A Characterization of L_p Intersection Bodies

Christoph Haberl and Monika Ludwig

Dedicated to Prof. Peter M. Gruber on the occasion of his sixty-fifth birthday

Abstract

All $\mathrm{GL}(n)$ covariant L_p radial valuations on convex polytopes are classified for every $p > 0$. It is shown that for $0 < p < 1$ there is a unique non-trivial such valuation with centrally symmetric images. This establishes a characterization of L_p intersection bodies.

2000 AMS subject classification: 52A20 (52B11, 52B45)

1 Introduction

Let $L \subset \mathbb{R}^n$ be a star body, that is, a compact set which is star-shaped with respect to the origin and has a continuous radial function, $\rho(L, u) = \max\{r \geq 0 : r\, u \in L\}$, $u \in S^{n-1}$. The *intersection body*, $\mathrm{I}\, L$, of L is the star body whose radial function in the direction $u \in S^{n-1}$ is equal to the $(n-1)$-dimensional volume of the section of L by $u^\perp$, the hyperplane orthogonal to u. So, for $u \in S^{n-1}$,

$$\rho(\mathrm{I}\, L, u) = \mathrm{vol}(L \cap u^\perp),$$

where vol denotes $(n-1)$-dimensional volume.

Intersection bodies which arise from centrally symmetric convex bodies first appeared in Busemann [5]. They are important in the theory of area in Finsler spaces. Intersection bodies of star bodies were defined and named by Lutwak [27]. The class of intersection bodies (as defined in [27]) turned out to be critical for the solution of the Busemann-Petty problem (see [7], [9], [39]) and are fundamental in geometric tomography (see e.g. [8]), in affine isoperimetric inequalities (see e.g. [19], [36]) and the geometry of Banach spaces (see e.g. [18], [37]).

Valuations allow us to obtain characterizations of many important functionals and operators on convex sets by their invariance or covariance properties with respect to suitable groups of transformations (see [12], [16], [33], [34] for information on the classical theory and [1]–[4], [14], [15], [20]–[22], [25] for some of the recent results). Here a function $\mathrm{Z} : \mathcal{L} \to \langle \mathcal{G}, + \rangle$, where $\mathcal{L}$ is a class of subsets of $\mathbb{R}^n$ and $\langle \mathcal{G}, + \rangle$ is an abelian semigroup, is called a *valuation* if

$$\mathrm{Z}\, K + \mathrm{Z}\, L = \mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L),$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{L}$.

In [24], intersection bodies were characterized as $\mathrm{GL}(n)$ covariant valuations. To state this result, we need some additional definitions. Let $\mathcal{P}_0^n$ denote the set of convex polytopes in $\mathbb{R}^n$ that contain the origin in their interiors and let $P^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for every } y \in P\}$ denote the polar body of $P \in \mathcal{P}_0^n$. We write $\mathcal{S}^n$ for the set of star bodies in $\mathbb{R}^n$. For $p > 0$, the *L_p-radial sum* $K \tilde{+}_p L$ of $K, L \in \mathcal{S}^n$ is defined by

$$\rho(K \tilde{+}_p L, \cdot)^p = \rho(K, \cdot)^p + \rho(L, \cdot)^p.$$

An operator $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$ is called *trivial*, if it is a linear combination with respect to $\tilde{+}_p$ of the identity and central reflection. An operator Z is called $\mathrm{GL}(n)$ *covariant of weight q*, $q \in \mathbb{R}$, if for all $\phi \in \mathrm{GL}(n)$ and all bodies Q,

$$\mathrm{Z}(\phi Q) = |\det \phi|^q \, \phi \, \mathrm{Z}\, Q,$$

where $\det \phi$ is the determinant of ϕ. An operator Z is called $\mathrm{GL}(n)$ *covariant*, if Z is $\mathrm{GL}(n)$ covariant of weight q for some $q \in \mathbb{R}$.

Theorem ([24])**.** *An operator $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_1 \rangle$ is a non-trivial $\mathrm{GL}(n)$ covariant valuation if and only if there is a constant $c \geq 0$ such that*

$$\mathrm{Z}\, P = c \, \mathrm{I}\, P^*$$

for every $P \in \mathcal{P}_0^n$.

This theorem establishes a classification of $\mathrm{GL}(n)$ covariant valuations within the dual Brunn-Minkowski theory. Intersection bodies were introduced in [27] as an analogue in this dual theory of the classical projection bodies in the Brunn-Minkowski theory. In recent years, the Brunn-Minkowski theory was extended using Firey's L_p Minkowski addition (see [28], [29]). In particular, Lutwak, Yang, and Zhang introduced L_p projection bodies and obtained important affine isoperimetric inequalities (see [30], [31]). In [23], a valuation theoretic characterization of L_p projection bodies was obtained.

Here we ask the corresponding question within the dual Brunn-Minkowski theory. The notion corresponding to L_p Minkowski addition is L_p radial addition. So we ask for a classification of L_p radial valuations. A complete answer for the planar case is given in Theorem 3 in Section 3.3. For $n \geq 3$, we obtain the following result.

Theorem 1. *For $0 < p < 1$, an operator $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$ is a non-trivial $\mathrm{GL}(n)$ covariant valuation if and only if there are constants $c_1, c_2 \geq 0$ such that*

$$\mathrm{Z}\, P = c_1 \, \mathrm{I}_p^+ \, P^* \, \tilde{+}_p \, c_2 \, \mathrm{I}_p^- \, P^*$$

for every $P \in \mathcal{P}_0^n$. For $p > 1$, all $\mathrm{GL}(n)$ covariant valuations $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$ are trivial.

Here, for $Q \in \mathcal{P}_0^n$, the star body $\mathrm{I}_p^+ \, Q$ is defined for $u \in S^{n-1}$ by

$$\rho(\mathrm{I}_p^+ \, Q, u)^p = \int_{Q \cap u^+} |u \cdot x|^{-p} dx,$$

where $u^+ = \{x \in \mathbb{R}^n : u \cdot x \geq 0\}$. We define $\mathrm{I}_p^- \, Q = \mathrm{I}_p^+(-Q)$.

As a consequence, we obtain the following characterization of L_p intersection bodies. For $p < 1$, we call the centrally symmetric star body $\mathrm{I}_p\, Q = \mathrm{I}_p^+\, Q \,\tilde{+}_p\, \mathrm{I}_p^-\, Q$ the L_p *intersection body* of $Q \in \mathcal{P}_0^n$. So, for $u \in S^{n-1}$,

$$\rho(\mathrm{I}_p\, Q, u)^p = \int_Q |u \cdot x|^{-p} dx. \tag{1}$$

Since

$$\mathrm{vol}(Q \cap u^\perp) = \lim_{\varepsilon \to 0+} \frac{\varepsilon}{2} \int_Q |u \cdot x|^{-1+\varepsilon} dx$$

(cf. [18], p. 9),

$$\rho(\mathrm{I}\, Q, u) = \lim_{p \to 1-} \frac{1-p}{2}\, \rho(\mathrm{I}_p\, Q, u)^p,$$

that is, the intersection body of Q is obtained as a limit of L_p intersection bodies of Q. Also note that a change to polar coordinates in (1) shows that up to a normalization factor $\rho(\mathrm{I}_p\, Q, u)^p$ equals the L_p cosine transform of $\rho(Q, \cdot)^{n-p}$.

We denote by $\mathcal{S}_c^n$ the set of centrally symmetric star bodies in $\mathbb{R}^n$ and classify $\mathrm{GL}(n)$ covariant valuations $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}_c^n, \tilde{+}_p \rangle$. The planar case is contained in Theorem 4 in Section 3.3. For $n \geq 3$, we obtain the following result.

Theorem 2. *For $0 < p < 1$, an operator $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}_c^n, \tilde{+}_p \rangle$ is a non-trivial $\mathrm{GL}(n)$ covariant valuation if and only if there is a constant $c \geq 0$ such that*

$$\mathrm{Z}\, P = c\ \mathrm{I}_p\, P^*$$

for every $P \in \mathcal{P}_0^n$. For $p > 1$, all $\mathrm{GL}(n)$ covariant valuations $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}_c^n, \tilde{+}_p \rangle$ are trivial.

Up to multiplication with a suitable power of the volume of Q, the L_p intersection body of Q is just the polar L_q centroid body of Q. For $q > 1$, L_q centroid bodies were introduced by Lutwak and Zhang [32]. They led to important affine isoperimetric inequalities (see [6], [10], [30], [32]). Yaskin and Yaskina [38] introduced polar L_q centroid bodies for $-1 < q < 1$ and solved the corresponding Busemann-Petty problem. For applications connected with embeddings in L_q spaces, see [13], [17], and [35]. For a detailed discussion of the operators I_p^+ and I_p, we refer to [11].

2 Notation and Preliminaries

We work in n-dimensional Euclidean space $\mathbb{R}^n$ and write $x = (x_1, x_2, \ldots, x_n)$ for vectors $x \in \mathbb{R}^n$. The standard basis in $\mathbb{R}^n$ will be denoted by $e_1, e_2, \ldots, e_n$. We use $x \cdot y$ to denote the usual scalar product $x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$ of two vectors $x, y \in \mathbb{R}^n$, and define the norm $\|x\| = \sqrt{x \cdot x}$. The unit sphere $\{x \in \mathbb{R}^n : \|x\| = 1\}$ is denoted by S^{n-1}. Given $A, A_1, A_2, \ldots, A_k \subset \mathbb{R}^n$, we write $[A_1, A_2, \ldots, A_k]$ for the convex hull of $A_1, A_2, \ldots, A_k$, we write $\mathrm{lin}\, A$ for the linear hull of A, and set $A^\perp = \{x \in \mathbb{R}^n : x \cdot y = 0 \text{ for all } y \in A\}$.

For $L \in \mathcal{S}^n$, we extend the radial function to a homogeneous function defined on $\mathbb{R}^n \backslash \{0\}$ by $\rho(L, x) = \|x\|^{-1} \rho(L, x/\|x\|)$. Then it follows immediately from the definition that

$$\rho(\phi L, x) = \rho(L, \phi^{-1} x), \qquad x \in \mathbb{R}^n \backslash \{0\} \tag{2}$$

for $\phi \in \mathrm{GL}(n)$.

We call a valuation $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$ an L_p *radial valuation*. A valuation Z with $\mathrm{Z}\, P = \{0\}$ for every P having dimension less than n is called *simple*. A valuation is called $\mathrm{GL}(n)$ *contravariant of weight* q, $q \in \mathbb{R}$, if

$$\mathrm{Z}\, \phi P = |\det \phi|^q \phi^{-t}\, \mathrm{Z}\, P$$

for every $\phi \in \mathrm{GL}(n)$ and every $P \in \mathcal{P}_0^n$. Here ϕ^{-t} denotes the transpose of the inverse of ϕ. For $0 < p < 1$, the operators $\mathrm{I}_p^{\pm} : \mathcal{P}_0^n \to \mathcal{S}^n$ and $\mathrm{I}_p : \mathcal{P}_0^n \to \mathcal{S}^n$ are L_p radial valuations and $\mathrm{GL}(n)$ contravariant operators of weight $1/p$.

