STAR BODY VALUED VALUATIONS

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Abstract. All linearly intertwining L_p harmonic valuations on convex polytopes are completely classified for nonzero p. The only symmetric ones turn out to be the polar L_p centroid bodies.

1. Introduction

L_p centroid bodies and their polars are fundamental concepts in modern convex geometry. For $p \geq 1$, they have been intensively studied; see e.g. [3], [5], [10], [24], [29], [31], [33], [37]. In addition, these bodies led to sharp information theoretic inequalities [32] and deep results concerning concentration of mass [38]. Moreover, they give rise to strong affine isoperimetric inequalities; see e.g. [3], [7], [31]. A recent result of Ludwig and the author [13] showed that for $-1 < p < 0$ polar L_p centroid bodies are the symmetric L_p analogs of intersection bodies. The class of intersection bodies turned out to be crucial for the solution of the Busemann-Petty problem; see [6], [9], [28], [45]. Current research is concerned with the analogy between the intersection body operator and polar L_p centroid bodies for $-1 < p < 1$; see e.g. [12], [15], [16], [44].

Let $p > -1$ be nonzero and let $K \subset \mathbb{R}^n$ be a convex body with nonempty interior. The Minkowski functional $\|\cdot\|_{\Gamma_p^* K}$ of the *polar* L_p *centroid body* $\Gamma_p^* K$ is defined by

$$\|u\|_{\Gamma_p^* K}^p = \int_K |x \cdot u|^p \, dx, \quad u \in S^{n-1}.$$

For $p \geq 1$, $\|\cdot\|_{\Gamma_p^* K}$ defines a norm with unit ball $\Gamma_p^* K$ and this body is in fact the polar of a unique convex body $\Gamma_p K$ called the L_p *centroid body* of K. Petty [39] introduced L_1 centroid bodies in the middle of the last century. For bodies of unit volume, the boundary of $\Gamma_1 K$ is just the locus of centroids of halves of K formed by slicing K by hyperplanes through the origin. For $p > 1$, L_p centroid bodies were defined by Lutwak and Zhang [33]. Subsequently, the definition of polar L_p centroid bodies was extended to nonzero p with $|p| < 1$ by Gardner and Giannopoulos [8].

In this paper, polar L_p centroid bodies are shown to be the unique nontrivial star body valued valuations which are compatible with the general linear group and have symmetric images.

A *valuation* is a function $\mathrm{Z} : \mathcal{Q} \to \langle \mathcal{G}, + \rangle$ defined on a class of subsets of $\mathbb{R}^n$ with values in an abelian semigroup $\langle \mathcal{G}, + \rangle$ which satisfies

$$\mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L) = \mathrm{Z}\, K + \mathrm{Z}\, L,$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{Q}$. The theory of real valued valuations lies at the very core of geometry; see e.g. [11], [19], [35], and [36]. Many important real valued

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functions in convex geometry are valuations and are characterized by their invariance or covariance behaviour with respect to certain groups of transformations; see e.g. [1], [2], [17], [18], [22], [26], [27]. Ludwig's seminal work [21], [23]–[25] showed that fundamental operators in convexity and geometric tomography can be classified as convex or star body valued valuations which are compatible with the general linear group (see also [13], [42]).

We are concerned with valuations defined on $\mathcal{P}_o^n$, i.e. convex polytopes containing the origin, which have star bodies as images. A compact starshaped subset S of $\mathbb{R}^n$ containing the origin in its interior is called a *star body* if its Minkowski functional $\|\cdot\|_S$ is a continuous function on the unit sphere. The set $\mathcal{S}^n$ of star bodies is turned into an abelian semigroup if it is endowed with L_p harmonic addition. The *L_p harmonic sum* $K \hat{+}_p L$ of two star bodies $K, L \in \mathcal{S}^n$ is defined for nonzero p by

$$\|\cdot\|_{K \hat{+}_p L}^p = \|\cdot\|_K^p + \|\cdot\|_L^p.$$

We denote by $\langle \mathcal{S}^n, \hat{+}_p \rangle$ the monoid obtained from the semigroup $\mathcal{S}^n$ (equipped with $\hat{+}_p$) by adjoining an identity element. A valuation $\mathrm{Z} : \mathcal{P}_o^n \to \langle \mathcal{S}^n, \hat{+}_p \rangle$ is called an *L_p harmonic valuation*.

Let $\mathrm{Z} : \mathcal{P}_o^n \to \langle \mathcal{S}^n, \hat{+}_p \rangle$ and denote by $\mathrm{SL}(n)$ the special linear group. The map Z is called $\mathrm{SL}(n)$ *covariant*, if for all $\phi \in \mathrm{SL}(n)$ and all polytopes $P \in \mathcal{P}_o^n$,

$$\mathrm{Z}(\phi P) = \phi \, \mathrm{Z} \, P.$$

It is called $\mathrm{SL}(n)$ *contravariant*, if for all $\phi \in \mathrm{SL}(n)$ and all polytopes $P \in \mathcal{P}_o^n$,

$$\mathrm{Z}(\phi P) = \phi^{-t} \, \mathrm{Z} \, P,$$

where ϕ^{-t} is the inverse of the transpose of ϕ. We call Z *homogeneous*, if there exists an $r \in \mathbb{R}$ such that $\mathrm{Z} \, \lambda P = \lambda^r \, \mathrm{Z} \, P$ for every $\lambda > 0$ and all $P \in \mathcal{P}_o^n$. All group actions on $\langle \mathcal{S}^n, \hat{+}_p \rangle$ in the above definitions are the unique extensions of the corresponding actions on $\mathcal{S}^n$. Finally, Z is called *linearly intertwining* if it is co- or contravariant and homogeneous. We remark that the notions of co- and contravariance were crucial for previous classifications of body valued valuations; see [13], [21]–[25].

Our main result is the following classification theorem.

Theorem 1. *Suppose $n \geq 3$, $p \neq 0$, and let $\mathrm{Z} : \mathcal{P}_o^n \to \langle \mathcal{S}^n, \hat{+}_p \rangle$ be a linearly intertwining valuation. For $p > -1$, the operator Z is nontrivial if and only if there exist positive constants c_1, c_2 such that*

$$\mathrm{Z} \, P = c_1 \, \Gamma_p^{+,*} P \, \hat{+}_p \, c_2 \, \Gamma_p^{-,*} P$$

for every $P \in \mathcal{P}_o^n$. If $p \leq -1$, then Z is trivial.

Here, a valuation is called trivial, if it maps every element of its domain to the identity element. The operators $\Gamma_p^{\pm,*}$ are closely related to the generalized Minkowski-Funk transform [40] and are the asymmetric L_p analogs of the intersection body operator for negative p; see [12], [13]. A detailed definition of $\Gamma_p^{\pm,*}$ will be given in Section 2.

If we denote centrally symmetric star bodies by $\mathcal{S}_c^n$, then we obtain from the above theorem the following characterization of polar L_p centroid bodies.

Theorem 2. *Suppose $n \geq 3$, $p \neq 0$, and let $\mathrm{Z} : \mathcal{P}_o^n \to \langle \mathcal{S}_c^n, \hat{+}_p \rangle$ be a linearly intertwining valuation. For $p > -1$, the operator Z is nontrivial if and only if there* *exists a positive constant c such that*

$$\mathrm{Z}\, P = c\, \Gamma_p^* P$$

for every $P \in \mathcal{P}_o^n$. If $p \le -1$, then Z is trivial.

Here, the polar L_p centroid body of a lower-dimensional polytope is defined as the identity element.

In Section 3 we prove the above results and we also provide complete classifications in dimension $n = 2$. Moreover, we address the question whether polar L_p centroid bodies are characterized as linearly intertwining valuations defined on general convex bodies containing the origin. As will be shown in Section 5, this is not the case in general. We construct a counterexample which is closely related to affine surface area, a classical notion from affine differential geometry. This counterexample is a nontrivial linearly intertwining body valued valuation which vanishes on polytopes. However, we establish characterizations of polar L_p centroid bodies as monotone valuations on arbitrary convex bodies in Section 4.

2. Preliminaries

We work in Euclidean n-space $\mathbb{R}^n$. The canonical basis vectors are denoted by $e_1, e_2, \ldots, e_n$. We write $\cdot$ for the usual inner product on $\mathbb{R}^n$ and $\|\cdot\|$ for the induced norm. Let $x^\perp$ be the hyperplane through the origin orthogonal to x. The closed halfspaces determined by a hyperplane H are denoted by $H^\pm$. We write $B(a, r)$ for the ball with center $a \in \mathbb{R}^n$ and radius r. The boundary of $B(o, 1)$ is denoted by S^{n-1}.

As usual, int, bd, lin, conv stand for interior, boundary, linear hull, and convex hull, respectively.

