{c}\Pi K}(u) \text{ is constant } \mu_C\text{-almost everywhere} \\
\iff \\
\exists c_0 \in \mathbb{S}^1 : h_{\bar{c}\Pi K}(u) = h_{\bar{c}_0 \Pi K}(u) \quad \forall u \in \mathbb{S}^n \text{ and } \mu_C\text{-almost all } c \in \mathbb{S}^1.
\end{gathered} \tag{41}$$

Obviously, the second condition implies the first one. So let us assume that the first condition holds. Then for each $u \in \mathbb{S}^n$ there exist a $c_u \in \mathbb{S}^1$ and a Borel set $N_u \subset \mathbb{S}^1$ with

$$\mu_C(N_u) = 0 \qquad \text{and} \qquad h_{\bar{c}\Pi K}(u) = h_{\bar{c}_u \Pi K}(u) \text{ for all } c \in N_u^c. \tag{42}$$

Let $u \in \mathbb{S}^n$ and $b \in \operatorname{supp}(\mu_C)$. By definition, each open neighborhood of b has positive μ_C measure and therefore non-empty intersection with N_u^c. So we can find a sequence $(b_k)_{k \in \mathbb{N}}$ with $b_k \in N_u^c$ and $b_k \to b$. By the continuity of $c \mapsto h_{\bar{c}\Pi K}(u)$ and (42) we get

$$h_{\bar{b}\Pi K}(u) = \lim_{k\to\infty} h_{\bar{b}_k \Pi K}(u) = h_{\bar{c}_u \Pi K}(u) = h_{\bar{c}\Pi K}(u)$$

for all $c \in N_u^c$. Since C is at least one dimensional, there exists a $c_0 \in \operatorname{supp}(\mu_C)$ and $N_u^c \neq \emptyset$. So for all $u \in \mathbb{S}^n$ and $c \in \operatorname{supp}(\mu_C)$ we have

$$h_{\bar{c}\Pi K}(u) = h_{\bar{c}_0 \Pi K}(u).$$

But $\mu_C(\operatorname{supp}(\mu_C)^c) = 0$, which concludes the proof of equivalence (41) .
Now, assume that equality holds in (40). Inspecting the above derivation of (40), this happens if and only if for all $u \in \mathbb{S}^n$ equality holds when Jensen's inequality is applied. The equality condition of Jensen's inequality implies that this is the case if and only if for all $u \in \mathbb{S}^n$ the map $c \mapsto h_{\bar{c}\Pi K}(u)$ is constant μ_C-almost everywhere. But (41) reveals that this happens precisely if there exists a $c_0 \in \mathbb{S}^1$ with $h_{\bar{c}\Pi K} = h_{\bar{c}_0\Pi K}$ for μ_C-almost every c, or equivalently, by (33), if $\Pi(\bar{c}K) = \Pi(\bar{c}_0 K)$ for μ_C-almost every c. From (37) we know that this happens if and only if $\nabla(\bar{c}K) = \nabla(\bar{c}_0 K)$ for μ_C-almost every c. Set $d := \bar{c}_0$. An application of (16) concludes the proof of the equality conditions. □

Before we continue, let us recall the classical real Petty projection inequality.

5.5 Theorem (Petty's projection inequality)**.** *Let $K \subset \mathbb{R}^{2n}$ be a convex body with nonempty interior. Then*

$$|K|^{2n-1}|\Pi^* K| \leq |\iota B|^{2n-1}|\Pi^* \iota B|$$

with equality if and only if K is an ellipsoid.

Now, we are in a position to prove the complex Petty projection inequality. We first establish a version where $C \in \mathcal{K}(\mathbb{C})$ is origin-symmetric and $K \in \mathcal{K}(\mathbb{C}^n)$ is arbitrary. Theorem 1, formulated for arbitrary C and centrally symmetric K, will be an easy consequence.

5.6 Theorem. *Let $C \in \mathcal{K}(\mathbb{C})$ be origin-symmetric and $K \in \mathcal{K}(\mathbb{C}^n)$. Then*

$$|K|^{2n-1}|\Pi_C^* K| \leq |B|^{2n-1}|\Pi_C^* B|. \tag{43}$$

If $\dim C = 1$, equality holds if and only if K is an ellipsoid. If $\dim C = 2$, equality holds if and only if K is an Hermitian ellipsoid.