The following lemma guarantees that a classification of all L_p radial valuations which are $\mathrm{GL}(n)$ covariant with negative weight follows from a classification of all L_p radial valuations which are $\mathrm{GL}(n)$ contravariant with positive weight. Moreover, if we know all L_p radial valuations, $\mathrm{GL}(n)$ covariant of arbitrary weight, we know all $\mathrm{GL}(n)$ contravariant L_p radial valuations and vice versa.

Lemma 1. *Let* Z *be an* L_p *radial valuation and define another* L_p *radial valuation* Z^* *by* $\mathrm{Z}^*\, P = \mathrm{Z}\, P^*$ *for every* $P \in \mathcal{P}_0^n$*. Then* Z *is* $\mathrm{GL}(n)$ *covariant of weight* q *if and only if* Z^* *is* $\mathrm{GL}(n)$ *contravariant of weight* $-q$*.*

Proof. That Z^* satisfies the valuation property is a consequence of

$$(P \cup Q)^* = P^* \cap Q^*, \quad (P \cap Q)^* = P^* \cup Q^*$$

for polytopes $P, Q \in \mathcal{P}_0^n$ having convex union (see, for example, [36]). The statement of the lemma follows from the fact that $(\phi P)^* = \phi^{-t} P^*$ holds for every $P \in \mathcal{P}_0^n$ and every $\phi \in \mathrm{GL}(n)$. □

2.1 Extension

Given an L_p radial valuation Z, we define another valuation $\mathrm{Y} : \mathcal{P}_0^n \to \mathcal{C}_+(S^{n-1})$ by $\mathrm{Y}\, P(\cdot) = \rho(\mathrm{Z}\, P, \cdot)^p$. Here $\mathcal{C}_+(S^{n-1})$ is the set of non-negative continuous functions on the sphere. We want to extend this valuation to the set $\overline{\mathcal{P}}_0^n$ of convex polytopes which are either in $\mathcal{P}_0^n$ or are the intersection of a polytope in $\mathcal{P}_0^n$ and a polyhedral cone with at most n facets having its apex at the origin. The following preparations will show when such extensions exist. For $1 \leq j \leq n$, let $\overline{\mathcal{P}}_j^n$ denote the set of polytopes which are intersections of polytopes in $\mathcal{P}_0^n$ and j halfspaces bounded by hyperplanes $H_1, \dots, H_j$ containing the origin and having linearly independent normals. We need some more notation. For a hyperplane $H \subset \mathbb{R}^n$, $\mathcal{P}_0^n(H)$ is the set of convex polytopes in H containing the origin in their interiors relative to H. Let $\overline{\mathcal{C}}_+(S^{n-1})$ denote the superset of $\mathcal{C}_+(S^{n-1})$ consisting of all non-negative functions defined almost everywhere (with respect to spherical Lebesgue measure) on S^{n-1} which are continuous almost everywhere. We write H^+, H^- for the closed halfspaces bounded by H.

For $P \in \mathcal{P}_0^n(H)$ and $A \subset S^{n-1}$, we say that $\mathrm{Y} : \mathcal{P}_0^n \to \mathcal{C}_+(S^{n-1})$ *vanishes on* A *at* P if for $u \in H^- \backslash H$, $v \in H^+ \backslash H$ and every $w \in A$, there exists a neighbourhood $A(w)$ of w such that

$$\lim_{u,v \to 0} \mathrm{Y}[P, u, v] = 0 \quad \text{uniformly on } A(w)$$

holds. If there exists a constant $c \in \mathbb{R}$ such that $\mathrm{Y}[P,u,v] \leq c$ for $\|u\|, \|v\| \leq 1$, $u \in H^-\backslash H$, $v \in H^+\backslash H$ and $[P,u,v] = [P,u] \cup [P,v]$, then we say that $\mathrm{Y} : \mathcal{P}_0^n \to \mathcal{C}_+(S^{n-1})$ is *bounded at* P.

Now we are able to formulate the following lemma proved in [24].

Lemma 2. *Let* $\mathrm{Y} : \mathcal{P}_0^n \to \mathcal{C}_+(S^{n-1})$ *be a valuation.*

1. *If* Y *vanishes on* S^{n-1} *at* P *for every hyperplane* H *containing the origin and every* $P \in \mathcal{P}_0^n(H)$*, then* Y *can be extended to a simple valuation* $\overline{\mathrm{Y}} : \overline{\mathcal{P}}_0^n \to \mathcal{C}_+(S^{n-1})$*.*

2. *If* Y *is bounded and vanishes on* $S^{n-1}\backslash H$ *at* P *for every hyperplane* H *containing the origin and every* $P \in \mathcal{P}_0^n(H)$*, then* Y *can be extended to a simple valuation* $\overline{\mathrm{Y}} : \overline{\mathcal{P}}_0^n \to \overline{\mathcal{C}}_+(S^{n-1})$ *and for* $P \in \overline{\mathcal{P}}_0^n$ *bounded by hyperplanes* $H_1, H_2, \ldots, H_n$ *containing the origin,* $\overline{\mathrm{Y}}\,P$ *is continuous and bounded on* $S^{n-1}\backslash(H_1 \cup \cdots \cup H_n)$*.*

3. *If* Y *is bounded and vanishes on* $S^{n-1}\backslash H^\perp$ *at* P *for every hyperplane* H *containing the origin and every* $P \in \mathcal{P}_0^n(H)$*, then* Y *can be extended to a simple valuation* $\overline{\mathrm{Y}} : \overline{\mathcal{P}}_0^n \to \overline{\mathcal{C}}_+(S^{n-1})$ *and for* $P \in \overline{\mathcal{P}}_0^n$ *bounded by hyperplanes* $H_1, H_2, \ldots, H_n$ *containing the origin,* $\overline{\mathrm{Y}}\,P$ *is continuous and bounded on* $S^{n-1}\backslash(H_1^\perp \cup \cdots \cup H_n^\perp)$*.*

4. *If* Y *vanishes on* $S^{n-1}\backslash H^\perp$ *at* P *for every hyperplane* H *containing the origin and every* $P \in \mathcal{P}_0^n(H)$*, then* Y *can be extended to a simple valuation* $\overline{\mathrm{Y}} : \overline{\mathcal{P}}_0^n \to \overline{\mathcal{C}}_+(S^{n-1})$ *and for* $P \in \overline{\mathcal{P}}_0^n$ *bounded by hyperplanes* $H_1, H_2, \ldots, H_n$ *containing the origin,* $\overline{\mathrm{Y}}\,P$ *is continuous on* $S^{n-1}\backslash(H_1^\perp \cup \cdots \cup H_n^\perp)$*.*

The extension is defined inductively for $j = 1, \ldots, n$, and convex polytopes $P = P_0 \cap H_1^+ \cap \cdots \cap H_j^+$ with $P_0 \in \mathcal{P}_0^n$ and hyperplanes having linearly independent normals: For $u \in H_1 \cap \cdots \cap H_{j-1}$, $u \in H_j^-\backslash H$, set

$$\overline{\mathrm{Y}}\,P = \lim_{u \to 0} \overline{\mathrm{Y}}[P,u]$$

on S^{n-1}, $S^{n-1}\backslash(H_1 \cup \cdots \cup H_j)$ or $S^{n-1}\backslash(H_1^\perp \cup \cdots \cup H_j^\perp)$ if Y vanishes on S^{n-1}, $S^{n-1}\backslash H$ or $S^{n-1}\backslash H^\perp$, respectively.

The proof of the following lemma is omitted since it is nearly the same as the proof of Lemma 5 and Lemma 8 in [24].

Lemma 3. *Let* $\mathrm{Z} : \mathcal{P}_0^n \to \mathcal{S}^n$ *be an* L_p *radial valuation and define* $\mathrm{Y} : \mathcal{P}_0^n \to \mathcal{C}_+(S^{n-1})$ *by* $\mathrm{Y}\,P(\cdot) = \rho(\mathrm{Z}\,P, \cdot)^p$*.*

1. *If* Z *is* $\mathrm{GL}(n)$ *covariant of weight* q*, then for every hyperplane* H *containing the origin and every* $P \in \mathcal{P}_0(H)$*, the following holds: If* $q = 0$*, then* Y *vanishes on* $S^{n-1}\backslash H$ *at* P *and if* $q > 0$*, then* Y *vanishes on* S^{n-1} *at* P*. In both cases,* Y *is bounded at* P*.*

2. *If* Z *is* $\mathrm{GL}(n)$ *contravariant of weight* q*, then for every hyperplane* H *containing the origin and every* $P \in \mathcal{P}_0(H)$*, the following holds: If* $q > 0$*, then* Y *vanishes on* $S^{n-1}\backslash H^\perp$ *at* P *and if* $q > 1$*, then* Y *vanishes on* S^{n-1} *at* P*. For* $q \geq 1$*,* Y *is bounded at* P*.*

Let Z be an L_p radial valuation which is $\mathrm{GL}(n)$ contravariant of weight q. For $q > 0$, Lemma 2 and Lemma 3 guarantee the existence of an extension of $\mathrm{Y}\,P(\cdot) = \rho(\mathrm{Z}\,P,\cdot)^p$ to $\overline{\mathcal{P}}_0^n$ for which we write $\overline{\mathrm{Y}}$. We extend these functions from S^{n-1} to $\mathbb{R}^n\backslash\{0\}$ by making them homogeneous of degree $-p$. From the definition of this extension it follows for $\phi \in \mathrm{GL}(n)$ and $P \in \overline{\mathcal{P}}_0^n$ bounded by hyperplanes $H_1, H_2, \ldots, H_j$ that

$$\overline{\mathrm{Y}}\,\phi P(x) = |\det\phi|^{pq}\,\overline{\mathrm{Y}}\,P(\phi^t x) \tag{3}$$

on $S^{n-1}\backslash\phi^{-t}(H_1^\perp \cup \cdots \cup H_j^\perp)$ for $0 < q \le 1$ and on S^{n-1} for $q > 1$.

If Z is an L_p radial valuation which is $\mathrm{GL}(n)$ covariant of weight q, we proceed as above. For $q \ge 0$, Lemma 2 and Lemma 3 guarantee the existence of an extension of $\mathrm{Y}\,P(\cdot) = \rho(\mathrm{Z}\,P,\cdot)^p$ to $\overline{\mathcal{P}}_0^n$ for which we write $\overline{\mathrm{Y}}$ and which we extend from S^{n-1} to $\mathbb{R}^n\backslash\{0\}$ by making it homogeneous of degree $-p$. From the definition of this extension it follows for $\phi \in \mathrm{GL}(n)$ and $P \in \overline{\mathcal{P}}_0^n$ bounded by hyperplanes $H_1, H_2, \ldots, H_j$ that

$$\overline{\mathrm{Y}}\,\phi P(x) = |\det\phi|^{pq}\,\overline{\mathrm{Y}}\,P(\phi^{-1} x) \tag{4}$$

on $S^{n-1}\backslash\phi(H_1 \cup \cdots \cup H_j)$ for $q = 0$ and on S^{n-1} for $q > 0$.

3 Proof of the Classification Results

We first establish a classification of valuations which are $\mathrm{GL}(n)$ contravariant of weight $q > 0$ and then a classification of valuations which are $\mathrm{GL}(n)$ covariant of weight $q \ge 0$. By Lemma 1, combining these results gives a classification of $\mathrm{GL}(n)$ covariant valuations. The classification result for $n \ge 3$ is contained in Theorem 1. The result for $n = 2$ is stated in Section 3.3.