The set of *convex bodies* in $\mathbb{R}^n$, i.e. nonempty, compact convex subsets of $\mathbb{R}^n$, is denoted by $\mathcal{K}^n$. Associated to a convex body $K \in \mathcal{K}^n$ is its *support function* $h(K, x) = \max\{x \cdot y : y \in K\}$, for $x \in \mathbb{R}^n$. Convex bodies containing the origin are denoted by $\mathcal{K}_o^n$. We write $\mathcal{P}^n$ for the set of convex polytopes, i.e. the convex hull of finitely many points, and abbreviate $\mathcal{P}_o^n = \mathcal{P}^n \cap \mathcal{K}_o^n$.

Star bodies were already defined in the introduction. For $S \in \mathcal{S}^n$, the *Minkowski functional* $\|\cdot\|_S$ is defined by $\|x\|_S = \min\{\lambda \ge 0 :\ x \in \lambda S\}$ for all $x \in \mathbb{R}^n$. Note that the Minkowski functional is homogeneous of degree one. We defined the monoid $\langle \mathcal{S}^n, \hat{+}_p \rangle$ by adjoining an identity element $e \notin \mathcal{S}^n$ to $\mathcal{S}^n$ equipped with L_p harmonic addition. Geometrically, the identity element can be interpreted as the origin for $p < 0$ and $\mathbb{R}^n$ for $p > 0$. Up to isomorphisms, this extension is unique and is the smallest monoid containing the semigroup $\mathcal{S}^n$.

Let G be a subgroup of the general linear group $\mathrm{GL}(n)$. We work with extensions of group actions of G on $\mathcal{S}^n$ to actions of G on $\mathcal{S}^n \cup \{e\}$ where $e \notin \mathcal{S}^n$. Such extensions are unique because the orbit of e has to be the set containing only e. In the sequel, all actions are defined in this way and therefore we will not further comment on it.

Suppose that $q \in \mathbb{R}$ and let $\mathrm{Z} : \mathcal{Q} \to \langle \mathcal{S}^n, \hat{+}_p \rangle$ be an operator defined on a set $\mathcal{Q} \subset \mathcal{K}^n$. Denote by $\mathrm{GL}^+(n)$ the group of linear maps with positive determinant and suppose that $\mathcal{Q}$ is closed with respect to the standard action of $\mathrm{GL}^+(n)$. The map Z is called $\mathrm{GL}^+(n)$ *covariant of weight* q, if for all $\phi \in \mathrm{GL}^+(n)$ and all bodies $Q \in \mathcal{Q}$,

$$\mathrm{Z}(\phi Q) = (\det \phi)^q\, \phi\, \mathrm{Z}\, Q,$$

where $\det\phi$ denotes the determinant of ϕ. It is called $\mathrm{GL}^+(n)$ *contravariant of weight* q, if for all $\phi\in\mathrm{GL}^+(n)$ and all bodies $Q\in\mathcal{Q}$,

$$\mathrm{Z}(\phi Q)=(\det\phi)^q\,\phi^{-t}\,\mathrm{Z}\,Q.$$

We call a map $\mathrm{Z}:\mathcal{Q}\to\langle\mathcal{S}^n,\hat{+}_p\rangle$ *linearly intertwining* if it is $\mathrm{GL}^+(n)$ co- or contravariant of some weight. Note that this definition for $\mathcal{Q}=\mathcal{P}_o^n$ is equivalent to the one given in the introduction. In order to classify linearly intertwining valuations it is therefore enough to characterize $\mathrm{GL}^+(n)$ co- and contravariant valuations of arbitrary weight.

Now, we are going to define examples of linearly intertwining operators. For nonzero $p>-1$ and convex bodies $K\in\mathcal{K}^n$ which contain the origin in their interiors, we define $\Gamma_p^{\pm,*}K$ by

$$\|u\|^p_{\Gamma_p^{+,*}K}=\int_{K\cap u^+}|x\cdot u|^p\,dx,\quad u\in S^{n-1},$$

and $\Gamma_p^{-,*}K=\Gamma_p^{+,*}(-K)$. Here, u^+ denotes the halfspace $\{x\in\mathbb{R}^n:\ x\cdot u\geq 0\}$. As was noted in the introduction, the operators $\Gamma_p^{\pm,*}$ are closely related to the generalized Minkowski-Funk transform [40] and are the asymmetric L_p analogs of intersection bodies. We extend the operator $c_1\,\Gamma_p^{+,*}\,\hat{+}_p\,c_2\,\Gamma_p^{-,*}$ for positive constants c_1, c_2 from convex bodies containing the origin in their interiors to $\mathcal{K}_o^n$ as follows. For n-dimensional $K\in\mathcal{K}_o^n$ we set

$$\|u\|^p_{c_1\,\Gamma_p^{+,*}K\hat{+}_pc_2\,\Gamma_p^{-,*}K}=c_1\int_{K\cap u^+}|x\cdot u|^p\,dx+c_2\int_{K\cap u^-}|x\cdot u|^p\,dx,$$

where u^- denotes the halfspace $\{x\in\mathbb{R}^n:\ x\cdot u\leq 0\}$, and for bodies K with $\dim K<n$ we define $c_1\Gamma_p^{+,*}K\,\hat{+}_p\,c_2\Gamma_p^{-,*}K=e$. This is reasonable by our geometric interpretation of the identity element. The resulting map is in fact a linearly intertwining valuation on $\mathcal{K}_o^n$.

3. Proofs of polytopal classifications

3.1. **Reduction.** In this subsection we establish two lemmas which show that in order to derive Theorem 1 it is enough to know the values of the involved valuations on simplices. We start by investigating the behaviour of co- or contravariant operators on lower-dimensional bodies. Let $\langle M,+\rangle$ be a monoid with identity element e and $\mathcal{Q}\subset\mathcal{K}^n$. We call a valuation $\mathrm{Z}:\mathcal{Q}\to\langle M,+\rangle$ *simple*, if bodies of dimension less than n are mapped to e. From now on it is assumed throughout that $p\neq 0$.

Lemma 1. *Let $\mathcal{Q}$ be either $\mathcal{P}_o^n$ or $\mathcal{K}_o^n$ and suppose $\mathrm{Z}:\mathcal{Q}\to\langle\mathcal{S}^n,\hat{+}_p\rangle$ is a $\mathrm{GL}^+(n)$ co- or contravariant operator of arbitrary weight. Then Z is simple.*

Proof. Because of the assumed $\mathrm{GL}^+(n)$ co- or contravariance it is enough to prove that a body Q which is contained in $e_n^\perp$ is mapped to the identity element.
For $s>0$, we define $\phi\in\mathrm{GL}^+(n)$ by

$$\phi e_i=e_i,\quad i=1,\ldots,n-1,\quad\text{and}\quad\phi e_n=se_n.$$

First, assume that Z is $\mathrm{GL}^+(n)$ contravariant of weight q. Thus

$$\mathrm{Z}\,Q=\mathrm{Z}\,\phi Q=(\det\phi)^q\phi^{-t}\,\mathrm{Z}\,Q.$$

Suppose $\mathrm{Z}\,Q\neq e$. Then we have

$$\|x\|_{\mathrm{Z}\,Q}=\|(\det\phi)^{-q}\phi^t x\|_{\mathrm{Z}\,Q}=(\det\phi)^{-q}\|\phi x\|_{\mathrm{Z}\,Q}\tag{1}$$

for every $x \in \mathbb{R}^n$. $\mathrm{Z}\,Q$ contains a closed Euclidean ball with center at the origin and is contained in such a ball. Thus there exist positive constants c_1, c_2 with

$$c_1\|x\| \leq \|x\|_{\mathrm{Z}\,Q} \leq c_2\|x\|, \tag{2}$$

for every $x \in \mathbb{R}^n$. Together with (1), this implies

$$c_1\sqrt{\sum_{i=1}^{n-1} x_i^2 + (sx_n)^2} \leq c_2 s^q \sqrt{\sum_{i=1}^{n} x_i^2}, \tag{3}$$

for all $x \in \mathbb{R}^n$. Note that $s > 0$ was arbitrary. Evaluate (3) at e_1 and take the limit $s \to 0^+$. This yields a contradiction for positive q. If $q = 0$, then the limit $s \to \infty$ in relation (3) at e_n gives a contradiction. Finally, for negative q consider (3) at e_1 and let $s \to \infty$. We obtain again a contradiction. Thus we proved that for all weights $\mathrm{Z}\,Q = e$.