Proof. If K has no interior points, then $|K| = 0$ and by our convention $0 \cdot \infty = 0$, the left hand side of (43) is always zero, whereas the right hand side is non-negative. Hence (43) trivially holds true and we can assume that $K \in \mathcal{K}_o(\mathbb{C}^n)$.
First, suppose that $\dim C = 0$, i.e. $C = \{0\}$. Then, by (31), we have $\Pi_C K = \{0\}$. Hence $\Pi_C^* K = \mathbb{C}^n$ and therefore $|\Pi_C^* K| = \infty$. So inequality (43) is again trivially true.
Now, assume that $\dim C > 0$. Polarizing both sides of (35) and using $^* \circ \iota = \iota \circ {}^*$ gives

$$\iota \Pi^* K = \Pi^* \iota K. \tag{44}$$

From inequality (40), the definition of volume, relation (44), and Theorem 5.5 we get

$$\begin{aligned}|K|^{2n-1}|\Pi_C^* K| &\leq |\mu_C|^{-2n}|K|^{2n-1}|\Pi^* K| \\ &= |\mu_C|^{-2n}|\iota K|^{2n-1}|\iota\Pi^* K| \\ &= |\mu_C|^{-2n}|\iota K|^{2n-1}|\Pi^* \iota K| \\ &\leq |\mu_C|^{-2n}|B|^{2n-1}|\Pi^* B|.\end{aligned}$$

Plugging in the value of the total mass $|\mu_C|$ from Lemma 5.3 proves (43).

Let us turn towards the equality conditions. By Lemma 5.4 and the equality conditions of the real Petty projection inequality, equality occurs in (43) if and only if there exists a $d \in \mathbb{S}^1$ such that $\bar{c}\nabla K = d\nabla K$ for μ_C-almost all $c \in \mathbb{S}^1$ and ιK is an ellipsoid. By definition, the latter is equivalent to K being an ellipsoid.

First, suppose that $\dim C = 1$, i.e. C is a segment $[-c_0, c_0]$ for some $c_0 \in \mathbb{C}\backslash\{0\}$. It suffices to show that the first of the above equality conditions is always true. The generating measure μ_C of C is given by

$$\mu_C = \frac{|c_0|}{2}\Big(\delta_{-\langle c_0\rangle} + \delta_{\langle c_0\rangle}\Big),$$

where δ denotes the Dirac measure and $\langle c_0\rangle := c_0|c_0|^{-1}$ stands for the spherical projection of c_0 to the unit circle. Thus the first equality condition holds if and only if $-\overline{\langle c_0\rangle}\nabla K = \overline{\langle c_0\rangle}\nabla K$. But this is always true since ∇K is origin-symmetric.

Next, suppose that $\dim C = 2$. If K is an Hermitian ellipsoid, then Lemma 3.1 shows that the above equality conditions hold. It remains to prove that the above equality conditions imply that K is an Hermitian ellipsoid. The first equality condition implies the existence of a point $d \in \mathbb{S}^1$ and a Borel set $N \subset \mathbb{S}^1$ with $\mu_C(N) = 0$ such that $\bar{c}\nabla K = d\nabla K$ for all $c \in N^c$. Since $\dim C = 2$, N^c contains two non-antipodal points, i.e. there exist $c_0, c_1 \in N^c$ such that $c_0 \neq -c_1$ and $\bar{c}_0\nabla K = \bar{c}_1\nabla K$. Clearly, $\bar{c}_0$ and $\bar{c}_1$ are also non-antipodal. So for $c := \bar{c}_0\bar{c}_1^{-1}$ we have

$$c\nabla K = \nabla K \quad \text{ where } c \in \mathbb{S}^1 \text{ with } \Im[c] \neq 0.$$

Now Theorem 3.4 concludes the proof. □

Proof of Theorem 1.1. Since K is assumed to be centrally symmetric and B is centrally symmetric, Corollary 4.4 proves $\Pi_C K = \Pi_{\Delta C} K$ and $\Pi_C B = \Pi_{\Delta C} B$. So the desired inequality (1) is equivalent to

$$|K|^{2n-1}|\Pi_{\Delta C}^* K| \leq |B|^{2n-1}|\Pi_{\Delta C}^* B|.$$

By definition, ΔC is origin-symmetric and hence Theorem 5.6 immediately implies Theorem 1.1. □