Let $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$ be a valuation which is $\mathrm{GL}(n)$ contravariant of weight $q > 0$ and let $\overline{\mathrm{Z}}$ denote its extension to $\overline{\mathcal{P}}_0^n$. Let $T^n = [0, e_1, \ldots, e_n]$ be the standard simplex in $\mathbb{R}^n$. First, we show that $\overline{\mathrm{Z}}$ is determined on $\overline{\mathcal{P}}_0^n$ by its value on T^n.

Since $\overline{\mathrm{Z}}$ is a simple valuation on $\overline{\mathcal{P}}_0^n$, it suffices to show the statement for a polytope $P \in \overline{\mathcal{P}}_0^n$ contained in a simplicial cone C bounded by n hyperplanes containing the origin and with linearly independent normal vectors. We dissect $P = T_1 \cup \cdots \cup T_k$, where $T_i \in \overline{\mathcal{P}}_0^n$ are n-dimensional simplices with pairwise disjoint interiors. Let H be a suitable affine hyperplane such that $D = C \cap H$ and $S_i = T_i \cap H$ are $(n-1)$-dimensional simplices. We need the following notions (see [26]). A finite set of $(n-1)$-dimensional simplices αD is called a *triangulation* of D if the simplices have pairwise disjoint interiors and their union equals D. An *elementary move* applied to αD is one of the two following operations: a simplex $S \in \alpha D$ is dissected into two $(n-1)$-dimensional simplices S_1, S_2 by an $(n-2)$-dimensional plane containing an $(n-3)$-dimensional face of S; or the reverse, that is, two simplices $S_1, S_2 \in \alpha D$ are replaced by $S = S_1 \cup S_2$ if S is again a simplex. It is shown in [26] that for every triangulation αD there are finitely many elementary moves that transform αD into the trivial triangulation $\{D\}$. Note that to each $(n-1)$-dimensional simplex $S \in \alpha D$, there corresponds a polytope $Q \in \overline{\mathcal{P}}_0^n$ such that $Q \cap H = S$. If S is dissected by an $(n-2)$-dimensional plane $E \subset H$ corresponding to an elementary move into S_1, S_2, then Q is dissected by the cone generated by E into $Q_1, Q_2 \in \overline{\mathcal{P}}_0^n$. Since $\overline{\mathrm{Z}}$ is a simple valuation on $\overline{\mathcal{P}}_0^n$, we obtain $\overline{\mathrm{Z}}\,Q = \overline{\mathrm{Z}}\,Q_1 \,\tilde{+}_p\, \overline{\mathrm{Z}}\,Q_2$. The same argument applies for the reverse move. Thus, after finitely many steps, we obtain that $\overline{\mathrm{Z}}\,P = \overline{\mathrm{Z}}\,T_1 \tilde{+}_p \cdots \tilde{+}_p \overline{\mathrm{Z}}\,T_k$. Since $\overline{\mathrm{Z}}$ is $\mathrm{GL}(n)$ contravariant, this proves that $\overline{\mathrm{Z}}$ is determined on $\overline{\mathcal{P}}_0^n$ by $\overline{\mathrm{Z}}\,T^n$.

We set

$$f(x) = \rho(\overline{\mathrm{Z}}\,T^n, x)^p$$

almost everywhere on $\mathbb{R}^n$. Since Z is $\mathrm{GL}(n)$ contravariant and T^n does not change when the coordinates are permutated, we obtain

$$f(x_1, \ldots, x_n) = f(x_{k_1}, \ldots, x_{k_n}) \tag{5}$$

for every permutation $(k_1, \ldots, k_n)$ of $(1, \ldots, n)$. We derive a family of functional equations for f.

For $0 < \lambda_j < 1$ and $j = 2, 3, \ldots, n$, we define two families of linear maps by

$$\begin{aligned} \phi_j e_j &= \lambda_j e_j + (1-\lambda_j) e_1, \qquad & \phi_j e_k &= e_k \text{ for } k \neq j, \\ \psi_j e_1 &= \lambda_j e_j + (1-\lambda_j) e_1, \qquad & \psi_j e_k &= e_k \text{ for } k \neq 1. \end{aligned}$$

Note that

$$\begin{aligned} \phi_j^{-1} e_j &= \frac{1}{\lambda_j} e_j - \frac{1-\lambda_j}{\lambda_j} e_1, \qquad & \phi_j^{-1} e_k &= e_k \text{ for } k \neq j, \\ \psi_j^{-1} e_1 &= -\frac{\lambda_j}{1-\lambda_j} e_j + \frac{1}{1-\lambda_j} e_1, \qquad & \psi_j^{-1} e_k &= e_k \text{ for } k \neq 1. \end{aligned}$$

Let H_j be the hyperplane through 0 with normal vector $\lambda_j e_1 - (1-\lambda_j) e_j$. Then we have $T^n \cap H_j^+ = \phi_j T^n$ and $T^n \cap H_j^- = \psi_j T^n$. Since $\overline{\mathrm{Z}}$ is a simple valuation, it follows that

$$\overline{\mathrm{Z}}\,T^n = \overline{\mathrm{Z}}(\phi_j T^n) \,\tilde{+}_p\, \overline{\mathrm{Z}}(\psi_j T^n).$$

Since $\overline{\mathrm{Z}}$ is $\mathrm{GL}(n)$ contravariant, this and (3) imply

$$f(x) = \lambda_j^{pq} f(\phi_j^t x) + (1-\lambda_j)^{pq} f(\psi_j^t x) \tag{6}$$

almost everywhere on $\mathbb{R}^n$ where the set of exception depends on the value of q.

Similar observations can be made if the valuation Z is $\mathrm{GL}(n)$ covariant of weight $q \geq 0$. Then we have by (4)

$$f(x) = \lambda_j^{pq} f(\phi_j^{-1} x) + (1-\lambda_j)^{pq} f(\psi_j^{-1} x) \tag{7}$$

almost everywhere. Note that (5) holds in the covariant case, too.

3.1 The 2-dimensional Contravariant Case

Lemma 4. *Let* $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}^2, \tilde{+}_p \rangle$ *be a valuation which is* $\mathrm{GL}(2)$ *contravariant of weight* $q = 1$. *Then there exists a constant* $c \geq 0$ *such that*

$$\mathrm{Z}\,P = c\,\psi_{\frac{\pi}{2}}(P \,\tilde{+}_p\, (-P))$$

for every $P \in \mathcal{P}_0^2$, *where* $\psi_{\frac{\pi}{2}}$ *denotes the rotation by an angle* $\frac{\pi}{2}$.

Proof. Since $\rho(P\cup Q,\cdot)=\max\{\rho(P,\cdot),\rho(Q,\cdot)\}$, $\rho(P\cap Q,\cdot)=\min\{\rho(P,\cdot),\rho(Q,\cdot)\}$, (2) implies that the function $P\mapsto c\psi_{\frac{\pi}{2}}(P\,\tilde{+}_p\,(-P))$ is in fact an L_p radial valuation. Since

$$\psi_{\frac{\pi}{2}}\phi\psi_{\frac{\pi}{2}}^{-1}=(\det\phi)\phi^{-t} \tag{8}$$

holds for every $\phi\in\mathrm{GL}(2)$, we obtain by using (2)

$$\begin{aligned}\rho(c\psi_{\frac{\pi}{2}}(\phi P\,\tilde{+}_p\,(-\phi P)),x)^p &= c^p\rho((\det\phi)P,\psi_{\frac{\pi}{2}}^{-1}\phi^tx)^p+c^p\rho(-(\det\phi)P,\psi_{\frac{\pi}{2}}^{-1}\phi^tx)^p\\ &= c^p\rho(|\det\phi|P,\psi_{\frac{\pi}{2}}^{-1}\phi^tx)^p+c^p\rho(-|\det\phi|P,\psi_{\frac{\pi}{2}}^{-1}\phi^tx)^p\\ &= \rho(|\det\phi|\phi^{-t}c\psi_{\frac{\pi}{2}}(P\,\tilde{+}_p\,(-P)),x)^p.\end{aligned}$$

This proves the contravariance of weight 1.

From Lemma 2, Lemma 3, and (6) we know that

$$f(x)=\lambda_2^pf(\phi_2^tx)+(1-\lambda_2)^pf(\psi_2^tx) \tag{9}$$

holds for every $x\in\mathbb{R}^2$ which does not lie in the linear hull of e_1, e_2 or $\lambda_2e_1-(1-\lambda_2)e_2$. Thus it follows by induction that for $k=1,2,\dots$,

$$f((\psi_2^{-t})^kx)=\lambda_2^p\sum_{i=1}^{k}(1-\lambda_2)^{p(k-i)}f(\phi_2^t(\psi_2^{-t})^ix)+(1-\lambda_2)^{kp}f(x) \tag{10}$$

holds on $\mathbb{R}^2$ except on a set consisting of countably many lines. For suitable $\varepsilon>0$, we can evaluate (10) at $x=e_1-\varepsilon e_2$. From this we obtain, using the homogeneity and the non-negativity of f, that

$$f(e_1-(1-\lambda_2)^k\varepsilon(\psi_2^{-t})^ke_2)\geq\lambda_2^p\sum_{i=1}^{k}f(\phi_2^t(e_1-(1-\lambda_2)^i\varepsilon(\psi_2^{-t})^ie_2)). \tag{11}$$

Note that $(\psi_2^{-t})^ke_2=-\lambda_2\sum_{i=0}^{k-1}(1-\lambda_2)^{i-k}e_1+e_2$. Thus $\|e_1-(1-\lambda_2)^k\varepsilon(\psi_2^{-t})^ke_2\|\geq 1$. Let $k\to\infty$ in (11). By Lemma 2, f is uniformly bounded on $S^1\backslash\{\pm e_1,\pm e_2\}$. So $f(\phi_2^t(e_1-(1-\lambda_2)^i\varepsilon(\psi_2^{-t})^ie_2))\to 0$ as $i\to\infty$. It follows from the continuity properties of f, that $f((1+\varepsilon)(e_1+(1-\lambda_2)e_2))=0$. Taking the limit $\varepsilon\to 0$, we obtain

$$f(1,x_2)=0,\qquad\text{for }\ 0<x_2<1. \tag{12}$$

By (5), this implies

$$f(x_1,1)=0,\qquad\text{for }\ 0<x_1<1. \tag{13}$$

Relations (12), (13), and the homogeneity of f imply

$$f(x_1,x_2)=0,\qquad\text{for }\ x_1,x_2>0. \tag{14}$$

By evaluating (10) at $-e_1-\varepsilon e_2$ we get in a similar way

$$f(-x_1,-x_2)=0,\qquad\text{for }\ x_1,x_2>0. \tag{15}$$

Formula (9) gives

$$f(-1,1)=\lambda_2^pf(-1,-1+2\lambda_2)+(1-\lambda_2)^pf(-1+2\lambda_2,1).$$

In combination with (14) and (15) we obtain

$$
\begin{aligned}
f(-1,1) &= \lambda_2^p f(-1,-1+2\lambda_2) \quad \text{for} \quad \frac{1}{2} < \lambda_2 < 1, \\
f(-1,1) &= (1-\lambda_2)^p f(-1+2\lambda_2,1) \quad \text{for} \quad 0 < \lambda_2 < \frac{1}{2}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
f(-1,x_2) &= \frac{c^p}{(1+x_2)^p} \quad \text{for} \quad 0 < x_2 < 1, \\
f(-x_1,1) &= \frac{c^p}{(1+x_1)^p} \quad \text{for} \quad 0 < x_1 < 1,
\end{aligned}
$$