Second, suppose that Z is $\mathrm{GL}^+(n)$ covariant. Then, on the assumption that $\mathrm{Z}\,Q \in \mathcal{S}^n$, one derives from

$$\|x\|_{\mathrm{Z}\,Q} = (\det\phi)^{-q}\|\phi^{-1}x\|_{\mathrm{Z}\,Q}$$

and (2) that

$$c_1 s^q\sqrt{\sum_{i=1}^{n} x_i^2} \leq c_2\sqrt{\sum_{i=1}^{n-1} x_i^2 + \left(\frac{x_n}{s}\right)^2}$$

holds on $\mathbb{R}^n$. For $q > 0$, $q = 0$, $q < 0$ consider this inequality at points e_1, e_n, e_1 and take limits $s \to \infty$, $s \to \infty$ and $s \to 0^+$, respectively. As above, we obtain that $\mathrm{Z}\,Q$ has to be the identity element. □

Lemma 2. *Let $\langle M, + \rangle$ be an abelian monoid with cancellation law. Then a simple valuation $\mathrm{Z} : \mathcal{P}_o^n \to \langle M, + \rangle$ is uniquely determined by its values on n-dimensional simplices having one vertex at the origin.*

As usual, we say that an abelian monoid $\langle M, + \rangle$ satisfies the *cancellation law* if the equality $x + z = y + z$ for $x, y, z \in M$ implies $x = y$.

Proof. A finite set $\mathcal{T}_P$ of n-dimensional simplices is a *triangulation* of an n-dimensional polytope $P \subset \mathbb{R}^n$ if the union of all simplices in $\mathcal{T}_P$ equals P and no pair of simplices intersects in a set of dimension n. A *starring of P at x* is a triangulation such that there is an $x \in P$ for which every simplex in $\mathcal{T}_P$ has a vertex at x.

Let $P \in \mathcal{P}_o^n$ be an n-dimensional polytope. It is well-known that for arbitrary $x \in P$ there exists a starring of P at x. Indeed, for $n = 1$ it is trivial. Suppose that the assertion is true for $(n-1)$-dimensional polytopes and denote by F_j, $j = 1, \ldots, k$ the facets of an n-dimensional polytope P. We choose starrings $\mathcal{T}_{F_j}$ of F_j for those facets which do not contain the given point x. Thus the convex hulls of x and the $(n-1)$-dimensional simplices in $\mathcal{T}_{F_j}$ define the desired starring.

The proof of the lemma is finished if we can show that

$$P = P_1 \cup P_2 \cup \cdots \cup P_k, \quad P, P_1, \ldots, P_k \in \mathcal{P}_o^n, \quad \dim(P_i \cap P_j) < n \quad \text{for } i \neq j$$

implies

$$\mathrm{Z}\,P = \sum_{i=1}^{k} \mathrm{Z}\,P_i$$

for an n-dimensional polytope $P \in \mathcal{P}_o^n$. We proceed by induction on k. For $k = 1, 2$ this is trivial. It also holds true if $P_1 = P$ because then $\dim P_i < n$ for $i \neq 1$ and

Z is assumed to be simple. Suppose that our desired conclusion is true for at most $k-1$ polytopes. Without loss of generality assume that $\dim P_1 = n$ and that P_1 is a proper subpolytope of P. Then P_1 has a facet F containing the origin such that

$$P \cap \operatorname{int}(\operatorname{lin} F)^+ \neq \emptyset, \quad \text{and} \quad P \cap \operatorname{int}(\operatorname{lin} F)^- \neq \emptyset.$$

For simplicity, we write $H := \operatorname{lin} F$ and assume that $P_1 \subset H^-$. Define

$$P^- := P \cap H^-, \quad P^+ := P \cap H^+, \quad P_i^- := P_i \cap H^-, \quad P_i^+ := P_i \cap H^+,$$

for $i = 1, \ldots, k$. From the fact that $P^+ = P_2^+ \cup \cdots \cup P_k^+$, the induction hypothesis, and the simplicity of Z we obtain

$$\mathrm{Z}\, P^+ = \sum_{i=2}^{k} \mathrm{Z}\, P_i^+ = \sum_{i=1}^{k} \mathrm{Z}\, P_i^+,$$

and therefore

$$\begin{aligned} \mathrm{Z}\, P + \sum_{i=1}^{k} \mathrm{Z}\, P_i^- &= \mathrm{Z}\, P^+ + \mathrm{Z}\, P^- + \sum_{i=1}^{k} \mathrm{Z}\, P_i^- \\ &= \sum_{i=1}^{k} (\mathrm{Z}\, P_i^+ + \mathrm{Z}\, P_i^-) + \mathrm{Z}\, P^- \\ &= \sum_{i=1}^{k} \mathrm{Z}\, P_i + \mathrm{Z}\, P^-. \end{aligned} \tag{4}$$

If $P_1^- = P^-$, we have $\sum_{i=1}^{k} \mathrm{Z}\, P_i^- = \mathrm{Z}\, P^-$ and we are done by the cancellation law. Otherwise, we can proceed as above but now for the polytope P^-. So cutting with a suitable hyperplane H_2 gives

$$P^{-,2} := P^- \cap H_2^-, \quad P^{+,2} := P^- \cap H_2^+, \quad P_i^{-,2} := P_i^- \cap H_2^-, \quad P_i^{+,2} := P_i^- \cap H_2^+,$$

and

$$\mathrm{Z}\, P^- + \sum_{i=1}^{k} \mathrm{Z}\, P_i^{-,2} = \sum_{i=1}^{k} \mathrm{Z}\, P_i^- + \mathrm{Z}\, P^{-,2}.$$

By (4) we therefore get

$$\begin{aligned} \mathrm{Z}\, P + \sum_{i=1}^{k} \mathrm{Z}\, P_i^- + \sum_{i=1}^{k} \mathrm{Z}\, P_i^{-,2} &= \sum_{i=1}^{k} \mathrm{Z}\, P_i + \mathrm{Z}\, P^- + \sum_{i=1}^{k} \mathrm{Z}\, P_i^{-,2} \\ &= \sum_{i=1}^{k} \mathrm{Z}\, P_i + \sum_{i=1}^{k} \mathrm{Z}\, P_i^- + \mathrm{Z}\, P^{-,2}. \end{aligned}$$

The cancellation law proves

$$\mathrm{Z}\, P + \sum_{i=1}^{k} \mathrm{Z}\, P_i^{-,2} = \sum_{i=1}^{k} \mathrm{Z}\, P_i + \mathrm{Z}\, P^{-,2}.$$

Repeating this procedure finitely many times (depending on the number of supporting hyperplanes of P_1 which contain the origin), we are in the situation that $P_1^{-,m} = P^{-,m}$. $\square$

Note that $\langle \mathcal{S}^n, \hat{+}_p \rangle$ satisfies the cancellation law. Indeed, suppose for a moment that S_1 is the identity element e whereas S_2 is contained in $\mathcal{S}^n$. Since $e \notin \mathcal{S}^n$ we obtain $S_1 \hat{+}_p e = e \neq S_2 = S_2 \hat{+}_p e$. But also for an arbitrary $R \in \mathcal{S}^n$ we would have $S_1 \hat{+}_p R = R \neq S_2 \hat{+}_p R$ because $\|\cdot\|^p_{S_2 \hat{+}_p R} = \|\cdot\|^p_{S_2} + \|\cdot\|^p_R > \|\cdot\|^p_R$ and two bodies in $\mathcal{S}^n$ coincide precisely when their Minkowski functionals are equal. If we interchange the roles of S_1 and S_2 in the above lines we deduce that the relation

$$S_1 \hat{+}_p R = S_2 \hat{+}_p R, \qquad S_1,\, S_2,\, R \in \mathcal{S}^n \cup \{e\}. \tag{5}$$

implies that S_1 and S_2 are either both contained in $\mathcal{S}^n$ or both equal to e. If S_1 as well as S_2 are star bodies, then (5) yields $\|\cdot\|^p_{S_1} + \|\cdot\|^p_R = \|\cdot\|^p_{S_2} + \|\cdot\|^p_R$ and hence $\|\cdot\|_{S_1} = \|\cdot\|_{S_2}$. This shows $S_1 = S_2$.

From Lemmas 1 and 2 we therefore conclude that a $\mathrm{GL}^+(n)$ co- or contravariant valuation $\mathrm{Z} : \mathcal{P}^n_o \to \langle \mathcal{S}^n, \hat{+}_p \rangle$ is uniquely determined by its value on the standard simplex $T^n := \mathrm{conv}\{o, e_1, \ldots, e_n\}$.

