GENERAL L_p AFFINE ISOPERIMETRIC INEQUALITIES

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Abstract. Sharp L_p affine isoperimetric inequalities are established for the entire class of L_p projection bodies and the entire class of L_p centroid bodies. These new inequalities strengthen the L_p Petty projection and the L_p Busemann–Petty centroid inequality.

1. Introduction

Projection bodies were introduced by Minkowski at the turn of the previous century and have since become a central notion in convex geometry. They arise naturally in a number of different areas such as functional analysis, stochastic geometry and geometric tomography, see e.g., [**5**, **9**, **12**, **19**, **44**, **49**, **50**]. The fundamental affine isoperimetric inequality for projection bodies is the *Petty projection inequality* [**38**]: Among convex bodies of given volume, the ones whose polar projection bodies have maximal volume are precisely the ellipsoids. This inequality turned out to be far stronger than the classical isoperimetric inequality. Lutwak, Yang, and Zhang [**30**] (see also Campi and Gronchi [**6**]) recently established an important L_p Petty projection inequality for the (symmetric) L_p analog of the projection operator. This extension is the core of a sharp affine L_p Sobolev inequality which is significantly stronger than the classical L_p Sobolev inequality, see [**32**, **52**].

Recent advances in valuation theory by Ludwig [**21**] revealed that the L_p projection operator used in [**30**] is only one representative of an entire class of L_p extensions of the classical projection operator. In this article we establish the L_p Petty projection inequality for each member of the family of L_p projection operators. It is shown that each of these new inequalities strengthens and directly implies the previously known L_p Petty projection inequality. Moreover, the two strongest inequalities are identified. We also obtain similar results for the L_p Busemann–Petty centroid inequality.

The celebrated *Blaschke–Santaló inequality* is by far the best known affine isoperimetric inequality (see e.g., [**9**, **14**, **42**]): The product of the volumes of polar reciprocal convex bodies is maximized precisely by ellipsoids. Lutwak and Zhang [**34**] obtained an important L_p version of the Blaschke–Santaló inequality. Their inequality includes as a limiting case the classical inequality for origin-symmetric convex bodies. For convex bodies which are not origin-symmetric this L_p extension yields an inequality which is weaker than the Blaschke–Santaló inequality. As an application of our work, we establish the correct L_p analog of the Blaschke–Santaló inequality, one that includes as a limiting case the classical inequality for all convex bodies.

For a convex body K (i.e., a nonempty, compact convex subset of $\mathbb{R}^n$) denote by $h(K,x) = \max\{x \cdot y : y \in K\}$, for $x \in \mathbb{R}^n$, the support function of K. The *projection body* ΠK of K is the convex body whose support function in the direction u is equal to the $(n-1)$-dimensional volume of the projection of K onto the hyperplane orthogonal to u. A recent important result by Ludwig [**21**] has demonstrated the special place of projection bodies in the affine theory of convex bodies: The projection operator was characterized as the unique Minkowski valuation which is contravariant with respect to nondegenerate linear transformations.

A function Φ defined on a subset $\mathcal{L}$ of the set of convex bodies $\mathcal{K}^n$ and taking values in an abelian semigroup is called a *valuation* if

$$\Phi(K \cup L) + \Phi(K \cap L) = \Phi K + \Phi L, \tag{1}$$

whenever $K, L, K \cap L, K \cup L \in \mathcal{L}$. The theory of real valued valuations lies at the core of geometry. They were the critical ingredient in Dehn's solution of Hilbert's third problem. For information on the classical theory of valuations, see [**18**] and [**35**]. For some of the more recent results, see [**1–4**, **19–24**].

First results on convex body valued valuations were obtained by Schneider [**41**] in the 1970s, where the addition of convex bodies in (1) is Minkowski addition defined by $h(K+L,\cdot) = h(K,\cdot) + h(L,\cdot)$, see also [**17**, **43**, **45**]. In recent years the investigations of these *Minkowski valuations* gained momentum through a series of articles by Ludwig [**19**, **21**]. She obtained complete classifications of Minkowski valuations compatible with nondegenerate linear transformations (see Section 3 for precise definitions).

Projection bodies are part of the classical Brunn–Minkowski theory which is the result of joining the notion of volume with the usual vector addition of convex sets. The books by Gardner [**9**], Gruber [**14**] and Schneider [**42**] form an excellent introduction to the subject. In a series of articles [**27**, **28**], Lutwak showed that merging the notion of volume with the L_p Minkowski addition of convex sets, introduced by Firey, leads to a Brunn–Minkowski theory for each $p \geq 1$. Since Lutwak's seminal work, the topic has been the focus of intense study, see e.g., [**7**, **10**, **11**, **21**, **24**, **29–34**, **40**, **46–48**].