with $c^p = 2^p f(-1,1)$. Since f is homogeneous of degree $-p$, we get

$$
f(-x_1,x_2) = \frac{c^p}{(x_1+x_2)^p} \qquad \text{for } x_1,x_2 > 0,
$$

and by (5)

$$
f(x_1,-x_2) = \frac{c^p}{(x_1+x_2)^p} \qquad \text{for } x_1,x_2 > 0.
$$

Combining these results finally yields

$$
f(x) = c^p \rho(\psi_{\frac{\pi}{2}} T^2, x)^p + c^p \rho(\psi_{\frac{\pi}{2}}(-T^2), x)^p
$$

almost everywhere on $\mathbb{R}^2$. □

For given $p,q \in \mathbb{R}$, we define the function $g_{p,q}$ on $\mathbb{R}^2$ by

$$
g_{p,q}(x_1,x_2) = \begin{cases} (x_1^{pq-p} - x_2^{pq-p})/(x_1-x_2)^{pq} & \text{for } 0 \le x_2 < x_1, \\ x_1^{pq-p}/(x_1-x_2)^{pq} & \text{for } x_1 > 0, x_2 < 0, \\ 0 & \text{otherwise.} \end{cases}
$$

Define the linear transformations γ_i, $i = 0,1,2$, by

$$
\gamma_0(x_1,x_2) = (-x_1,-x_2),\ \gamma_1(x_1,x_2) = (x_2,x_1),\ \gamma_2(x_1,x_2) = (-x_2,-x_1),
$$

that is, γ_0, γ_1, and γ_2 are the reflections with respect to the origin, the first median, and the second median, respectively.

Lemma 5. *Let $f : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}$ be a function positively homogeneous of degree $-p$ such that*

$$
f(x_1,x_2) = \lambda^{pq} f(x_1,(1-\lambda)x_1 + \lambda x_2) + (1-\lambda)^{pq} f((1-\lambda)x_1 + \lambda x_2, x_2) \tag{16}
$$

holds on $\mathbb{R}^2 \backslash \{0\}$ for every $0 < \lambda < 1$. Then

$$
f = f(1,0)\, g_{p,q} + f(-1,0)\, g_{p,q} \circ \gamma_0 + f(0,1)\, g_{p,q} \circ \gamma_1 + f(0,-1)\, g_{p,q} \circ \gamma_2 \tag{17}
$$

on $\mathbb{R}^2 \backslash \{(x_1,x_2) : x_1 = x_2\}$.

Proof. Equation (16) evaluated at the points $\pm(1,0)$, $\pm(0,1)$, $\pm(-\lambda, 1-\lambda)$ and the homogeneity of f yield

$$\begin{aligned}
f(1,1-\lambda) &= \frac{1-(1-\lambda)^{pq-p}}{\lambda^{pq}} f(1,0), & (18)\\
f(-1,\lambda-1) &= \frac{1-(1-\lambda)^{pq-p}}{\lambda^{pq}} f(-1,0), & (19)\\
f(\lambda,1) &= \frac{1-\lambda^{pq-p}}{(1-\lambda)^{pq}} f(0,1), & (20)\\
f(-\lambda,-1) &= \frac{1-\lambda^{pq-p}}{(1-\lambda)^{pq}} f(0,-1), & (21)\\
f(-\lambda,1-\lambda) &= \lambda^{pq-p} f(-1,0) + (1-\lambda)^{pq-p} f(0,1), & (22)\\
f(\lambda,\lambda-1) &= \lambda^{pq-p} f(1,0) + (1-\lambda)^{pq-p} f(0,-1). & (23)
\end{aligned}$$

First, suppose that $x_1 > x_2 \geq 0$. If $x_2 = 0$, it follows from the homogeneity of f that $f(x_1,0) = x_1^{-p} f(1,0) = f(1,0)\, g_{p,q}(x_1,0)$. For $x_1 > x_2 > 0$ we obtain by (18)

$$\begin{aligned}
f(x_1,x_2) &= x_1^{-p} f(1, 1-(1-x_2/x_1)) = x_1^{-p} \frac{1-(x_2/x_1)^{pq-p}}{(1-(x_2/x_1))^{pq}} f(1,0)\\
&= \frac{x_1^{pq-p} - x_2^{pq-p}}{(x_1-x_2)^{pq}} f(1,0) = f(1,0)\, g_{p,q}(x_1,x_2).
\end{aligned}$$

Since $g_{p,q} \circ \gamma_0$, $g_{p,q} \circ \gamma_1$, and $g_{p,q} \circ \gamma_2$ are zero for $x_1 > x_2 \geq 0$, (17) holds in this part of the plane. (19) gives

$$\begin{aligned}
f(-x_1,-x_2) &= x_1^{-p} f(-1, (1-x_2/x_1)-1) = x_1^{-p} \frac{1-(x_2/x_1)^{pq-p}}{(1-(x_2/x_1))^{pq}} f(-1,0)\\
&= \frac{x_1^{pq-p} - x_2^{pq-p}}{(x_1-x_2)^{pq}} f(-1,0) = f(-1,0)(g_{p,q} \circ \gamma_0)(-x_1,-x_2).
\end{aligned}$$

But $g_{p,q}$, $g_{p,q} \circ \gamma_1$ as well as $g_{p,q} \circ \gamma_2$ vanish for $x_1 < x_2 < 0$ and therefore (17) is true if $x_1 < x_2 < 0$. Using the homogeneity we obtain that (17) is correct for $x_1 < 0$, $x_2 = 0$.

Now, assume $x_2 > x_1 \geq 0$. If $x_1 = 0$, then we have

$$\begin{aligned}
f(0,x_2) &= x_2^{-p} f(0,1) = f(0,1)(g_{p,q} \circ \gamma_1)(0,x_2),\\
f(0,-x_2) &= x_2^{-p} f(0,-1) = f(0,-1)(g_{p,q} \circ \gamma_2)(0,-x_2).
\end{aligned}$$

Formulae (20) and (21) for $x_2 > x_1 > 0$ yield

$$\begin{aligned}
f(x_1,x_2) &= x_2^{-p} f(x_1/x_2, 1) = x_2^{-p} \frac{1-(x_1/x_2)^{pq-p}}{(1-(x_1/x_2))^{pq}} f(0,1)\\
&= \frac{x_2^{pq-p} - x_1^{pq-p}}{(x_2-x_1)^{pq}} f(0,1) = f(0,1)(g_{p,q} \circ \gamma_1)(x_1,x_2),\\
f(-x_1,-x_2) &= x_2^{-p} f(-x_1/x_2, -1) = x_2^{-p} \frac{1-(x_1/x_2)^{pq-p}}{(1-(x_1/x_2))^{pq}} f(0,-1)\\
&= \frac{x_2^{pq-p} - x_1^{pq-p}}{(x_2-x_1)^{pq}} f(0,-1) = f(0,-1)(g_{p,q} \circ \gamma_2)(-x_1,-x_2).
\end{aligned}$$

Since $g_{p,q}$, $g_{p,q}\circ\gamma_0$, $g_{p,q}\circ\gamma_2$ are zero for $x_2 > x_1 \geq 0$ and $g_{p,q}$, $g_{p,q}\circ\gamma_0$, $g_{p,q}\circ\gamma_1$ vanish for $x_2 < x_1 \leq 0$, it remains to prove identity (17) if the coordinates have different signs.

Finally, let x_1 and x_2 be greater than zero. By (22) and (23) we have

$$\begin{aligned}
f(-x_1,x_2) &= (x_1+x_2)^{-p} f(-x_1/(x_1+x_2), 1-x_1/(x_1+x_2))\\
&= \frac{x_2^{pq-p}}{(x_1+x_2)^{pq}} f(0,1) + \frac{x_1^{pq-p}}{(x_1+x_2)^{pq}} f(-1,0)\\
&= f(0,1)(g_{p,q}\circ\gamma_1)(-x_1,x_2) + f(-1,0)(g_{p,q}\circ\gamma_0)(-x_1,x_2),\\
f(x_1,-x_2) &= (x_1+x_2)^{-p} f(x_1/(x_1+x_2), x_1/(x_1+x_2)-1)\\
&= \frac{x_2^{pq-p}}{(x_1+x_2)^{pq}} f(0,-1) + \frac{x_1^{pq-p}}{(x_1+x_2)^{pq}} f(1,0)\\
&= f(0,-1)(g_{p,q}\circ\gamma_2)(x_1,-x_2) + f(1,0) g_{p,q}(x_1,-x_2).
\end{aligned}$$

The fact that $g_{p,q}$ and $g_{p,q}\circ\gamma_2$ are zero in the second quadrant and $g_{p,q}\circ\gamma_0$, $g_{p,q}\circ\gamma_1$ are zero in the fourth quadrant completes the proof. $\square$

In the following, we have $q > 1$. Therefore Lemma 2 and Lemma 3 imply that f is continuous on S^{n-1}. Thus (6) holds on $\mathbb{R}\backslash\{0\}$ and f satisfies the conditions of Lemma 5. Combined with (5) this implies that

$$f = f(1,0)(g_{p,q} + g_{p,q}\circ\gamma_1) + f(-1,0)(g_{p,q}\circ\gamma_0 + g_{p,q}\circ\gamma_2) \tag{24}$$

on $\mathbb{R}^2\backslash\{0\}$.

Lemma 6. *Let* $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}^2, \tilde{+}_p\rangle$ *be a valuation which is* $\mathrm{GL}(2)$ *contravariant of weight* q*. Let* $p > 1$*,* $q > 1$ *or* $0 < p < 1$*,* $q > 1/p$*. Then*

$$\mathrm{Z}\, P = \{0\}$$

for every $P \in \mathcal{P}_0^2$*.*

Proof. For $x_2 > 0$ fixed, we obtain by (24) that

$$\lim_{x_1\to x_2+} f(x_1,x_2) = \lim_{x_1\to x_2+} \frac{x_1^{pq-p} - x_2^{pq-p}}{(x_1-x_2)^{pq}} f(1,0)$$

has to be finite. This implies that $f(1,0)$ has to be zero.