3.2. **The 2-dimensional case.** For $0 < \lambda < 1$, we define two families of linear maps by

$$\begin{aligned} \phi e_2 &= (1-\lambda)e_1 + \lambda e_2, \qquad & \phi e_k &= e_k \text{ for } k \neq 2, \\ \psi e_1 &= (1-\lambda)e_1 + \lambda e_2, \qquad & \psi e_k &= e_k \text{ for } k \neq 1. \end{aligned}$$

Note that

$$\begin{aligned} \phi^{-1} e_2 &= -\frac{1-\lambda}{\lambda} e_1 + \frac{1}{\lambda} e_2, \qquad & \phi^{-1} e_k &= e_k \text{ for } k \neq 2, \\ \psi^{-1} e_1 &= \frac{1}{1-\lambda} e_1 - \frac{\lambda}{1-\lambda} e_2, \qquad & \psi^{-1} e_k &= e_k \text{ for } k \neq 1. \end{aligned}$$

Let H be the hyperplane through o with normal vector $\lambda e_1 - (1-\lambda)e_2$. Then we have $T^n \cap H^+ = \phi T^n$ and $T^n \cap H^- = \psi T^n$. So for a simple valuation $\mathrm{Z} : \mathcal{P}^n_o \to \langle \mathcal{S}^n, \hat{+}_p \rangle$ we obtain

$$\mathrm{Z}\, T^n = \mathrm{Z}(\phi T^n) \,\hat{+}_p\, \mathrm{Z}(\psi T^n). \tag{6}$$

If $\mathrm{Z} : \mathcal{P}^2_o \to \langle \mathcal{S}^2, \hat{+}_p \rangle$ is a nontrivial $\mathrm{GL}^+(2)$ contravariant valuation, then $\mathrm{Z}\, T^2 \in \mathcal{S}^2$ and $f(x_1, x_2) := \|(x_1, x_2)\|^p_{\mathrm{Z}\, T^2}$ is a continuous function on $\mathbb{R}^2 \backslash \{o\}$ which is positively homogeneous of degree p. Moreover, equation (6) reads as

$$f(x_1, x_2) = \lambda^{-pq} f(x_1, (1-\lambda)x_1 + \lambda x_2) + (1-\lambda)^{-pq} f((1-\lambda)x_1 + \lambda x_2, x_2) \tag{7}$$

for arbitrary $(x_1, x_2) \in \mathbb{R}^2 \backslash \{o\}$. It turns out in the next lemma, that such functions f are of very special form. Indeed, for given $q \in \mathbb{R}$, we define a function $g_{p,q}$ on $\mathbb{R}^2$ by

$$g_{p,q}(x_1, x_2) = \begin{cases} (x_1^{p-pq} - x_2^{p-pq})(x_1 - x_2)^{pq} & \text{for } 0 < x_2 < x_1, \\ x_1^{p-pq}(x_1 - x_2)^{pq} & \text{for } x_1 > 0, x_2 \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Define linear transformations γ_i, $i = 0, 1, 2$, by

$$\gamma_0(x_1, x_2) = (-x_1, -x_2), \;\; \gamma_1(x_1, x_2) = (x_2, x_1), \;\; \gamma_2(x_1, x_2) = (-x_2, -x_1),$$

that is, γ_0, γ_1, and γ_2 are the reflections with respect to the origin, the line $x_1 = x_2$, and the line $x_1 = -x_2$, respectively.

Lemma 3. *Let $f : \mathbb{R}^2\backslash\{o\} \to \mathbb{R}$ be a function which is positively homogeneous of degree p and satisfies (7) for every $0 < \lambda < 1$. Then*

$$f = f(1,0)\, g_{p,q} + f(-1,0)\, g_{p,q} \circ \gamma_0 + f(0,1)\, g_{p,q} \circ \gamma_1 + f(0,-1)\, g_{p,q} \circ \gamma_2 \tag{8}$$

on $\mathbb{R}^2\backslash\{(x_1,x_2) : x_1 = x_2\}$.

Proof. For points $(x_1,0)$, $(0,x_2)$ with $x_1,x_2 \in \mathbb{R}\backslash\{0\}$, equality (8) immediately follows from the homogeneity of f and the definition of $g_{p,q}$.
Evaluate equation (7) at the point $(1,0)$. This gives

$$f(1,0) = \lambda^{-pq} f(1,1-\lambda) + (1-\lambda)^{-pq} f(1-\lambda,0).$$

Since f is positively homogeneous of degree p we obtain

$$f(1,0) = \lambda^{-pq} f(1,1-\lambda) + (1-\lambda)^{p-pq} f(1,0),$$

and therefore

$$f(1,1-\lambda) = \lambda^{pq}(1-(1-\lambda)^{p-pq}) f(1,0). \tag{9}$$

Similarly, equation (7) evaluated at the points $-(1,0)$, $\pm(0,1)$, $\pm(-\lambda,1-\lambda)$ and the homogeneity of f yield

$$f(-1,\lambda-1) = \lambda^{pq}(1-(1-\lambda)^{p-pq}) f(-1,0), \tag{10}$$

$$f(\lambda,1) = (1-\lambda)^{pq}(1-\lambda^{p-pq}) f(0,1), \tag{11}$$

$$f(-\lambda,-1) = (1-\lambda)^{pq}(1-\lambda^{p-pq}) f(0,-1), \tag{12}$$

$$f(-\lambda,1-\lambda) = \lambda^{p-pq} f(-1,0) + (1-\lambda)^{p-pq} f(0,1), \tag{13}$$

$$f(\lambda,\lambda-1) = \lambda^{p-pq} f(1,0) + (1-\lambda)^{p-pq} f(0,-1). \tag{14}$$

First, suppose that $x_1 > x_2 > 0$. By (9) we obtain

$$\begin{aligned} f(x_1,x_2) &= x_1^p f(1, 1-(1-x_2/x_1)) = x_1^p \left(1-\frac{x_2}{x_1}\right)^{pq} \left(1-\left(\frac{x_2}{x_1}\right)^{p-pq}\right) f(1,0) \\ &= f(1,0)\, g_{p,q}(x_1,x_2). \end{aligned}$$

Since $g_{p,q}\circ\gamma_0$, $g_{p,q}\circ\gamma_1$, and $g_{p,q}\circ\gamma_2$ are zero for $x_1 > x_2 > 0$, equation (8) holds in this part of the plane. Relation (10) gives

$$\begin{aligned} f(-x_1,-x_2) &= x_1^p f(-1, (1-x_2/x_1)-1) \\ &= x_1^p \left(1-\frac{x_2}{x_1}\right)^{pq} \left(1-\left(\frac{x_2}{x_1}\right)^{p-pq}\right) f(-1,0) \\ &= f(-1,0)(g_{p,q}\circ\gamma_0)(-x_1,-x_2). \end{aligned}$$

But $g_{p,q}$, $g_{p,q}\circ\gamma_1$ as well as $g_{p,q}\circ\gamma_2$ vanish at points $(-x_1,-x_2)$ with $x_1 > x_2 > 0$ and therefore (8) is true at these points.
Second, assume $x_2 > x_1 > 0$. Formulae (11) and (12) yield

$$\begin{aligned} f(x_1,x_2) &= x_2^p f(x_1/x_2, 1) = x_2^p \left(1-\frac{x_1}{x_2}\right)^{pq} \left(1-\left(\frac{x_1}{x_2}\right)^{p-pq}\right) f(0,1) \\ &= f(0,1)(g_{p,q}\circ\gamma_1)(x_1,x_2), \\ f(-x_1,-x_2) &= x_2^p f(-x_1/x_2, -1) = x_2^p \left(1-\frac{x_1}{x_2}\right)^{pq} \left(1-\left(\frac{x_1}{x_2}\right)^{p-pq}\right) f(0,-1) \\ &= f(0,-1)(g_{p,q}\circ\gamma_2)(-x_1,-x_2). \end{aligned}$$

Since $g_{p,q}$, $g_{p,q}\circ\gamma_0$, $g_{p,q}\circ\gamma_2$ are zero for $x_2>x_1>0$ and $g_{p,q}$, $g_{p,q}\circ\gamma_0$, $g_{p,q}\circ\gamma_1$ vanish at points $(-x_1,-x_2)$ with $x_2>x_1>0$, it remains to prove identity (8) at points with coordinates which have different signs.
So let x_1 and x_2 be greater than zero. By (13) and (14) we have

$$\begin{aligned}
f(-x_1,x_2) &= (x_1+x_2)^p f(-x_1/(x_1+x_2), 1-x_1/(x_1+x_2))\\
&= (x_1+x_2)^p\left(\left(\frac{x_1}{x_1+x_2}\right)^{p-pq} f(-1,0)+\left(\frac{x_2}{x_1+x_2}\right)^{p-pq} f(0,1)\right)\\
&= f(0,1)(g_{p,q}\circ\gamma_1)(-x_1,x_2)+f(-1,0)(g_{p,q}\circ\gamma_0)(-x_1,x_2),\\
f(x_1,-x_2) &= (x_1+x_2)^p f(x_1/(x_1+x_2), x_1/(x_1+x_2)-1)\\
&= (x_1+x_2)^p\left(\left(\frac{x_1}{x_1+x_2}\right)^{p-pq} f(1,0)+\left(\frac{x_2}{x_1+x_2}\right)^{p-pq} f(0,-1)\right)\\
&= f(0,-1)(g_{p,q}\circ\gamma_2)(x_1,-x_2)+f(1,0)g_{p,q}(x_1,-x_2).
\end{aligned}$$

The fact that $g_{p,q}$ and $g_{p,q}\circ\gamma_2$ are zero in the second quadrant and $g_{p,q}\circ\gamma_0$, $g_{p,q}\circ\gamma_1$ are zero in the fourth quadrant completes the proof. □

Now, we are in the position to establish the 2-dimensional classifications. We start with the contravariant case.