6 Affine vs. unitary inequalities

The isoperimetric inequality states that among bodies of given perimeter, balls have largest volume. This can be written as

$$\left(\frac{|\partial K|}{|\partial B|}\right)^{-2n}|K|^{2n-1} \leq |B|^{2n-1} \tag{45}$$

for all $K \in \mathcal{K}(\mathbb{C}^n)$, where the surface area $|\partial K|$ of K is defined as $\mathcal{H}^{2n-1}(\iota\partial K)$. We will show that each of our new Petty projection inequalities (1) is stronger and directly implies the isoperimetric inequality.

Note that the isoperimetric inequality is only invariant with respect to the unitary group, whereas the complex Petty projection inequalities are invariant with respect to the larger affine group $\mathrm{SL}(n,\mathbb{C})$. Nevertheless, the affine inequalities turn out to be stronger. This fact will be an easy consequence of the following theorem.

6.1 Theorem. *Suppose that $C \in \mathcal{K}(\mathbb{C})$ has $\dim C > 0$ and is normalized such that $\Pi_C B = \Pi B$. For centrally symmetric $K \in \mathcal{K}(\mathbb{C}^n)$*

$$\left(\frac{|\partial K|}{|\partial B|}\right)^{-2n}|\Pi^* B| \leq |\Pi^*_C K| \leq |\Pi^* K|. \tag{46}$$

Equality holds in the first inequality if and only if $\Pi_C K$ is a ball. Equality holds in the second inequality if and only if there exists a point $d \in \mathbb{S}^1$ such that $\overline{c}\nabla K = d\nabla K$ for $\mu_{\Delta C}$-almost every $c \in \mathbb{S}^1$.

Proof. First, our normalization $\Pi_C B = \Pi B$ is possible due to Theorem 4.2 and equation (32). Next, let $v \in \mathbb{S}^n$. We write $\overline{v}$ for the vector obtained from v by componentwise conjugation. As a special case of Theorem 4.2 we deduce that ΠB is a ball and hence $\Pi B = h_{\Pi B}(\overline{v})B$. By our normalization assumption we have $h_{\Pi B}(\overline{v}) = h_{\Pi_C B}(\overline{v})$, and thus $\Pi B = h_{\Pi_C B}(\overline{v})B$. If we polarize both sides of the last equation, keep (24) in mind, take the volume, and solve for $h_{\Pi_C B}(\overline{v})$, we obtain

$$h_{\Pi_C B}(\overline{v}) = \left(\frac{|B|}{|\Pi^* B|}\right)^{1/2n}. \tag{47}$$

Lemma 5.1 and the fact that σ is invariant with respect to componentwise conjugation show

$$\int_{\mathbb{S}^n} h_{Cu}(v)d\sigma(u) = \int_{\mathbb{S}^n} h_C(u\cdot v)d\sigma(u) = \int_{\mathbb{S}^n} h_C(\overline{u}\cdot v)d\sigma(u).$$

By the relation $\overline{u}\cdot v = \overline{v}\cdot u$, Lemma 5.1 again, and (31) we have

$$\int_{\mathbb{S}^n} h_{Cu}(v)d\sigma(u) = \int_{\mathbb{S}^n} h_C(\overline{v}\cdot u)d\sigma(u) = \int_{\mathbb{S}^n} h_{C\overline{v}}(u)d\sigma(u) = 2h_{\Pi_C B}(\overline{v}).$$

If we combine this with (47) then

$$\frac{1}{2}\int_{\mathbb{S}^n} h_{Cu}(v)d\sigma(u) = \left(\frac{|B|}{|\Pi^*B|}\right)^{1/2n}. \tag{48}$$