Considering $\lim_{x_1\to x_2+} f(-x_1,-x_2)$ proves $f(-1,0)=0$. So by (24), f vanishes on $\mathbb{R}^2\backslash\{0\}$. $\square$

Lemma 7. *For* $p < 1$*, let* $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}^2, \tilde{+}_p\rangle$ *be a valuation which is* $\mathrm{GL}(2)$ *contravariant of weight* $q = 1/p$*. Then there exist constants* $c_1, c_2 \geq 0$ *such that*

$$\mathrm{Z}\, P = c_1\, \mathrm{I}_p^+ P \,\tilde{+}_p\, c_2\, \mathrm{I}_p^- P$$

for every $P \in \mathcal{P}_0^2$*.*

Proof. A simple calculation shows

$$\rho(\mathrm{I}_p^+ T^2, \cdot)^p = (p^2 - 3p + 2)^{-1}(g_{p,1/p} + g_{p,1/p} \circ \gamma_1) \tag{25}$$

almost everywhere. Therefore

$$\rho(\mathrm{I}_p^- T^2, \cdot)^p = \rho(\mathrm{I}_p^+ T^2, \gamma_0(\cdot))^p = (p^2 - 3p + 2)^{-1}(g_{p,1/p} \circ \gamma_0 + g_{p,1/p} \circ \gamma_2). \tag{26}$$

Combined with (24), these equations complete the proof. □

Finally, we consider the case $p < 1$ and $q \in (1, 1/p)$. We define

$$\rho(\mathrm{I}_{p,q}^+ T^2, x)^p = (g_{p,q} + g_{p,q} \circ \gamma_1)(x).$$

The restrictions on q show that $\rho(\mathrm{I}_{p,q}^+ T^2, \cdot)$ is continuous and non-negative on $\mathbb{R}^2 \backslash \{0\}$. By definition, $\rho(\mathrm{I}_{p,q}^+ T^2, \cdot)$ is positively homogeneous of degree -1 and thus the radial function of a star body.

We extend this definition to all simplices in $\mathbb{R}^2$ having one vertex at the origin (we denote this set by $\mathcal{T}_0^2$):

$$\mathrm{I}_{p,q}^+ S = \begin{cases} |\det \phi|^q \phi^{-t}\, \mathrm{I}_{p,q}^+ T^2 & \text{if } S \text{ is 2-dimensional and } S = \phi T^2, \\ \{0\} & \text{otherwise.} \end{cases}$$

Note that $\mathrm{I}_{p,q}^+$ is well defined on $\mathcal{T}_0^2$ since $\rho(\mathrm{I}_{p,q}^+ T^2, \cdot)$ does not change if the coordinates are interchanged. We claim that $\mathrm{I}_{p,q}^+$ is a valuation on $\mathcal{T}_0^2$. To prove this, it suffices to check the valuation property if the two involved simplices coincide in an edge. Since by definition $\mathrm{I}_{p,q}^+$ is GL(2) contravariant, it suffices to check the valuation property for the standard simplex. Thus it suffices to show that

$$\mathrm{I}_{p,q}^+ T^2 = \mathrm{I}_{p,q}^+ (T^2 \cap H^+) \,\tilde{+}_p\, \mathrm{I}_{p,q}^+ (T^2 \cap H^-)$$

where H is the line with normal vector $\lambda e_1 - (1-\lambda)e_2$, $0 < \lambda < 1$. Therefore we have to prove

$$\begin{aligned} \rho(\mathrm{I}_{p,q}^+ T^2, (x_1, x_2))^p &= \lambda^{pq} \rho(\mathrm{I}_{p,q}^+ T^2, (x_1, (1-\lambda)x_1 + \lambda x_2))^p \\ &\quad + (1-\lambda)^{pq} \rho(\mathrm{I}_{p,q}^+ T^2, ((1-\lambda)x_1 + \lambda x_2, x_2))^p. \end{aligned} \tag{27}$$

The case $x_1, x_2 < 0$ is trivial. So assume $x_1 > x_2 \geq 0$. Then $x_1 > (1-\lambda)x_1 + \lambda x_2 \geq 0$, $(1-\lambda)x_1 + \lambda x_2 > x_2 \geq 0$, and the right hand side of (27) equals

$$\lambda^{pq} \frac{x_1^{pq-p} - ((1-\lambda)x_1 + \lambda x_2)^{pq-p}}{(x_1 - (1-\lambda)x_1 - \lambda x_2)^{pq}} + (1-\lambda)^{pq} \frac{((1-\lambda)x_1 + \lambda x_2)^{pq-p} - x_2^{pq-p}}{((1-\lambda)x_1 + \lambda x_2 - x_2)^{pq}}$$

which is nothing else than $\rho(\mathrm{I}_{p,q}^+ T^2, (x_1, x_2))^p$. Similar, we obtain (27) for points $x_2 > x_1 \geq 0$. To check (27) for $(x_1, -x_2)$, $x_1, x_2 > 0$, we first assume that $(1-\lambda)x_1 - \lambda x_2 > 0$. Then $0 < (1-\lambda)x_1 - \lambda x_2 < x_1$ and the sum appearing in (27) equals

$$\lambda^{pq} \frac{x_1^{pq-p} - ((1-\lambda)x_1 - \lambda x_2)^{pq-p}}{(x_1 - (1-\lambda)x_1 + \lambda x_2)^{pq}} + (1-\lambda)^{pq} \frac{((1-\lambda)x_1 - \lambda x_2)^{pq-p}}{((1-\lambda)x_1 - \lambda x_2 + x_2)^{pq}}.$$

If $(1-\lambda)x_1 - \lambda x_2 < 0$, the right hand side of (27) is

$$\lambda^{pq} \frac{x_1^{pq-p}}{(x_1 - (1-\lambda)x_1 + \lambda x_2)^{pq}}.$$

These two expressions are equal to $\rho(\mathrm{I}_{p,q}^+ T^2, (x_1, -x_2))^p$. The case $(1-\lambda)x_1 - \lambda x_2 = 0$ is simple and the remaining part can be treated in an analogous way.

Now, we extend the valuation $\mathrm{I}_{p,q}^+$ to $\overline{\mathcal{P}}_0^2$ by setting

$$\rho(\mathrm{I}_{p,q}^+ P, x)^p = \sum_{i \in I} \rho(\mathrm{I}_{p,q}^+ S_i, x)^p,$$

where $\{S_i : i \in I, \dim S_i = 2\} \subset \mathcal{T}_0^2$ is a dissection of P, that is, I is finite, $P = \bigcup_{i \in I} S_i$ and no pair of simplices intersects in a set of dimension 2.

Given two different dissections, it is always possible to obtain one from the other by a finite number of the following operations: a simplex is dissected into two 2-dimensional simplices by a line through the origin, or the converse, that is, two simplices whose union is again a simplex are replaced by their union (We remark that the corresponding result holds true for $n \geq 3$, see [26]). Since $\mathrm{I}_{p,q}^+$ is a valuation on $\mathcal{T}_0^2$, this shows that $\mathrm{I}_{p,q}^+$ is well defined on $\overline{\mathcal{P}}_0^2$.

We have to prove that $\mathrm{I}_{p,q}^+$ is a valuation. To do so, let $P, Q \in \overline{\mathcal{P}}_0^2$ be two 2-dimensional convex polytopes such that their union is again convex. We dissect $\mathbb{R}^2$ into 2-dimensional convex cones with apex 0 in such a way that each vertex of $P, Q, P \cap Q, P \cup Q$ lies on the boundary of some cone in this dissection. The intersection of such a cone with the boundary of P and Q are line segments which are either identical, do not intersect, or intersect in their endpoints only. Therefore $\mathrm{I}_{p,q}^+$ is a valuation and obviously it is $\mathrm{GL}(2)$ contravariant of weight q.

We define the L_p radial valuation $\mathrm{I}_{p,q}^-$ by setting $\mathrm{I}_{p,q}^- P = \mathrm{I}_{p,q}^+(-P)$. Now, (24) implies the following result.

Lemma 8. *For $p < 1$, let $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}^2, \tilde{+}_p \rangle$ be a valuation which is $\mathrm{GL}(2)$ contravariant of weight $1 < q < 1/p$. Then there exist constants $c_1, c_2 \geq 0$ such that*

$$\mathrm{Z}\, P = c_1 \, \mathrm{I}_{p,q}^+ P \,\tilde{+}_p\, c_2 \, \mathrm{I}_{p,q}^- P$$

for every $P \in \mathcal{P}_0^2$.

3.2 The 2-dimensional Covariant Case

Let $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}^2, \tilde{+}_p \rangle$ be a valuation which is $\mathrm{GL}(2)$ covariant of weight q. Let $q > 0$. As before, let $\mathrm{Y}\, P(\cdot) = \rho(\mathrm{Z}\, P, \cdot)^p$ and denote the extension of Y to $\overline{\mathcal{P}}_0^2$ by $\overline{\mathrm{Y}}$. Note that Lemma 2 and Lemma 3 imply that $\overline{\mathrm{Y}}$ is continuous on S^1. We define the valuation $\hat{\mathrm{Y}}$ by $\hat{\mathrm{Y}}\, P(\cdot) = \overline{\mathrm{Y}}\, P(\psi_{\frac{\pi}{2}}^{-1}(\cdot))$ for every $P \in \overline{\mathcal{P}}_0^2$. From (4) and (8) it follows that for $\phi \in \mathrm{GL}(2)$ with $\det \phi > 0$

$$\hat{\mathrm{Y}}\, \phi P(x) = |\det \phi|^{pq} \; \overline{\mathrm{Y}}\, P(\phi^{-1} \psi_{\frac{\pi}{2}}^{-1} x) = |\det \phi|^{pq+p} \; \hat{\mathrm{Y}}\, P(\phi^t x)$$

for every $P \in \overline{\mathcal{P}}_0^2$. So $\hat{\mathrm{Y}}\, T^2$ satisfies (16) with $q+1$ instead of q. From the $\mathrm{GL}(2)$ covariance it follows that $\hat{\mathrm{Y}}\, T^2(x_1,x_2) = \hat{\mathrm{Y}}\, T^2(-x_2,-x_1)$. Thus Lemma 5 shows that

$$\hat{\mathrm{Y}}\, T^2 = \hat{\mathrm{Y}}\, T^2(1,0)(g_{p,q+1} + g_{p,q+1}\circ\gamma_2) + \hat{\mathrm{Y}}\, T^2(0,1)(g_{p,q+1}\circ\gamma_0 + g_{p,q+1}\circ\gamma_1). \quad (28)$$

Considering the limit

$$\lim_{x_1\to x_2+} \frac{x_1^{pq} - x_2^{pq}}{(x_1 - x_2)^{pq+p}}$$

for fixed $x_2 > 0$, we derive for $p > 1$ that $\hat{\mathrm{Y}}\, T^2(1,0) = \hat{\mathrm{Y}}\, T^2(0,1) = 0$ since $\hat{\mathrm{Y}}\, T^2$ is continuous on $\mathbb{R}^2\backslash\{0\}$ and has to be finite on the first median. This limit also proves that $\hat{\mathrm{Y}}\, T^2(1,0) = \hat{\mathrm{Y}}\, T^2(0,1) = 0$ for $p < 1$ and $q > 1/p - 1$. Now, (28) implies the following result.

Lemma 9. *Let* $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}^2, \tilde{+}_p\rangle$ *be a valuation which is* $\mathrm{GL}(2)$ *covariant of weight* $q > 0$. *Let* $p > 1$, $q > 0$ *or* $0 < p < 1$, $q > 1/p - 1$. *Then*

$$\mathrm{Z}\, P = \{0\}$$

for every $P \in \mathcal{P}_0^2$.