Lemma 4. *Let* $\mathrm{Z}:\mathcal{P}_o^2\to\langle\mathcal{S}^2,\hat{+}_p\rangle$ *be a valuation which is* $\mathrm{GL}^+(2)$ *contravariant of weight* q. *For* $p>-1$, $q=-1/p$, *and nontrivial* Z, *there exist positive constants* c_1, c_2 *such that*

$$\mathrm{Z}\,P=c_1\Gamma_p^{+,*}P\;\hat{+}_p\;c_2\Gamma_p^{-,*}P,\quad \textit{for every } P\in\mathcal{P}_o^2.$$

In all other cases, Z *is trivial.*

Proof. Assume that Z is nontrivial. Thus $\mathrm{Z}\,T^2\in\mathcal{S}^2$. As before, we set $f(x_1,x_2):=\|(x_1,x_2)\|^p_{\mathrm{Z}\,T^2}$. So f is positively homogeneous of degree p and Lemma 3 implies

$$f=f(1,0)\,g_{p,q}+f(-1,0)\,g_{p,q}\circ\gamma_0+f(0,1)\,g_{p,q}\circ\gamma_1+f(0,-1)\,g_{p,q}\circ\gamma_2$$

on $\mathbb{R}^2\backslash\{(x_1,x_2):x_1=x_2\}$. Let $p>-1$ and $q=-1/p$. Since f is continuous, the latter representation yields

$$f(x_1,x_1)=\lim_{x_2\to x_1^-}\frac{x_1^{1+p}-x_2^{1+p}}{x_1-x_2}f(1,0)=\lim_{x_2\to x_1^+}\frac{x_2^{1+p}-x_1^{1+p}}{x_2-x_1}f(0,1)$$

for positive x_1. Thus $f(1,0)=f(0,1)$ and an analogous observation for negative values x_1 proves $f(-1,0)=f(0,-1)$. So f satisfies

$$f=f(1,0)\,(g_{p,-1/p}+\,g_{p,-1/p}\circ\gamma_1)+f(-1,0)(\,g_{p,-1/p}\circ\gamma_0+\,g_{p,-1/p}\circ\gamma_2).$$

We claim that

$$\int_{T^2\cap u^+}|x\cdot u|^p\,dx=(p^2+3p+2)^{-1}(g_{p,-1/p}(u)+g_{p,-1/p}\circ\gamma_1(u)) \tag{15}$$

almost everywhere. Indeed, let $u=(u_1,u_2)$. If $u_1\le 0$ and $u_2\le 0$, then the left hand side of (15) vanishes. By the definition of $g_{p,-1/p}$ this is also the case for the right hand side of (15). For other u, use Fubini's theorem to rewrite the integral in

(15) as a certain double integral. Then one can calculate this integral and observes that (15) holds. Moreover, we have

$$\begin{aligned}\int_{T^2\cap u^-} |x\cdot u|^p\,dx &= \int_{T^2\cap(\gamma_0(u))^+} |x\cdot\gamma_0(u)|^p\,dx\\ &= (p^2+3p+2)^{-1}(g_{p,-1/p}\circ\gamma_0(u)+g_{p,-1/p}\circ\gamma_2(u))\end{aligned}$$

almost everywhere. Thus the first part of the lemma is settled.
Still under the assumption that $\mathrm{Z}\,T^2\in\mathcal{S}^2$, we investigate the relation

$$f(x_1,x_2)=(x_1^{p-pq}-x_2^{p-pq})(x_1-x_2)^{pq}f(1,0) \tag{16}$$

for other weights q and $x_1>x_2>0$. For $pq>-1$, the right hand side of (16) converges to zero when $x_2\to x_1^-$. If $pq<-1$ and $p-pq>0$, it assumes arbitrary large values as $x_2\to x_1^-$. For $pq<-1$ and $p-pq\le 0$, or $q=-1/p$ and $p\le -1$, the right hand side of (16) is less or equal than zero. But f is a positive, continuous function on $\mathbb{R}^n\backslash\{o\}$, a contradiction. □

We define the rotation

$$\psi_{\pi/2}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

Then the characterization of covariant valuations reads as follows.

Lemma 5. *Let* $\mathrm{Z}:\mathcal{P}_o^2\to\langle\mathcal{S}^2,\hat{+}_p\rangle$ *be a valuation which is* $\mathrm{GL}^+(2)$ *covariant of weight* q. *For* $p>-1$, $q=-1/p-1$, *and nontrivial* Z, *there exist positive constants* c_1, c_2 *such that*

$$\mathrm{Z}\,P=\psi_{\pi/2}^{-1}(c_1\Gamma_p^{+,*}P\,\hat{+}_p\,c_2\Gamma_p^{-,*}P),\quad \textit{for every } P\in\mathcal{P}_o^2.$$

In all other cases, Z *is trivial.*

Proof. Define an operator $\overline{\mathrm{Z}}$ by

$$\overline{\mathrm{Z}}P:=\psi_{\pi/2}\,\mathrm{Z}\,P$$

for every $P\in\mathcal{P}_o^2$. Since $\psi_{\pi/2}(S_1\hat{+}_pS_2)=\psi_{\pi/2}S_1\hat{+}_p\psi_{\pi/2}S_2$ for all $S_1,S_2\in\mathcal{S}^2\cup\{e\}$, $\overline{\mathrm{Z}}$ is a valuation. For every $\phi\in\mathrm{GL}^+(2)$ we have

$$\psi_{\pi/2}\phi\psi_{\pi/2}^{-1}=(\det\phi)\phi^{-t}.$$

So if Z is $\mathrm{GL}^+(2)$ covariant of weight q, then $\overline{\mathrm{Z}}$ is contravariant of weight $q+1$. This and the already established characterization result of Lemma 4 finish the proof. □

If we combine the last two lemmas we arrive at a complete classification in dimension two.

Theorem 3. *Suppose* $p\neq 0$ *and let* $\mathrm{Z}:\mathcal{P}_o^2\to\langle\mathcal{S}^2,\hat{+}_p\rangle$ *be a linearly intertwining valuation. For* $p>-1$, *the operator* Z *is nontrivial if and only if there exist positive constants* c_1, c_2 *such that*

$$\mathrm{Z}\,P=c_1\Gamma_p^{+,*}P\,\hat{+}_p\,c_2\Gamma_p^{-,*}P$$

or

$$\mathrm{Z}\,P=\psi_{\pi/2}^{-1}(c_1\Gamma_p^{+,*}P\,\hat{+}_p\,c_2\Gamma_p^{-,*}P),$$

for every $P\in\mathcal{P}_o^2$. *If* $p\le -1$, *then* Z *is trivial.*

3.3. The n-dimensional case, $n \geq 3$.

Lemma 6. *Let $n \geq 3$ and $\mathrm{Z} : \mathcal{P}_o^n \to \langle \mathcal{S}^n, \hat{+}_p \rangle$ be a valuation which is $\mathrm{GL}^+(n)$ contravariant of weight q. For $p > -1$, $q = -1/p$, and Z nontrivial, there exist positive constants c_1, c_2 such that*

$$\mathrm{Z}\, P = c_1 \Gamma_p^{+,*} P \;\hat{+}_p\; c_2 \Gamma_p^{-,*} P, \quad \textit{for every } P \in \mathcal{P}_o^n.$$

In all other cases, Z is trivial.

Proof. Assume that $\mathrm{Z}\, T^n \in \mathcal{S}^n$ and set

$$f(x) = \|x\|_{\mathrm{Z}\, T^n}^p, \quad x \in \mathbb{R}^n \backslash \{o\}.$$