Now, by the polar formula for volume together with the equality $2n|B| = |\partial B|$, Jensen's inequality, integral representation (31), and Fubini's theorem

$$\begin{aligned}
\frac{|\Pi_C^*K|}{|B|} &= \frac{1}{|\partial B|}\int_{\mathbb{S}^n} h_{\Pi_C K}(u)^{-2n} d\sigma(u) \\
&\geq \left(\frac{1}{|\partial B|}\int_{\mathbb{S}^n} h_{\Pi_C K}(u) d\sigma(u)\right)^{-2n} \\
&= \left(\frac{1}{2|\partial B|}\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} h_{Cu}(v)\, dS_K(v) d\sigma(u)\right)^{-2n} \\
&= \left(\frac{1}{2|\partial B|}\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} h_{Cu}(v)\, d\sigma(u) dS_K(v)\right)^{-2n}.
\end{aligned}$$

Plugging (48) into the last term and using the fact that the total mass of S_K equals $|\partial K|$ yields the first inequality of (46). Equality holds if and only if there is equality in the above application of Jensen's inequality, i.e. $h_{\Pi_C K}$ is constant. This is equivalent to $\Pi_C K$ being a ball.

Let us now prove the second inequality of (46). By Corollary 4.4, $\Pi_C K = \Pi_{\Delta C} K$. Hence it suffices to prove

$$|\Pi_{\Delta C}^* K| \leq |\Pi^* K|.$$

But this is an immediate consequence of Lemma 5.4, since $|\mu_{\Delta C}| = 1$ by our normalization and Lemma 5.3. $\square$

If we multiply the inequalities (46) with $|K|^{2n-1}$ and apply Theorem 1.1, then

$$\left(\frac{|\partial K|}{|\partial B|}\right)^{-2n} |\Pi^*B||K|^{2n-1} \leq |\Pi_C^*K||K|^{2n-1} \leq |\Pi^*K||K|^{2n-1} \leq |B|^{2n-1}|\Pi^*B|.$$

Keeping our normalization $|\Pi_C^*B| = |\Pi^*B|$ in mind, a glance at Theorem 1.1 and (45) reveals that each complex Petty projection inequality strengthens and directly implies the isoperimetric inequality. Moreover, the classical Petty projection inequality turns out to be the strongest inequality. So roughly speaking, the more invariance properties the inequality has, the stronger it gets.

7 Some properties of the complex centroid body operator

In this section we collect some basic properties of complex centroid bodies which will be used in the proof of the complex Busemann-Petty centroid inequality. Throughout this section let $C \in \mathcal{K}(\mathbb{C})$ and $K \in \mathcal{K}_o(\mathbb{C}^n)$. We begin with rewriting the definition of Γ_C. If

K contains the origin in its interior, then a change to polar coordinates in the definition of h_{Γ_C} shows

$$h_{\Gamma_C K}(u) = \frac{1}{(2n+1)|K|} \int_{\mathbb{S}^n} h_{Cu}\, \rho_K^{2n+1}\, d\sigma \tag{49}$$

for every $u \in \mathbb{S}^n$. Next, we will show that Γ_C intertwines with the complex general linear group $\mathrm{GL}(n, \mathbb{C})$. More precisely, Γ_C turns out to be $\mathrm{GL}(n, \mathbb{C})$-covariant.

7.1 Theorem. *Let $\phi \in \mathrm{GL}(n, \mathbb{C})$. Then $\Gamma_C(\phi K) = \phi \Gamma_C K$.*

Proof. From the definition of Γ_C and relation (6) together with the transformation formula we get

$$h_{\Gamma_C(\phi K)}(u) = \frac{1}{|\phi K|} \int_{\phi K} h_{Cu}(x)\, dx = \frac{1}{|K|} \int_K h_{Cu}(\phi x)\, dx$$

for all $u \in \mathbb{S}^n$. Now (10), the equality $\phi^* Cu = C(\phi^* u)$, and the definition of Γ_C yield

$$h_{\Gamma_C(\phi K)}(u) = \frac{1}{|K|} \int_K h_{\phi^* Cu}(x)\, dx = \frac{1}{|K|} \int_K h_{C(\phi^* u)}(x)\, dx = h_{\Gamma_C K}(\phi^* u).$$

By (10) again we have $h_{\Gamma_C(\phi K)} = h_{\phi \Gamma_C K}$, and hence $\Gamma_C(\phi K) = \phi \Gamma_C K$. □

The following theorem describes the image of Γ_C on origin-symmetric balls.