For $p < 1$ and $q \in (0, 1/p - 1)$, we define $\mathrm{J}_{p,q}^+$ by

$$\rho(\mathrm{J}_{p,q}^+\, T^2, x)^p = (g_{p,q+1} + g_{p,q+1}\circ\gamma_2)(\psi_{\frac{\pi}{2}} x).$$

Similar to the contravariant case, $\mathrm{J}_{p,q}^+$ can be extended to a covariant valuation on $\mathcal{P}_0^2$. We define $\mathrm{J}_{p,q}^-$ by $\mathrm{J}_{p,q}^-\, P = \mathrm{J}_{p,q}^+(-P)$. Now, (28) implies the following result.

Lemma 10. *For* $p < 1$, *let* $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}^2, \tilde{+}_p\rangle$ *be a valuation which is* $\mathrm{GL}(2)$ *covariant of weight* $0 < q < 1/p - 1$. *Then there exist constants* $c_1, c_2 \geq 0$ *such that*

$$\mathrm{Z}\, P = c_1\, \mathrm{J}_{p,q}^+\, P \mathbin{\tilde{+}_p} c_2\, \mathrm{J}_{p,q}^-\, P$$

for every $P \in \mathcal{P}_0^2$.

For $q = 1/p - 1$, the continuity of $\hat{\mathrm{Y}}\, T^2$ at the first median and (28) yield that $\hat{\mathrm{Y}}\, T^2(1,0) = \hat{\mathrm{Y}}\, T^2(0,1)$. Therefore we obtain the following lemma by using (25), (26) and the identity $\psi_{\frac{\pi}{2}}\, \mathrm{I}_p\, P = \psi_{\frac{\pi}{2}}^{-1}\, \mathrm{I}_p\, P$.

Lemma 11. *Let* $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}^2, \tilde{+}_p\rangle$ *be a valuation which is* $\mathrm{GL}(2)$ *covariant of weight* $q = 1/p - 1$. *Then there exists a constant* $c \geq 0$ *such that*

$$\mathrm{Z}\, P = c\, \psi_{\frac{\pi}{2}}\, \mathrm{I}_p\, P$$

for every $P \in \mathcal{P}_0^2$.

3.3 The 2-dimensional Classification Theorems

Using the lemmas of the preceding sections and the planar case of Lemma 12 and Lemma 17, we obtain the following result.

Theorem 3. *For $0 < p < 1$, an operator $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}^2, \tilde{+}_p \rangle$ is a non-trivial valuation which is* $\mathrm{GL}(2)$ *covariant of weight q if and only if there are constants $c_1, c_2 \geq 0$ such that*

$$\mathrm{Z}\, P = \begin{cases} c_1\, \mathrm{I}_p^+ \, P^* \,\tilde{+}_p\, c_2\, \mathrm{I}_p^- \, P^* & \text{for } q = -1/p \\ c_1\, \mathrm{I}_{p,q}^+ \, P^* \,\tilde{+}_p\, c_2\, \mathrm{I}_{p,q}^- \, P^* & \text{for } -1/p < q < -1 \\ c_1 \psi_{\frac{\pi}{2}} (P^* \,\tilde{+}_p\, (-P^*)) & \text{for } q = -1 \\ c_1\, \mathrm{J}_{p,q}^+ \, P \,\tilde{+}_p\, c_2\, \mathrm{J}_{p,q}^- \, P & \text{for } 0 < q < 1/p - 1 \\ c_1 \psi_{\frac{\pi}{2}}\, \mathrm{I}_p \, P & \text{for } q = 1/p - 1 \end{cases}$$

for every $P \in \mathcal{P}_0^2$. For $p > 1$, an operator $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}^2, \tilde{+}_p \rangle$ is a non-trivial $\mathrm{GL}(2)$ *covariant valuation if and only if there is a constant $c \geq 0$ such that*

$$\mathrm{Z}\, P = c\, \psi_{\frac{\pi}{2}} (P^* \,\tilde{+}_p\, (-P^*))$$

for every $P \in \mathcal{P}_0^2$.

Next, we consider an operator Z with centrally symmetric images. Note that in this case also the extended operator $\overline{\mathrm{Z}}$ has centrally symmetric images. Using again the lemmas of the preceding sections and the planar case of Lemma 12 and Lemma 17, we obtain the following result. Here $\mathrm{I}_{p,q}\, P = \mathrm{I}_{p,q}^+ \, P \,\tilde{+}_p\, \mathrm{I}_{p,q}^- \, P$.

Theorem 4. *For $0 < p < 1$, an operator $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}_c^2, \tilde{+}_p \rangle$ is a non-trivial valuation which is* $\mathrm{GL}(2)$ *covariant of weight q if and only if there is a constant $c \geq 0$ such that*

$$\mathrm{Z}\, P = \begin{cases} c\; \mathrm{I}_p \, P^* & \text{for } q = -1/p \\ c\; \mathrm{I}_{p,q} \, P^* & \text{for } -1/p < q < -1 \\ c\, \psi_{\frac{\pi}{2}} (P^* \,\tilde{+}_p\, (-P^*)) & \text{for } q = -1 \\ c\, \psi_{\frac{\pi}{2}}\, \mathrm{I}_{p,q} \, P & \text{for } 0 < q < 1/p - 1 \\ c\, \psi_{\frac{\pi}{2}}\, \mathrm{I}_p \, P & \text{for } q = 1/p - 1 \end{cases}$$

for every $P \in \mathcal{P}_0^2$. For $p > 1$, an operator $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}_c^2, \tilde{+}_p \rangle$ is a non-trivial $\mathrm{GL}(2)$ *covariant valuation if and only if there is a constant $c \geq 0$ such that*

$$\mathrm{Z}\, P = c\, \psi_{\frac{\pi}{2}} (P^* \,\tilde{+}_p\, (-P^*))$$

for every $P \in \mathcal{P}_0^2$.

3.4 The Contravariant Case for $n \geq 3$

Lemma 12. *Let* $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$, $n \geq 2$, *be a valuation which is* $\mathrm{GL}(n)$ *contravariant of weight* $0 < q < 1$. *Then*

$$\mathrm{Z}\, P = \{0\}$$

for every $P \in \mathcal{P}_0^n$.

Proof. From (6) we deduce that for $x \notin \operatorname{lin} e_1 \cup \cdots \cup \operatorname{lin} e_n \cup \operatorname{lin}(\lambda_j e_1 - (1-\lambda_j) e_j)$

$$f(x) = \lambda_j^{pq} f(\phi_j^t x) + (1-\lambda_j)^{pq} f(\psi_j^t x) \tag{29}$$

holds. First, we want to show that f is uniformly bounded on $S^{n-1} \backslash \{\pm e_1, \dots, \pm e_n\}$. To do so, note that since f is positive, equation (29) for $j = 2$ at $(x_1, 1-\lambda_2, x_3, \dots, x_n)$ and $(x_1, -(1-\lambda_2), x_3, \dots, x_n)$ gives

$$\begin{aligned} f(x_1, (1-\lambda_2)(x_1+\lambda_2), x_3, \dots, x_n) &\leq \lambda_2^{-pq} f(x_1, 1-\lambda_2, x_3, \dots, x_n), & (30)\\ f(x_1, (1-\lambda_2)(x_1-\lambda_2), x_3, \dots, x_n) &\leq \lambda_2^{-pq} f(x_1, -(1-\lambda_2), x_3, \dots, x_n). & (31) \end{aligned}$$

Let $x_1 \to -\lambda_2$, $x_2, \dots, x_n \to 0$ in (30). Since f is continuous on $S^{n-1} \backslash \{\pm e_1, \dots, \pm e_n\}$ and homogeneous, it is bounded in a suitable neighbourhood of $-\lambda_2 e_1 + (1-\lambda_2) e_2$. Thus f is also bounded in a suitable neighbourhood of $-\lambda_2 e_1$. From (5) we conclude that f is bounded in a neighbourhood of every $-e_i, i = 1 \dots, n$. Proceeding in an analogous way but taking the limit $x_1 \to \lambda_2$ and taking (31) into account we obtain the boundedness in suitable neighbourhoods of $e_i, i = 1 \dots, n$.

From (29) we know that

$$f(\phi_2^{-t} x) = \lambda_2^{pq} f(x) + (1-\lambda_2)^{pq} f(\psi_2^t \phi_2^{-t} x)$$

for $x \notin \operatorname{lin} e_1 \cup \cdots \cup \operatorname{lin} e_n \cup \operatorname{lin}(e_1 + (1-\lambda_2) e_2)$. Thus we obtain for $(-1, 1, x_3, \dots, x_n)$ by using the homogeneity and the non-negativity of f that

$$\lambda_2^{pq-p} f(-1, 1, x_3, \dots, x_n) \leq f(-\lambda_2, 2-\lambda_2, \lambda_2 x_3, \dots, \lambda_2 x_n).$$

Since $pq - p < 0$ and f is bounded, this yields

$$f(-1, 1, x_3, \dots, x_n) = 0, \quad x_3, \dots, x_n \in \mathbb{R}.$$

Evaluating (29) at $(-1, 1, x_3, \dots, x_n)$ proves

$$0 = \lambda_2^{pq} f(-1, 2\lambda_2 - 1, x_3, \dots, x_n) + (1-\lambda_2)^{pq} f(2\lambda_2 - 1, 1, x_3, \dots, x_n).$$

for $\lambda_2 \neq 1/2$. Since f is non-negative,

$$\begin{aligned} f(-1, x_2, x_3, \dots, x_n) = 0, &\qquad -1 < x_2 < 1, \quad x_2 \neq 0, \quad x_3, \dots, x_n \in \mathbb{R},\\ f(x_1, 1, x_3, \dots, x_n) = 0, &\qquad -1 < x_1 < 1, \quad x_1 \neq 0, \quad x_3, \dots, x_n \in \mathbb{R}. \end{aligned}$$

Because of (3) we also have for $-1 < x_1 < 1$, $-1 < x_2 < 1$, $x_1, x_2 \neq 0$ and arbitrary $x_3, \dots, x_n$

$$\begin{aligned} f(x_1, -1, x_3, \dots, x_n) &= 0,\\ f(1, x_2, x_3, \dots, x_n) &= 0. \end{aligned}$$

These last four equations prove that f is equal to zero almost everywhere on $\mathbb{R}^n$. $\square$

Lemma 13. *Let* $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$, $n \geq 3$, *be a valuation which is* $\mathrm{GL}(n)$ *contravariant of weight* $q = 1$. *Then*

$$\mathrm{Z}\, P = \{0\}$$

for every $P \in \mathcal{P}_0^n$.