We further define a function $\bar{f} : \mathbb{R}^2 \backslash \{o\} \to \mathbb{R}$ by

$$\bar{f}(x_1, x_2) = f(x_1 e_1 + x_2 e_2).$$

From (6) we obtain

$$f(x) = \lambda^{-pq} f(\phi^t x) + (1-\lambda)^{-pq} f(\psi^t x)$$

for arbitrary $x \in \mathbb{R}^n \backslash \{o\}$. Thus $\bar{f}$ satisfies (7) and the proof of Lemma 4 shows that a nontrivial valuation can only exist for $p > -1$ and $q = -1/p$. Moreover,

$$\bar{f}(x_1, x_2) = \|(x_1, x_2)\|^p_{\bar{c}_1 \Gamma_p^{+,*} T^2 \hat{+}_p \bar{c}_2 \Gamma_p^{-,*} T^2}$$

for $p > -1$ and $q = -1/p$. Denote by $\mathbb{I}_A$ the indicator function of a set A. Since there exists a positive constant c with

$$\int_{T^2} |x_1 y_1 + x_2 y_2|^p \, \mathbb{I}_{(0,\infty)}(x_1 y_1 + x_2 y_2) \, dy_1 dy_2$$
$$= c \int_{T^n \cap (x_1 e_1 + x_2 e_2)^+} |(x_1 e_1 + x_2 e_2) \cdot y|^p \, dy,$$

we obtain that

$$f(x_1 e_1 + x_2 e_2) = \|x_1 e_1 + x_2 e_2\|^p_{c_1 \Gamma_p^{+,*} T^n \hat{+}_p c_2 \Gamma_p^{-,*} T^n}$$

for suitable positive constants c_1, c_2. For simplicity we will write

$$f_1(x) = \|x\|^p_{c_1 \Gamma_p^{+,*} T^n \hat{+}_p c_2 \Gamma_p^{-,*} T^n}$$

in the sequel.
Hence we arrived at the following situation: $f, f_1 : \mathbb{R}^n \backslash \{o\} \to \mathbb{R}$ are two continuous functions satisfying

$$f(x) = \lambda f(\phi^t x) + (1-\lambda) f(\psi^t x). \tag{17}$$

They are invariant under even permutations of indices and are equal on the set $\mathrm{lin}\{e_1, e_2\} \backslash \{o\}$. We will show that f coincides with f_1 on $\mathbb{R}^n \backslash \{o\}$. Because of the invariance properties it is enough to prove
(18)

$$f(x) = f_1(x),\ \forall\, x \in \mathrm{lin}\{e_{i_1}, \dots, e_{i_k}\} \Longrightarrow f(x) = f_1(x),\ \forall\, x \in \mathrm{lin}\{e_1, \dots, e_{k+1}\}$$

for $2 \leq k \leq n-1$. Indeed, let x be contained in $\mathrm{lin}\{e_1, \dots, e_{k+1}\}$. Suppose $0 < x_1/x_2 < 1$ and let $\lambda := x_1/x_2$. Then

$$(\psi^{-t} x)_1 = (\phi^t \psi^{-t} x)_1 = \frac{x_1}{1-\lambda} - \frac{\lambda}{1-\lambda} x_2 = 0,$$
$$(\psi^{-t} x)_i = (\phi^t \psi^{-t} x)_i = 0, \quad i = k+2, \dots, n.$$

By (17) it follows that

$$f(\psi^{-t}x) = \lambda f(\phi^t \psi^{-t} x) + (1-\lambda) f(x).$$

But the analogous relation holds true for f_1, too. Therefore, implication (18) is correct for $0 < x_1 < x_2$ and $x_2 < x_1 < 0$.
For $0 < \lambda := (x_1 - x_2)/x_1 < 1$, we obtain

$$\begin{aligned}(\phi^{-t}x)_2 = (\psi^t \phi^{-t} x)_2 &= -\frac{1-\lambda}{\lambda} x_1 + \frac{1}{\lambda} x_2 = 0,\\ (\phi^{-t}x)_i = (\psi^t \phi^{-t} x)_i &= 0, \quad i = k+2, \dots, n.\end{aligned}$$

Since

$$f(\phi^{-t}x) = \lambda f(x) + (1-\lambda) f(\psi^t \phi^{-t} x)$$

and f_1 satisfies the same identity, (18) holds for $0 < x_2 < x_1$ and $x_1 < x_2 < 0$.
For $x_1, x_2 \neq 0$ and $\operatorname{sgn}(x_1) \neq \operatorname{sgn}(x_2)$ define $0 < \lambda := x_1/(x_1 - x_2) < 1$. Then

$$(\phi^t x)_2 = (\phi^t x)_i = (\psi^t x)_1 = (\psi^t x)_i = 0, \quad i = k+2, \dots, n.$$

As before we conclude that (17) implies (18) for $x_1 < 0, x_2 > 0$ and $x_1 > 0, x_2 < 0$.
The continuity of f and f_1 concludes the proof of (18). $\square$

In contrast to the two dimensional case, the next lemma shows that nontrivial $\mathrm{GL}^+(n)$ covariant valuations do not exist.

Lemma 7. *Every* $\mathrm{GL}^+(n)$ *covariant valuation* $\mathrm{Z} : \mathcal{P}_o^n \to \langle \mathcal{S}^n, \hat{+}_p \rangle$ *for* $n \geq 3$ *is trivial.*

Proof. Assume $\mathrm{Z}\, T^n \neq e$ and set $f(x) := \|x\|^p_{\mathrm{Z}\, T^n}$. Then f is a positive, continuous function on $\mathbb{R}^n \backslash \{o\}$. Thus (6) implies

$$f(x) = \lambda^{-pq} f(\phi^{-1}x) + (1-\lambda)^{-pq} f(\psi^{-1}x) \tag{19}$$

on $\mathbb{R}^n \backslash \{o\}$. Since e_3 is an eigenvector with eigenvalue 1 of ϕ^{-1} and ψ^{-1}, we get

$$1 = \lambda^{-pq} + (1-\lambda)^{-pq}, \quad \text{for every} \quad 0 < \lambda < 1.$$

If $q \neq -1/p$ this is not possible. For $q = -1/p$, evaluate (19) at e_1 to obtain

$$f(e_1) = (1-\lambda)^{-p} f(e_1 - \lambda e_2), \quad \text{for every} \quad 0 < \lambda < 1.$$

Taking the limit $\lambda \to 1$ yields a contradiction. $\square$

Combining the last two lemmas we obtain Theorem 1. Theorem 2 follows immediately from it.

4. Monotone valuations

As announced in the introduction, we want to establish characterizations for valuations $\mathrm{Z} : \mathcal{K}_o^n \to \langle \mathcal{S}^n, \hat{+}_p \rangle$. To do so, we initially extend the usual partial ordering on $\mathcal{S}^n$ (which is induced by the partial ordering by inclusion on $\mathbb{R}^n$) to $\mathcal{S}^n \cup \{e\}$. Motivated by the geometric interpretation of the identity element e we set $S \subset e$ for $p > 0$ and every $S \in \mathcal{S}^n \cup \{e\}$. If $p < 0$, then we define $e \subset S$ for every $S \in \mathcal{S}^n \cup \{e\}$. We investigate operators which are compatible with this partial ordering. A map $\mathrm{Z} : \mathcal{K}_o^n \to \langle \mathcal{S}^n, \hat{+}_p \rangle$ is called *increasing* if $K \subset L$ implies $\mathrm{Z}\, K \subset \mathrm{Z}\, L$, and *decreasing* if $K \subset L$ implies $\mathrm{Z}\, L \subset \mathrm{Z}\, K$. If Z is either increasing or decreasing, we simply call it *monotonic.*

Theorem 4. *Suppose $n \geq 3$, $p \neq 0$, and let $\mathrm{Z} : \mathcal{K}_o^n \to \langle \mathcal{S}^n, \hat{+}_p \rangle$ be a monotonic, linearly intertwining valuation. For $p > -1$, the operator Z is nontrivial if and only if there exist positive constants c_1, c_2 such that*

$$\mathrm{Z}\, K = c_1 \Gamma_p^{+,*} K \,\hat{+}_p\, c_2 \Gamma_p^{-,*} K, \tag{20}$$

for every $K \in \mathcal{K}_o^n$. If $p \leq -1$, then Z is trivial.

Proof. We start with the case $p \leq -1$. The restriction of Z to $\mathcal{P}_o^n$ satisfies the conditions of Theorem 1. Thus $\mathrm{Z}\, P = e$ for every $P \in \mathcal{P}_o^n$. Obviously, for each $K \in \mathcal{K}_o^n$ there exist polytopes P_1, $P_2 \in \mathcal{P}_o^n$ such that $P_1 \subset K \subset P_2$. The monotonicity of Z implies $\mathrm{Z}\, P_1 \subset \mathrm{Z}\, K \subset \mathrm{Z}\, P_2$ or $\mathrm{Z}\, P_2 \subset \mathrm{Z}\, K \subset \mathrm{Z}\, P_1$ and hence $e \subset \mathrm{Z}\, K \subset e$. This proves $\mathrm{Z}\, K = e$. Consequently, Z is simple. Let $p > -1$. We know from Lemmas 6 and 7 that unless Z is contravariant of degree $q = -1/p$, it is trivial on polytopes. For $q \neq -1/p$ we can therefore conclude that $\mathrm{Z}\, K = e$ for all $K \in \mathcal{K}_o^n$ as above. In order to prove the theorem we can therefore restrict ourselves to nontrivial $\mathrm{GL}^+(n)$ contravariant valuations of weight $-1/p$ for $p > -1$.