7.2 Theorem. *If $\dim C > 0$, then Γ_C maps origin-symmetric balls to origin-symmetric balls.*

Proof. Let $r > 0$. From representation (49), the fact that $\rho_{rB} = r$, and (7), we get

$$h_{\Gamma_C(rB)}(u) = \frac{r}{(2n+1)|B|} \int_{\mathbb{S}^n} h_{Cu}\, d\sigma$$

for every $u \in \mathbb{S}^n$. As in the proof of Theorem 4.2 if follows that the right hand side is independent from u and, since $\dim C > 0$, greater than zero. Hence $\Gamma_C(rB)$ is an origin-symmetric ball. □

7.3 Theorem. *If K is origin-symmetric, then $\Gamma_C K = \Gamma_{\Delta C} K$.*

Proof. Since K is assumed to be origin-symmetric we have

$$h_{\Gamma_C K}(u) = \frac{1}{2|K|} \int_K h_{Cu}(x) + h_{Cu}(-x)\, dx.$$

From (10) we know that $h_{Cu}(-v) = h_{-Cu}(v)$. Thus the definition of Δ implies

$$h_{\Gamma_C K}(u) = \frac{1}{|K|} \int_K h_{\Delta(Cu)}(x)\, dx.$$

Since, by (11), $\Delta(Cu) = (\Delta C)u$, we are done. □

We conclude this section by relating Γ_C to the classical centroid body operator. Define $\Gamma K := \Gamma_{[-1,1]}K$. Let $u \in \mathbb{R}^{2n}$. By (8), the definition of $\Gamma_{[-1,1]}$, and relation $h_{[-1,1]\iota^{-1}u}(x) = |\Re[\iota^{-1}u \cdot x]|$,

$$h_{\iota\Gamma K}(u) = h_{\Gamma K}(\iota^{-1}u) = \frac{1}{|K|}\int_K h_{[-1,1]\iota^{-1}u}(x)\, dx = \frac{1}{|K|}\int_K |\Re[\iota^{-1}u \cdot x]|\, dx.$$

But by the definition of volume, equality (3), and the definition of the real centroid body

$$h_{\iota\Gamma K}(u) = \frac{1}{|\iota K|}\int_{\iota K} |\Re[\iota^{-1}u \cdot \iota^{-1}x]|\, dx = \frac{1}{|\iota K|}\int_{\iota K} |u \cdot x|\, dx = h_{\Gamma\iota K}(u).$$

Hence we arrive at

$$\iota\Gamma K = \Gamma\iota K, \tag{50}$$

which justifies our notation Γ for $\Gamma_{[-1,1]}$. Γ is actually the basis for all operators Γ_C, provided that C is 1-dimensional. This is the content of our next result.

7.4 Theorem. *If C is origin-symmetric and $\dim C = 1$, then $\Gamma_C = c\,\Gamma$ for some $c \in \mathbb{C}$.*

Proof. By our assumption, C is an origin-symmetric line segment. Thus there exists a $d \in \mathbb{C}\backslash\{0\}$ with $C = [-d, d]$ and therefore

$$h_{\Gamma_C K}(u) = \frac{1}{|K|}\int_K h_{[-1,1](du)}(x)\, dx = h_{\Gamma K}(du).$$

So (10) proves $h_{\Gamma_C K} = h_{\overline{d}\Gamma K}$. If we set $c := \overline{d}$, the assertion is proved. □

8 Proof of the complex Busemann-Petty centroid inequality

It was shown in [28] that once the real Petty projection inequality is established, the real Busemann-Petty centroid inequality can be obtained as an almost effortless consequence. We will adapt this clever argument to prove Theorem 1.2.

Let $K, L \in \mathcal{K}(\mathbb{C}^n)$ contain the origin in their interiors. The dual mixed volume $\tilde{V}(K, L)$ of K and L is defined by

$$\tilde{V}(K, L) = \frac{1}{2n}\int_{\mathbb{S}^n} \rho_K^{2n+1}\rho_L^{-1} d\sigma.$$

An application of Hölder's inequality and the polar formula for volume proves the dual Minkowski inequality

$$\tilde{V}(K, L) \geq |K|^{1+1/2n}|L|^{-1/2n}, \tag{51}$$

with equality if and only if K and L are real dilates.