Proof. By Lemma 2 and Lemma 3, f is continuous and uniformly bounded on S^{n-1} except on $\operatorname{lin} e_1 \cup \cdots \cup \operatorname{lin} e_n$. By (6), we have for $2 \leq j \leq n$

$$f(x) = \lambda_j^p f(\phi_j^t x) + (1 - \lambda_j)^p f(\psi_j^t x) \tag{32}$$

on $\mathbb{R}^n$ except on a finite union of lines. Using this repeatedly, we get

$$\begin{aligned} f(x) \quad &= \quad \lambda_2^p \cdots \lambda_n^p f(\phi_n^t \cdots \phi_2^t x) + \sum_{j=3}^{n} \lambda_2^p \cdots \lambda_{j-1}^p (1 - \lambda_j)^p f(\psi_j^t \phi_{j-1}^t \cdots \phi_2^t x) \\ &\quad\quad + (1 - \lambda_2)^p f(\psi_2^t x) \\ &\geq \quad \lambda_2^p \cdots \lambda_n^p f(\phi_n^t \cdots \phi_2^t x) + (1 - \lambda_2)^p f(\psi_2^t x). \end{aligned}$$

This implies for $k = 1, 2, \ldots,$

$$f((\psi_2^{-t})^k x) \geq \lambda_2^p \cdots \lambda_n^p \sum_{i=1}^{k} (1 - \lambda_2)^{p(k-i)} f(\phi_n^t \cdots \phi_2^t (\psi_2^{-t})^i x) \tag{33}$$

except on countably many lines. Define $x' = x_3 e_3 + \cdots + x_n e_n$. Evaluating (33) at suitable $e_1 + x'$ and multiplying by $(1 - \lambda_2)^{-pk}$ shows that

$$f(e_1 + (1 - \lambda_2)^k x') \geq \lambda_2^p \cdots \lambda_n^p \sum_{i=1}^{k} f(\phi_n^t \cdots \phi_2^t (e_1 + (1 - \lambda_2)^i x')).$$

Let $k \to \infty$. Since f is uniformly bounded and continuous at $\phi_n^t \cdots \phi_2^t e_1 = e_1 + (1 - \lambda_2) e_2 + \cdots + (1 - \lambda_n) e_n$, it follows that $f(\phi_n^t \cdots \phi_2^t e_1) = 0$. So we get

$$f(1, x_2, \ldots, x_n) = 0, \quad 0 < x_2, \ldots, x_n < 1.$$

From (5) we obtain (using the homogeneity of f) that

$$f(1, x_2, \ldots, x_n) = 0, \quad x_2, \ldots, x_k > 0, \quad 0 < x_{k+1}, \ldots, x_n < 1.$$

So $f(x_1, \ldots, x_n) = 0$ for $x_1, \ldots, x_n > 0$. Considering $-e_1 + x'$ and (33) like before shows $f(-x_1, \ldots, -x_n) = 0$ for $x_1, \ldots, x_n > 0$.

Note that (32) for $j = 2$ and arbitrary $c \geq 1$ at $(c, c, -1, c, \ldots, c)$ proves (since $p \neq 1$) that $f(c, c, -1, c, \ldots, c) = 0$. Let $x_1 < 0$, $x_2, \ldots, x_n > 0$, and $(1 - \lambda_j) x_1 + \lambda_j x_j > 0$. By (32) and the fact that f vanishes at points having all coordinates greater than zero we get

$$f(\phi_n^t \cdots \phi_2^t x) = \frac{1}{\lambda_2^p \cdots \lambda_n^p} f(x)$$

except on finitely many lines. Thus we obtain

$$\begin{aligned} &\lambda_2^{-p} \cdots \lambda_n^{-p} f(-1, c - \varepsilon, c, \ldots, c) = f(\phi_n^t \cdots \phi_2^t (-1, c - \varepsilon, c, \ldots, c)) \\ &= f(-1, -1 + \lambda_2 (1 + c - \varepsilon), -1 + \lambda_3 (1 + c), \ldots, -1 + \lambda_n (1 + c)) \end{aligned}$$

for suitable $\varepsilon > 0$ and $\lambda_2, \ldots, \lambda_n > 1/(1 + c - \varepsilon)$. The continuity of f shows

$$f(-1, x_2, \ldots, x_n) = 0, \quad 0 < x_2, \ldots, x_n < c.$$

But $c \geq 1$ was arbitrary, so $f(-1, x_2, \ldots, x_n) = 0$ for $x_2, \ldots, x_n > 0$. The homogeneity yields $f(-x_1, x_2, \ldots, x_n) = 0$ for $x_1, x_2, \ldots, x_n > 0$. In conclusion, $f(x_1, \ldots, x_n) = 0$ if at most one coordinate is negative. Suppose $f(x_1, \ldots, x_n) = 0$ where at most $1 \leq k < n-1$ coordinates are negative. Let x be chosen such that $x_1, \ldots, x_{k+1} < 0$ and $x_{k+2}, \ldots, x_n > 0$. Suppose $x_2 < x_1 < 0$. Choose λ_2 with $0 < x_1/x_2 < \lambda_2 < 1$. Then

$$\begin{aligned} (\psi_2^{-t} x)_1 = (\phi_2^t \psi_2^{-t} x)_1 &= \frac{x_1}{1 - \lambda_2} - \frac{\lambda_2}{1 - \lambda_2} x_2 > 0, \\ (\psi_2^{-t} x)_i = (\phi_2^t \psi_2^{-t} x)_i > 0, & \qquad i = k+2, \ldots, n. \end{aligned}$$

Since $f(\psi_2^{-t} x) = \lambda_2^{pq} f(\phi_2^t \psi_2^{-t} x) + (1 - \lambda_2)^{pq} f(x)$ we obtain $f(x) = 0$. By (5) we conclude $f(x) = 0$ for the case $x_1 < x_2 < 0$. $\square$

In the following, we have $q > 1$. Therefore Lemma 2 and Lemma 3 imply that f is continuous on $\mathbb{R}^n \backslash \{0\}$. In the proof of Lemmas 14 to 16, we use the following remark. Suppose we have two functions f_1, f_2 which are continuous on $\mathbb{R}^n \backslash \{0\}$ satisfying (5) and such that for $0 < \lambda_j < 1$, $j = 2, \ldots, n$,

$$f_i(x) = \lambda_j^{pq} f_i(\phi_j^t x) + (1 - \lambda_j)^{pq} f_i(\psi_j^t x)$$

holds on $\mathbb{R}^n$. Further assume that these functions are equal for all points where at most two coordinates do not vanish. Then an argument similar to that at the end of the last proof shows that these functions have to be equal.

Lemma 14. *For $p > 1$, let $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$ be a valuation which is $\mathrm{GL}(n)$ contravariant of weight $q > 1$. Then*

$$\mathrm{Z}\, P = \{0\}$$

for every $P \in \mathcal{P}_0^n$.

Proof. Define $\tilde{f}(x_1, x_2) = f(x_1 e_1 + x_2 e_2)$. Then $\tilde{f}$ is continuous and satisfies the conditions of Lemma 5. The proof of Lemma 6 shows $\tilde{f} = 0$. By (5) this implies that $f(x_i e_i + x_j e_j) = 0$ for arbitrary $1 \leq i, j \leq n$. Thus f vanishes on $\mathbb{R}^n \backslash \{0\}$. $\square$

Lemma 15. *For $p < 1$, let $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$ be a valuation which is $\mathrm{GL}(n)$ contravariant of weight $q = 1/p$. Then there exist constants $c_1, c_2 \geq 0$ such that*

$$\mathrm{Z}\, P = c_1 \, \mathrm{I}_p^+ \, P \; \tilde{+}_p \; c_2 \, \mathrm{I}_p^- \, P$$

for every $P \in \mathcal{P}_0^n$.

Proof. For $x = x_1 e_1 + x_2 e_2$, note that $\rho(\mathrm{I}_p^{\pm}\, T^n, x)$ is a multiple of $\rho(\mathrm{I}_p^{\pm}\, T^2, (x_1, x_2))$. This and an analogous argument as before proves the lemma. $\square$

Lemma 16. *For $p < 1$, let $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$ be a valuation which is* $\mathrm{GL}(n)$ *contravariant of weight $q > 1$, $q \neq 1/p$. Then*

$$\mathrm{Z}\, P = \{0\}$$

for every $P \in \mathcal{P}_0^n$.

Proof. By (6) we have

$$f(x) = \lambda^{pq} f(\phi_2^t x) + (1-\lambda)^{pq} f(\psi_2^t x)$$

on $\mathbb{R}^n \backslash \{0\}$. Since e_3 is an eigenvector of ϕ_2^t and ψ_2^t with eigenvalue 1, we get $f(\pm e_i) = 0$ for $i = 1, \dots, n$. For $\tilde{f}(x_1, x_2) = f(x_1 e_1 + x_2 e_2)$ this implies $\tilde{f}(1,0) = \tilde{f}(-1,0) = 0$. Lemma 5 proves $\tilde{f} = 0$. $\square$

3.5 The Covariant Case for $n \geq 3$

Lemma 17. *Let $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$, $n \geq 2$, be a valuation which is* $\mathrm{GL}(n)$ *covariant of weight $q = 0$. Then there exist constants $c_1, c_2 \geq 0$ such that*

$$\mathrm{Z}\, P = c_1\, P \,\tilde{+}_p\, c_2(-P)$$

for every $P \in \mathcal{P}_0^n$.

Proof. By Lemma 2 and Lemma 3, f is continuous and uniformly bounded on $S^{n-1} \backslash (e_1^\perp \cup \dots \cup e_n^\perp)$. By (7), the equation

$$f(x) = f(\phi_j^{-1} x) + f(\psi_j^{-1} x) \tag{34}$$

holds for $x \notin e_1^\perp \cup \dots \cup e_n^\perp \cup (\lambda_j e_1 - (1-\lambda_j) e_j)^\perp$. Using this, we get by induction for $k = 1, 2, \dots$,

$$f(\phi_2^k x) = \sum_{i=1}^{k} f(\psi_2^{-1} \phi_2^i\, x) + f(x) \tag{35}$$

for $x \notin e_2^\perp \cup \dots \cup e_n^\perp \bigcup_{i=1}^\infty (e_1 + a_i e_2)^\perp$ and a suitable sequence (a_i). Define $x' = x_1 e_1 + x_3 e_3 + x_4 e_4 + \dots + x_n e_n$ where $x_1, x_3, x_4, \dots, x_n \neq 0$ and $x_1 \neq 1 - a_i$ for every i. Then (35) at $e_2 - e_1 + x'$ and the non-negativity of f show

$$f(\lambda_2^k (e_2 - e_1) + x') \geq \sum_{i=1}^{k} f(\psi_2^{-1}(\lambda_2^i (e_2 - e_1) + x')).$$

Let $k \to \infty$. Since f is uniformly bounded, $\lim_{i\to\infty} f(\psi_2^{-1}(\lambda_2^i(e_2 - e_1) + x')) = 0$. The continuity properties of f yield

$$f\left(\frac{x_1}{1-\lambda_2}, \frac{-\lambda_2 x_1}{1-\lambda_2}, x_3, \dots, x_n \right) = 0, \quad \text{for } x_1, x_3, \dots, x_n \neq 0. \tag{36}$$

From (36) we obtain that

$$f(x_1, x_2, \dots, x_n) = 0, \quad \text{for } x_1, x_2, \dots, x_n \neq 0 \text{ and not all } x_i \text{ have the same sign.}$$

For $j = 2, 3, \ldots, n$ and $x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$ where all the x_i have the same sign, it follows that at least two coordinates of $\psi_j^{-1}\phi_j x$ have different signs. Thus (34) gives

$$f(\phi_n \cdots \phi_2 x) = f(x), \quad \text{for } x \notin e_1^\perp \cup \cdots \cup e_n^\perp \cup \bigcup_{k=0}^{n-2} (e_1 + \sum_{i=0}^{k} (1-\lambda_{2+i}) e_{2+i})^\perp. \tag{37}$$