From Lemma 1 we deduce that $\mathrm{Z}\, K = e$ for bodies K of dimension less than n. Thus Z coincides with $c_1 \Gamma_p^{+,*} \,\hat{+}_p\, c_2 \Gamma_p^{-,*}$ on lower-dimensional bodies. It is therefore enough to prove (20) for n-dimensional bodies. Given a body $K \in \mathcal{K}_o^n$ with nonempty interior, there exists a vector $u \in \operatorname{int} K$ such that for every $\lambda > 1$ we can find polytopes P_λ which have nonempty interior, converge to $K - u$ with respect to Hausdorff distance as λ tends to one, and satisfy $P_\lambda + u \in \mathcal{P}_o^n$ as well as

$$P_\lambda + u \subset K \subset \lambda P_\lambda + u.$$

Assume $p > 0$. Since Z restricted to $\mathcal{P}_o^n$ coincides with $c_1 \Gamma_p^{+,*} \,\hat{+}_p\, c_2 \Gamma_p^{-,*}$ and is therefore decreasing on polytopes, it has to be decreasing on $\mathcal{K}_o^n$. Thus

$$\|x\|_{\mathrm{Z}(P_\lambda + u)} \leq \|x\|_{\mathrm{Z}\, K} \leq \lambda^{n/p+1} \|x\|_{\mathrm{Z}(P_\lambda + \lambda^{-1} u)}$$

for every $x \in \mathbb{R}^n$. Now take the limit $\lambda \to 1^+$. Then the already established classification of such valuations on polytopes together with the continuity properties of the operator $c_1 \Gamma_p^{+,*} \,\hat{+}_p\, c_2 \Gamma_p^{-,*}$ conclude the proof. For negative p, one proceeds in a similar way. □

As in the proof of Theorem 4, one establishes the next result.

Theorem 5. *Suppose $n \geq 3$, $p \neq 0$, and let $\mathrm{Z} : \mathcal{K}_o^n \to \langle \mathcal{S}_c^n, \hat{+}_p \rangle$ be a monotonic, linearly intertwining valuation. For $p > -1$, the operator Z is nontrivial if and only if there exists a positive constant c such that*

$$\mathrm{Z}\, K = c\, \Gamma_p^* K,$$

for every $K \in \mathcal{K}_o^n$. If $p \leq -1$, then Z is trivial.

5. Affine surface area and star body valued valuations

In the sequel we present an example of a valuation $\mathrm{Z} : \mathcal{K}_o^n \to \langle \mathcal{S}_c^n, \hat{+}_p \rangle$ which is linearly intertwining and nontrivial on $\mathcal{K}_o^n$, but is trivial if restricted to $\mathcal{P}_o^n$. This shows that no extension of Theorems 1 and 2 to arbitrary convex bodies holds without further assumptions.

We write $\mathcal{H}^{n-1}$ for $(n-1)$-dimensional Hausdorff measure. Let $K \in \mathcal{K}^n$. The *surface area measure $S(K, \cdot)$ of K* is a finite Borel measure on the sphere and defined as follows (see, e.g. [41, Chapter 4]). For a Borel set $\omega \subseteq S^{n-1}$, $S(K, \omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K

for which there exists a normal vector of K belonging to ω. For every Borel subset ω of the sphere S^{n-1}, we write

$$S(K,\omega)=\int_\omega f_K\,d\mathcal{H}^{n-1}+S^\perp(K,\omega),$$

for the Lebesgue decomposition of $S(K,\cdot)$ with respect to $\mathcal{H}^{n-1}$. So the density $f_K : S^{n-1}\to\mathbb{R}$ is a nonnegative, Borel measurable function, and $S^\perp(K,\cdot)\perp\mathcal{H}^{n-1}$, i.e. $S^\perp(K,\cdot)$ is singular with respect to $\mathcal{H}^{n-1}$. If the body K is smooth, f_K is just the curvature function of K. For $-1/n<p<1$, we define a function on $\mathcal{K}^n\times\mathbb{R}^n\backslash\{o\}$ by

$$a_p(K,u)=\int_{S^{n-1}}|u\cdot v|^p f_K(v)^{\frac{n+p}{n+1}}\,d\mathcal{H}^{n-1}(v). \tag{21}$$

Hölder's inequality immediately shows that the integral in (21) is finite. We remark that for $-1<p<-1/n$ there exist by Minkowski's existence theorem (see e.g. [41, Theorem 7.1.2]) convex bodies for which this integral is not finite for some u.

Taking the limit $p\to 0^+$ in (21), we obtain that $a(K,\cdot)$ converges pointwise to

$$\int_{S^{n-1}} f_K(v)^{\frac{n}{n+1}}\,d\mathcal{H}^{n-1}(v).$$

For smooth bodies K, this is the *affine surface area of* K.

From now on, we assume $0<p<1$. Obviously, for fixed K the function $a_p(K,\cdot)$ is continuous and either constant with value zero or strictly positive on S^{n-1}. Now, we are going to investigate the behaviour of $a_p(\phi K,\cdot)$ for maps $\phi\in\mathrm{GL}(n)$.

Suppose $q>0$ and let $f,g:\mathbb{R}^n\backslash\{o\}\to[0,\infty)$ be functions such that f and g are positively homogeneous of degree q and $-(n+q)$, respectively, g is $\mathcal{H}^{n-1}$ integrable on the sphere and f is continuous. Then we claim that

$$|\det\phi|\int_{S^{n-1}} f(\phi v)g(v)\,d\mathcal{H}^{n-1}(v)=\int_{S^{n-1}} f(v)g(\phi^{-1}v)\,d\mathcal{H}^{n-1}(v), \tag{22}$$

for every $\phi\in\mathrm{GL}(n)$. For $q\geq 1$, linear maps ϕ with determinant one, and positive and continuous functions f and g, this result is due to Lutwak [30, Lemma 1.14]. Using Lutwak's ideas, we start with the proof of (22) for $q>0$, arbitrary $\phi\in\mathrm{GL}(n)$ and continuous, strictly positive functions f and g. For arbitrary star bodies K and L the polar formula for volume yields

$$-q\lim_{\varepsilon\to 0^+}\frac{V(K\hat{+}_q\varepsilon^{-1/q}L)-V(K)}{\varepsilon}=\int_{S^{n-1}}\|u\|_K^{-n-q}\|u\|_L^q\,d\mathcal{H}^{n-1}(u). \tag{23}$$

Since $\phi(K\hat{+}_qL)=\phi K\hat{+}_q\phi L$, formula (23) gives

$$|\det\phi|\int_{S^{n-1}}\|u\|_K^{-n-q}\|u\|_{\phi^{-1}L}^q\,d\mathcal{H}^{n-1}(u)=\int_{S^{n-1}}\|u\|_{\phi K}^{-n-q}\|u\|_L^q\,d\mathcal{H}^{n-1}(u).$$

Define special star bodies K and L by $\|\cdot\|_K^{-(n+q)}=g$ and $\|\cdot\|_L^q=f$. Then the definition of K and L proves (22) for positive and continuous functions. Subsequent approximation arguments conclude the proof of (22) in the general case. We extend f_K to a positively homogeneous function of degree $-(n+1)$ on $\mathbb{R}^n\backslash\{o\}$. Let L be an arbitrary body contained in $\mathcal{K}_o^n$. Since

$$\frac{1}{n}\int_{S^{n-1}} h(L,v)\,dS(K,v)=V(K,\dots,K,L),$$

where $V(K,\dots,K,L)$ denotes the mixed volume of $(n-1)$ copies of K and L (see, e.g. [7, Formula A.12]) and $V(\phi K_1,\dots,\phi K_n) = |\det\phi| V(K_1,\dots,K_n)$ (see e.g. [7, Formula A.17]), we obtain by (22)

$$\int_{S^{n-1}} h(L,v) f_{\phi K}(v)\, d\mathcal{H}^{n-1}(v) + \int_{S^{n-1}} h(L,v)\, dS_{n-1}^{\perp}(\phi K, v) = \tag{24}$$
$$|\det\phi|^2 \int_{S^{n-1}} h(L,v) f_K(\phi^t v)\, d\mathcal{H}^{n-1}(v) + |\det\phi| \int_{S^{n-1}} h(L,\phi^{-t}v)\, dS_{n-1}^{\perp}(K,v).$$

There exists a Borel measure ν on S^{n-1} with

$$|\det\phi| \int_{S^{n-1}} h(L,\phi^{-t}v)\, dS_{n-1}^{\perp}(K,v) = \int_{S^{n-1}} h(L,v)\, d\nu \qquad \text{and} \qquad \nu \perp \mathcal{H}^{n-1}.$$

Since differences of support functions are dense in $C(S^{n-1})$ (see e.g. [41, Lemma 1.7.9]), we obtain that (24) holds for arbitrary continuous functions. By Riesz's representation theorem we therefore get

$$\int_\omega f_{\phi K}\, d\mathcal{H}^{n-1} + S^{\perp}(\phi K,\omega) = |\det\phi|^2 \int_\omega f_K \circ \phi^t\, d\mathcal{H}^{n-1} + \nu(\omega).$$

The uniqueness of the Lebesgue decomposition yields $f_{\phi K} = |\det\phi|^2 f_K \circ \phi^t$ almost everywhere on S^{n-1} with respect to $\mathcal{H}^{n-1}$. So by (22) we conclude that

$$a_p(\phi K, u) = |\det\phi|^{\frac{n+2p-1}{n+1}} a_p(K,\phi^{-1}u), \quad u \in S^{n-1}. \tag{25}$$