The following lemma provides a connection of Π_C and Γ_C in terms of mixed volumes and their duals. For $C \subset \mathbb{C}$ we write $\bar{C} := \{\bar{c} : c \in C\}$.

8.1 Lemma. *Let $C \in \mathcal{K}(\mathbb{C})$ have $\dim C > 0$. Then*

$$V(K, \Gamma_C L) = \frac{2}{(2n+1)|L|} \tilde{V}(L, \Pi^*_{\bar{C}} K)$$

for all $K, L \in \mathcal{K}(\mathbb{C}^n)$ containing the origin in their interiors.

Proof. Representation (22) of mixed volumes and (49) prove

$$\begin{aligned} V(K, \Gamma_C L) &= \frac{1}{2n} \int_{\mathbb{S}^n} h_{\Gamma_C L}(u) dS_K(u) \\ &= \frac{1}{2n(2n+1)|L|} \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} h_{Cu}(v) \rho_L(v)^{2n+1} d\sigma(v) dS_K(u). \end{aligned}$$

It follows directly from the definition of support functions that $h_{Cu}(v) = h_{\bar{C}v}(u)$. This together with Fubini's theorem, representation (31), equality (25), and the definition of dual mixed volumes gives

$$\begin{aligned} V(K, \Gamma_C L) &= \frac{1}{2n(2n+1)|L|} \int_{\mathbb{S}^n} \rho_L(v)^{2n+1} \int_{\mathbb{S}^n} h_{\bar{C}v}(u) dS_K(u) d\sigma(v) \\ &= \frac{1}{n(2n+1)|L|} \int_{\mathbb{S}^n} \rho_L(v)^{2n+1} h_{\Pi_{\bar{C}} K}(v) d\sigma(v) \\ &= \frac{1}{n(2n+1)|L|} \int_{\mathbb{S}^n} \rho_L(v)^{2n+1} \rho_{\Pi^*_{\bar{C}} K}(v)^{-1} d\sigma(v) \\ &= \frac{2}{(2n+1)|L|} \tilde{V}(L, \Pi^*_{\bar{C}} K). \end{aligned}$$ □

Let us introduce two abbreviations which contain all terms of the complex Petty projection and Busemann-Petty centroid inequality, respectively:

$$\mathrm{pp}(C, K) = \Big(|K|^{2n-1} |\Pi^*_C K|\Big)^{-1} \Big(|B|^{2n-1} |\Pi^*_C B|\Big) \tag{52}$$

and

$$\mathrm{bpc}(C, K) = \Big(|K|^{-1} |\Gamma_C K|\Big) \Big(|B|^{-1} |\Gamma_C B|\Big)^{-1}. \tag{53}$$

Note that the complex Petty projection inequality is equivalent to $\mathrm{pp}(C, K) \geq 1$, whereas the complex Busemann-Petty inequality is equivalent to $\mathrm{bpc}(C, K) \geq 1$.

8.2 Lemma. *For $C \in \mathcal{K}(\mathbb{C})$ with $\dim C > 0$ and origin-symmetric $K \in \mathcal{K}_o(\mathbb{C}^n)$,*

$$\mathrm{bpc}(C, K) \geq \mathrm{pp}(\bar{C}, \Gamma_C K),$$

with equality if and only if K and $\Pi^*_{\bar{C}} \Gamma_C K$ are real dilates.

Proof. It is enough to prove that

$$|K|^{-1} |\Gamma_C K| \geq \left(\frac{2}{2n+1}\right)^{2n} \Big(|\Gamma_C K|^{2n-1} |\Pi^*_{\bar{C}} \Gamma_C K|\Big)^{-1} \tag{54}$$

with equality if and only if K and $\Pi^*_{\bar{C}}\Gamma_C K$ are real dilates. Indeed, assume that (54) holds. By Theorems 4.2 and 7.2, as well as the fact that the polar of an origin-symmetric ball is an origin-symmetric ball, B and $\Pi^*_{\bar{C}}\Gamma_C B$ are real dilates. So there is equality in (54) for $K = B$. This together with the definitions of bpc and pp show that is indeed enough to prove (54) along with its equality conditions.
So let us turn towards the proof of (54). Note that by assumption K is origin-symmetric and contains the origin in its interior. Since $\dim C > 0$, also $\Gamma_C K$ contains the origin in its interior. So from (19) and Lemma 8.1 we get

$$|\Gamma_C K| = V(\Gamma_C K, \Gamma_C K) = \frac{2}{(2n+1)|K|}\tilde{V}(K, \Pi^*_{\bar{C}}\Gamma_C K).$$