Evaluating (37) at $(1, \ldots, 1)$ gives

$$f(1 + (1-\lambda_2) + \cdots + (1-\lambda_n), \lambda_2, \ldots, \lambda_n) = f(1, \ldots, 1),$$

from which we conclude

$$f(x_1, \ldots, x_n) = f(1, \ldots, 1), \quad \text{for } 0 < x_2, \ldots, x_n < 1,\ x_1 = n - x_2 - \cdots - x_n. \tag{38}$$

But (37) for positive $x_2, \ldots, x_n$ is nothing else than

$$f(x_1, \ldots, x_n) = f(x_1 + (1-\lambda_2)x_2 + \cdots + (1-\lambda_n)x_n, \lambda_2 x_2, \ldots, \lambda_n x_n).$$

Choosing sufficiently small $\lambda_2, \ldots, \lambda_n$, we obtain by (38)

$$f(x_1, \ldots, x_n) = f(1, \ldots, 1), \quad \text{for } x_1, \ldots, x_n > 0,\ x_1 = n - x_2 - \cdots - x_n.$$

Similarly, we derive

$$f(-x_1, \ldots, -x_n) = f(-1, \ldots, -1), \quad \text{for } x_1, \ldots, x_n > 0,\ x_1 = n - x_2 - \cdots - x_n.$$

This shows that $f(x) = c_1\, \rho(T^n, x)^p + c_2\, \rho(-T^n, x)^p$. $\square$

Lemma 18. *Let $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$, $n \geq 3$, be a valuation which is $\mathrm{GL}(n)$ covariant of weight $q > 0$. Then*

$$\mathrm{Z}\, P = \{0\}$$

for every $P \in \mathcal{P}_0^n$.

Proof. Since $q > 0$, Lemma 2 and Lemma 3 imply that f is continuous on S^{n-1}. By (7) we have

$$f(x) = \lambda_j^{pq} f(\phi_j^{-1} x) + (1-\lambda_j)^{pq} f(\psi_j^{-1} x) \tag{39}$$

on $\mathbb{R}^n \backslash \{0\}$. The vector e_3 is an eigenvector with eigenvalue 1 of ϕ_2^{-1} and ψ_2^{-1}. So for $pq \neq 1$, (39) and (5) imply $f(\pm e_k) = 0$ for $k = 1, 2, \ldots, n$. For $pq = 1$, (39) evaluated at e_j for $j > 1$ yields

$$f(e_j)\lambda_j^{-p} = f(e_j - (1-\lambda_j)e_1).$$

Since $f(e_j - e_1)$ has to be finite and f is continuous, $f(e_j)$ has to be zero. Thus also in this case $f(\pm e_k) = 0$ for $k = 1, 2, \ldots, n$.

Hence (39) gives

$$f((1-\lambda_j)e_1 + \lambda_j e_j) = \lambda^{pq} f(e_j) + (1-\lambda_j)^{pq} f(e_1) = 0.$$

Therefore $f(x_1e_1+e_j)=0$ for positive x_1. Using (39) again shows

$$f(-e_1)=\lambda_j^{pq}f(-e_1)+(1-\lambda_j)^{pq+p}f(-e_1+\lambda_je_j),$$

and so $f(x_1e_1+e_j)=0$ for $x_1\le -1$. But

$$f(e_j)=\lambda_j^{pq+p}f(-(1-\lambda_j)e_1+e_j)+(1-\lambda_j)^{pq}f(e_j),$$

which proves, together with the observations made before, that $f(x_1e_1+e_j)=0$ for all x_1. By (5) this implies that $f(e_1+x_je_j)=0$ for all x_j. The homogeneity of f shows $f(x_1e_1+x_je_j)=0$ for all x_1,x_j. Thus f vanishes on all points with at most two coordinates not equal to zero.

We use induction on the number of non-vanishing coordinates. We assume that f equals zero on points with $(j-1)$ non-vanishing coordinates. Set $x'=x_2e_2+\cdots+x_{j-1}e_{j-1}$. By (39),

$$f((1-\lambda_j)e_1+\lambda_je_j+x')=\lambda_j^{pq}f(e_j+x')+(1-\lambda_j)^{pq}f(e_1+x')=0,$$

which gives $f(x_1e_1+e_j+x'/\lambda_j)=0$ for $x_1>0$. Therefore $f(x_1e_1+e_j+x')=0$ for all $x_1>0$ and $x'=x_2e_2+\cdots+x_{j-1}e_{j-1}$. But by (39)

$$\begin{aligned}
f(-e_1+x') &= \lambda_j^{pq}f(-e_1+x')+(1-\lambda_j)^{pq+p}f(-e_1+\lambda_j\,e_j+(1-\lambda_j)x'),\\
f(e_j+x') &= \lambda_j^{pq+p}f(-(1-\lambda_j)e_1+e_j+x')+(1-\lambda_j)^{pq}f(e_j+\lambda_j\,x').
\end{aligned}$$

So $f(x_1e_1+e_j+x')=0$ for all x_1 and $x'=x_2e_2+\cdots+x_{j-1}e_{j-1}$. By (5), $f(e_1+x_je_j+x')=0$ for all x_j and $x'=x_2e_2+\cdots+x_{j-1}e_{j-1}$. The homogeneity of f finally shows that $f(x_1e_1+\cdots+x_je_j)=0$ for all $x_1,\dots,x_j$. $\square$

Acknowledgements

The authors would like to thank the referees for their careful reading and helpful remarks. The research of the first named author was carried out in part during a stay at the University Freiburg funded by the European Network PHD, MCRN-511953. He is grateful to Daniel Hug and Rolf Schneider for their warm hospitality during this stay. His research was also supported by FWF project P18308-N12. The second named author was supported in part by the European Network PHD, MCRN-511953.

References

[1] S. Alesker, *Continuous rotation invariant valuations on convex sets*, Ann. of Math. (2) **149** (1999), 977–1005.

[2] S. Alesker, *Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture*, Geom. Funct. Anal. **11** (2001), 244–272.

[3] S. Alesker, *Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations*, J. Differential Geom. **63** (2003), 63–95.

[4] S. Alesker, *The multiplicative structure on continuous polynomial valuations*, Geom. Funct. Anal. **14** (2004), 1–26.

[5] H. Busemann, *A theorem on convex bodies of the Brunn-Minkowski type*, Proc. Nat. Acad. Sci. U. S. A. **35** (1949), 27–31.

[6] S. Campi and P. Gronchi, *The L^p-Busemann-Petty centroid inequality*, Adv. Math. **167** (2002), 128–141.

[7] R. Gardner, *A positive answer to the Busemann-Petty problem in three dimensions*, Ann. of Math. (2) **140** (1994), 435–447.

[8] R. Gardner, *Geometric tomography*, Cambridge University Press, Cambridge, 1995.

[9] R. Gardner, A. Koldobsky, and T. Schlumprecht, *An analytic solution to the Busemann-Petty problem on sections of convex bodies*, Ann. of Math. (2) **149** (1999), 691–703.

[10] E. Grinberg and G. Zhang, *Convolutions, transforms, and convex bodies*, Proc. London Math. Soc. (3) **78** (1999), 77–115.

[11] C. Haberl, *L_p intersection bodies*, in preparation.

[12] H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer, Berlin, 1957.

[13] N. J. Kalton and A. Koldobsky, *Intersection bodies and L_p spaces*, Adv. Math. **196** (2005), 257–275.

[14] D. A. Klain, *Star valuations and dual mixed volumes*, Adv. Math. **121** (1996), 80–101.

[15] D. A. Klain, *Invariant valuations on star-shaped sets*, Adv. Math. **125** (1997), 95–113.

[16] D. A. Klain and G.-C. Rota, *Introduction to geometric probability*, Cambridge University Press, Cambridge, 1997.

[17] A. Koldobsky, *A functional analytic approach to intersection bodies*, Geom. Funct. Anal. **10** (2000), 1507–1526.

[18] A. Koldobsky, *Fourier analysis in convex geometry*, Mathematical Surveys and Monographs, vol. 116, American Mathematical Society, Providence, RI, 2005.

[19] K. Leichtweiß, *Affine geometry of convex bodies*, Johann Ambrosius Barth, Heidelberg, 1998.

[20] M. Ludwig, *Valuations on polytopes containing the origin in their interiors*, Adv. Math. **170** (2002), 239–256.

[21] M. Ludwig, *Projection bodies and valuations*, Adv. Math. **172** (2002), 158–168.

[22] M. Ludwig, *Ellipsoids and matrix valued valuations*, Duke Math. J. **119** (2003), 159–188.

[23] M. Ludwig, *Minkowski valuations*, Trans. Amer. Math. Soc. **357** (2005), 4191–4213.

[24] M. Ludwig, *Intersection bodies and valuations*, Amer. J. Math., to appear.

[25] M. Ludwig and M. Reitzner, *A characterization of affine surface area*, Adv. Math. **147** (1999), 138–172.

[26] M. Ludwig and M. Reitzner, *Elementary moves on triangulations*, Discrete Comput. Geom. **35**, 527 - 536 (2006).

[27] E. Lutwak, *Intersection bodies and dual mixed volumes*, Adv. Math. **71** (1988), 232–261.

[28] E. Lutwak, *The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem*, J. Differential Geom. **38** (1993), 131–150.

[29] E. Lutwak, *The Brunn-Minkowski-Firey theory. II: Affine and geominimal surface areas*, Adv. Math. **118** (1996), 244–294.

[30] E. Lutwak, D. Yang, and G. Zhang, *L_p affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), 111–132.

[31] E. Lutwak, D. Yang, and G. Zhang, *Sharp affine L_p Sobolev inequalities*, J. Differential Geom. **62** (2002), 17–38.

[32] E. Lutwak and G. Zhang, *Blaschke-Santaló inequalities*, J. Differential Geom. **47** (1997), 1–16.

[33] P. McMullen, *Valuations and dissections*, Handbook of Convex Geometry, Vol. B (P.M. Gruber and J.M. Wills, eds.), North-Holland, Amsterdam, 1993, 933–990.

[34] P. McMullen and R. Schneider, *Valuations on convex bodies*, Convexity and its applications (P.M. Gruber and J.M. Wills, eds.), Birkhäuser, 1983, 170–247.

[35] B. Rubin, *Generalized cosine transforms and classes of star bodies*, preprint (2006), arXiv:math.FA/0602540.

[36] R. Schneider, *Convex bodies: the Brunn–Minkowski theory*, Cambridge Univ. Press, Cambridge, 1993.

[37] A. C. Thompson, *Minkowski geometry*, Cambridge University Press, Cambridge, 1996.

[38] V. Yaskin and M. Yaskina, *Centroid bodies and comparison of volumes*, Indiana Univ. Math. J. **55** (2006), 1175–1194.

[39] G. Zhang, *A positive solution to the Busemann-Petty problem in $\mathbf{R}^4$*, Ann. of Math. (2) **149** (1999), 535–543.

Christoph Haberl: Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstraße 8-10/104, 1040 Wien, Austria
E-mail address: christoph.haberl@tuwien.ac.at

Monika Ludwig: Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstraße 8-10/104, 1040 Wien, Austria
E-mail address: monika.ludwig@tuwien.ac.at