Finally, we establish the valuation property

$$a_p(K\cup L,\cdot) + a_p(K\cap L,\cdot) = a_p(K,\cdot) + a_p(L,\cdot), \tag{26}$$

provided that $K, L, K\cup L \in \mathcal{K}_o^n$. Since a_p vanishes on lower dimensional sets, it suffices to prove (26) for n-dimensional bodies K and L. Let $\operatorname{reg} K$ denote the set of *regular points* on the boundary $\operatorname{bd} K$ of K, i.e. boundary points at which there exists a unique outward normal vector of K. For $x \in \operatorname{reg} K$, denote by $\sigma_K(x)$ the unique outward unit normal vector of K at x. We write $\mathcal{M}(K) \subset \operatorname{reg} K$ for the set of points at which the function which locally represents the boundary of K is twice differentiable. Let $H_{n-1}(K,x)$ denote the generalized Gauss curvature which can be defined $\mathcal{H}^{n-1}$ almost everywhere on the boundary of K. For n-dimensional K we will deduce the representation

$$a_p(K,u) = \int_{\operatorname{bd} K} |\sigma_K(x)\cdot u|^p H_{n-1}(K,x)^{\frac{1-p}{n+1}}\, d\mathcal{H}^{n-1}(x). \tag{27}$$

The proof of this relation is based on techniques developed by Hug [14]. For $r > 0$, define

$$(\operatorname{bd} K)_r = \{x \in \operatorname{bd} K \mid \exists a \in \mathbb{R}^n : \ x \in B(a,r) \subset K\}.$$

For $\mathcal{H}^{n-1}$ almost all $x \in (\operatorname{bd} K)_r$, the approximate Jacobian (see [4, Theorem 3.2.22]) $\operatorname{ap} J_{d-1}\sigma_K(x)$ equals $H_{n-1}(K,x)$ (see [14, Lemma 2.3]). Since σ_K is Lipschitz continuous on $(\operatorname{bd} K)_r$ (see [14, Lemma 2.1]), the coarea formula [4, Theorem

3.2.22] yields

$$\int_{(\operatorname{bd} K)_r} |\sigma_K(x)\cdot u|^p H_{n-1}(K,x)^{\frac{1-p}{n+1}}\, d\mathcal{H}^{n-1}(x) =$$
$$\int_{(\operatorname{bd} K)_r} \mathbb{I}_{\{H_{n-1}(K,\cdot)>0\}}(x)|\sigma_K(x)\cdot u|^p H_{n-1}(K,x)^{\frac{-p-n}{n+1}} \operatorname{ap} J_{d-1}\sigma_K(x)\, d\mathcal{H}^{n-1}(x) =$$
$$\int_{S^{n-1}}\int_{\sigma_K^{-1}(v)\cap(\operatorname{bd} K)_r} \mathbb{I}_{\{H_{n-1}(K,\cdot)>0\}}(x)|\sigma_K(x)\cdot u|^p H_{n-1}(K,x)^{\frac{-p-n}{n+1}}\, d\mathcal{H}^0(x) d\mathcal{H}^{n-1}(v).$$

For $\mathcal{H}^{n-1}$ almost every $u \in \sigma_K((\operatorname{bd} K)_r)$, the support function h_K is twice differentiable at u, $\nabla h_K(u) \in \mathcal{M}(K)$, and $H_{n-1}(K,\nabla h_K(u))D_{n-1}h_K(u) = 1$ (see [14, Lemma 2.6]). Here, $D_{n-1}h_K(u)$ denotes the sum of the principal minors of order $n-1$ of the Hessian of h_K at $u \in S^{n-1}$ (which can be defined $\mathcal{H}^{n-1}$ almost everywhere). Thus

$$\int_{(\operatorname{bd} K)_r} |\sigma_K(x)\cdot u|^p H_{n-1}(K,x)^{\frac{1-p}{n+1}}\, d\mathcal{H}^{n-1}(x) = \int_{\sigma_K((\operatorname{bd}\ K)_r)} |v\cdot u|^p D_{n-1}h_K(v)^{\frac{n+p}{n+1}}\, d\mathcal{H}^{n-1}(v).$$

Define $(\operatorname{bd}\ K)_+ := \bigcup_{r>0}(\operatorname{bd}\ K)_r$. Since $\mathcal{H}^{n-1}(\operatorname{bd}\ K\backslash(\operatorname{bd}\ K)_+) = 0$ (see [34]) the theorem of monotone convergence implies

$$\int_{\operatorname{bd} K} |\sigma_K(x)\cdot u|^p H_{n-1}(K,x)^{\frac{1-p}{n+1}}\, d\mathcal{H}^{n-1}(x) = \int_{\sigma_K((\operatorname{bd}\ K)_+)} |v\cdot u|^p D_{n-1}h_K(v)^{\frac{n+p}{n+1}}\, d\mathcal{H}^{n-1}(v).$$

But for $\mathcal{H}^{n-1}$ almost every $v \in S^{n-1}$ we have $v \in \sigma_K((\operatorname{bd}\ K)_+)$ if and only if $D_{n-1}h_K(v) > 0$ (see [14, Lemma 2.7]) and $\mathcal{H}^{n-1}$ almost everywhere $f_K = D_{n-1}h_K(\cdot)$ (see [20, Formula 781] and the references cited there). Consequently, we obtain

$$\int_{\operatorname{bd} K} |\sigma_K(x)\cdot u|^p H_{n-1}(K,x)^{\frac{1-p}{n+1}}\, d\mathcal{H}^{n-1}(x) = \int_{S^{n-1}} |v\cdot u|^p f_K(v)^{\frac{n+p}{n+1}}\, d\mathcal{H}^{n-1}(v).$$

This proves representation (27).

For $K, L \in \mathcal{K}_o^n$ with n-dimensional intersection and convex union we follow Schütt [43] and work with the decompositions

$$\begin{aligned}
\operatorname{bd}(K\cup L) &= (\operatorname{bd}\ K\cap \operatorname{bd}\ L)\cup(\operatorname{bd}\ K\cap L^c)\cup(\operatorname{bd}\ L\cap K^c),\\
\operatorname{bd}(K\cap L) &= (\operatorname{bd}\ K\cap \operatorname{bd}\ L)\cup(\operatorname{bd}\ K\cap \operatorname{int}\ L)\cup(\operatorname{bd}\ L\cap \operatorname{int}\ K),\\
\operatorname{bd}\ K &= (\operatorname{bd}\ K\cap \operatorname{bd}\ L)\cup(\operatorname{bd}\ K\cap L^c)\cup(\operatorname{bd}\ K\cap \operatorname{int}\ L),\\
\operatorname{bd}\ L &= (\operatorname{bd}\ K\cap \operatorname{bd}\ L)\cup(\operatorname{bd}\ L\cap K^c)\cup(\operatorname{bd}\ L\cap \operatorname{int}\ K),
\end{aligned} \tag{28}$$

where all unions are disjoint. By [43, Lemma 5] we have

$$\begin{aligned}
H_{n-1}(K\cup L,x) &= \min\{H_{n-1}(K,x), H_{n-1}(L,x)\},\\
H_{n-1}(K\cap L,x) &= \max\{H_{n-1}(K,x), H_{n-1}(L,x)\},
\end{aligned}$$

at points $x \in \text{bd}\, K \cap \text{bd}\, L$ where all involved generalized Gauss curvatures exist. For n-dimensional convex bodies K, L with convex union we have

$$\begin{aligned} \sigma_K(x) &= \pm\sigma_L(x), && x \in \text{reg}\, K \cap \text{reg}\, L, \\ \sigma_{K\cup L}(x) = \sigma_K(x) &= \sigma_L(x), && x \in \text{reg}\, K \cap \text{reg}\, L \cap \text{reg}\,(K \cup L), \\ \sigma_{K\cap L}(x) = \sigma_K(x) &= \sigma_L(x), && x \in \text{reg}\, K \cap \text{reg}\, L \cap \text{reg}\,(K \cap L). \end{aligned}$$

To prove (26), one uses representation (27), splits the involved integrals in the parts indicated by (28), and applies the above observations concerning curvature and spherical image. For bodies with intersection of dimension less than n, note that there exists a hyperplane H such that $(K \cup L) \cap H = K \cap L$ and that $H_{n-1}(K, \cdot) = H_{n-1}(L, \cdot) = 0$ on the relative interior of $K \cap H = L \cap H$.
So by (25) and (26) the function $\Omega : \mathcal{K}_o^n \to \langle \mathcal{S}_c^n, \hat{+}_p \rangle$ defined by

$$\|x\|_{\Omega K}^p = a_p(K, x)$$

for bodies K with $a_p(K, \cdot) > 0$ on S^{n-1}, and e otherwise, is a covariant valuation. Obviously, it vanishes on polytopes.

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