The dual Minkowski inequality (51) applied to the right hand side gives

$$|\Gamma_C K| \geq \frac{2}{2n+1}\Big(|K||\Pi^*_{\bar{C}}\Gamma_C K|^{-1}\Big)^{1/2n}$$

with equality if and only if K and $\Pi^*_{\bar{C}}\Gamma_C K$ are dilates. Rearranging terms yields (54). □

Before we prove the complex Busemann-Petty centroid inequality, let us state the classical real version.

8.3 Theorem (Busemann-Petty centroid inequality)**.** *Let $K \subset \mathbb{R}^{2n}$ be a convex body with nonempty interior. Then*

$$|K|^{-1}|\Gamma K| \geq |\iota B|^{-1}|\Gamma \iota B|$$

with equality if and only if K is an origin-symmetric ellipsoid.

Proof of Theorem 1.2. First, assume that $\dim C = 0$, i.e. $C = \{c\}$ for some $c \in \mathbb{C}$. Let $u \in \mathbb{S}^n$. Then $h_{Cu}(x) = \Re[x \cdot (cu)]$ for every $x \in \mathbb{C}^n$. So by the definition of complex centroid bodies and the assumption $K = -K$ together with the transformation formula

$$h_{\Gamma_C K}(u) = \frac{1}{|K|}\int_K \Re[x \cdot (cu)]\, dx = -\frac{1}{|K|}\int_K \Re[x \cdot (cu)]\, dx = -h_{\Gamma_C K}(u).$$

Thus $h_{\Gamma_C K} = 0$ and hence $\Gamma_C K = \{0\}$ for every origin-symmetric K. Consequently, inequality (2) holds trivially true.
Next, let $\dim C = 1$. By Theorem 7.3 we can assume that C is origin-symmetric. Moreover, by Theorem 7.4 and (7) it suffices to prove the assertion for Γ. But by (50), inequality (2) is equivalent to the classical Busemann-Petty centroid inequality. So Theorem 8.3 settles the case where $\dim C = 1$.
Finally, let $\dim C = 2$. From Lemma 8.2 and Theorem 1.1 we get

$$\operatorname{bpc}(C, K) \geq \operatorname{pp}(\bar{C}, \Gamma_C K) \geq 1. \tag{55}$$

By the definition of $\operatorname{bpc}(C, K)$, this immediately implies the complex Busemann-Petty centroid inequality (2).

Assume that there is equality in (2). Then there must be equality in both inequalities from (55). Thus K must be a dilate of $\Pi^*_{\bar{C}}\Gamma_C K$ and $\Gamma_C K$ must be an Hermitian ellipsoid. So there exists a $\phi \in \mathrm{GL}(n, \mathbb{C})$ and a $t \in \mathbb{C}^n$ with $\Gamma_C K = \phi B + t$. From Theorems 4.1 and 4.2 as well as (23) we deduce that $\Pi^*_{\bar{C}}\Gamma_C K$ is an origin-symmetric Hermitian ellipsoid. So K, being a dilate of $\Pi^*_{\bar{C}}\Gamma_C K$, is an origin-symmetric Hermitian ellipsoid as well.
It remains to show that the equality condition are also sufficient. So assume that K is an origin-symmetric Hermitian ellipsoid, i.e. $K = \phi B$ for some $\phi \in \mathrm{GL}(n, \mathbb{C})$. This, Theorem 7.1, and (6) prove

$$|K|^{-1}|\Gamma_C K| = |\phi B|^{-1}|\Gamma_C \phi B| = |\phi B|^{-1}|\phi \Gamma_C B| = |B|^{-1}|\Gamma_C B|. \qquad \square$$

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