For $p > 1$, Ludwig [**21**] introduced a family of convex bodies,

$$c_1 \cdot \Pi_p^+ K +_p c_2 \cdot \Pi_p^- K, \qquad K \in \mathcal{K}_o^n, \tag{2}$$

and established the L_p analog of her classification of the projection operator: She showed that the convex bodies defined in (2) constitute all of the L_p extensions of projection bodies. Here, $\mathcal{K}_o^n$ is the set of convex bodies which contain the origin in their interiors and $c_1, c_2 \geq 0$ (not both zero). The convex body defined by (2) is an L_p Minkowski combination of the *nonsymmetric L_p projection bodies* $\Pi_p^{\pm} K$ (see Sections 2 and 3 for definitions).

The (symmetric) *L_p projection body* $\Pi_p K$ of $K \in \mathcal{K}_o^n$, first defined in [**30**], is

$$\Pi_p K = \tfrac{1}{2} \cdot \Pi_p^+ K +_p \tfrac{1}{2} \cdot \Pi_p^- K.$$

As our main result we extend the L_p Petty projection inequality for Π_p by Lutwak, Yang, and Zhang to the entire class (2) of L_p projection bodies.

Let $K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}$ denote the polar body of $K \in \mathcal{K}_o^n$. We use $V(K)$ to denote the volume of K and we write B for the Euclidean unit ball. If $\Phi : \mathcal{K}_o^n \to \mathcal{K}_o^n$, we use $\Phi^* K$ to denote $(\Phi K)^*$.

Theorem 1. *Let $K \in \mathcal{K}_o^n$ and $p > 1$. If $\Phi_p K$ is the convex body defined by*

$$\Phi_p K = c_1 \cdot \Pi_p^+ K +_p c_2 \cdot \Pi_p^- K,$$

where $c_1, c_2 \geq 0$ are not both zero, then

$$V(K)^{n/p-1} V(\Phi_p^* K) \leq V(B)^{n/p-1} V(\Phi_p^* B),$$

with equality if and only if K is an ellipsoid centered at the origin.

The case $\Phi_p = \Pi_p$ of Theorem 1 is the L_p Petty projection inequality by Lutwak, Yang, and Zhang.

The natural problem arises to determine for fixed $K \in \mathcal{K}_o^n$ the extreme values of $V(\Phi_p^* K)$ among all suitably normalized (say, e.g., $\Phi_p B = B$) L_p projection bodies (2). Here, we will show that for $K \in \mathcal{K}_o^n$,

$$V(\Pi_p^* K) \leq V(\Phi_p^* K) \leq V(\Pi_p^{\pm,*} K).$$

If K is not origin-symmetric and p is not an odd integer, these inequalities are strict unless $\Phi_p = \Pi_p$, or $\Phi_p = \Pi_p^{\pm}$, respectively. This shows that each of the new inequalities established in Theorem 1 strengthens and implies the previously known L_p Petty projection inequality and that the nonsymmetric operators $\Pi_p^{\pm}$ (and their multiples) give rise to the strongest inequalities.

Centroid bodies (volume normalized moment bodies) are a classical notion from geometry which have attracted increased attention in recent years, see e.g., [**9**, **12**, **25**, **26**, **30**]. The *moment body* $\mathrm{M}K$ of a convex body K is the convex body defined by

$$h(\mathrm{M}K, u) = \int_K |u \cdot x| \, dx, \qquad u \in S^{n-1}.$$

If K has nonempty interior, then $\Gamma K = V(K)^{-1} \mathrm{M}K$ is the *centroid body* of K.

Petty established the Petty projection inequality as a consequence of the *Busemann–Petty centroid inequality* [**37**]: Among convex bodies of given volume, the ones whose centroid bodies have minimal volume are precisely the ellipsoids. Lutwak, Yang, and Zhang [**30**] (see also Campi and Gronchi [**6**]) established the L_p version of the Busemann–Petty centroid inequality: For $p > 1$ and convex bodies K containing the origin in their interiors,

$$V(K)^{n/p-1} V(\mathrm{M}_p K) \leq V(B)^{n/p}, \tag{3}$$

with equality if and only if K is an ellipsoid centered at the origin. Here, M_pK denotes the (symmetric) L_p *moment body*, defined in [**34**] by

$$\mathrm{M}_pK = \tfrac{1}{2} \cdot \mathrm{M}_p^+K +_p \tfrac{1}{2} \cdot \mathrm{M}_p^-K,$$

where $\mathrm{M}_p^{\pm}K$ are the *nonsymmetric L_p moment bodies* (see Section 3). Since their introduction L_p moment bodies have become the focus of intense study, see e.g., [**6**, **8**, **12**, **15**, **16**, **21**, **30**, **51**] and the noted paper [**36**].

Ludwig [**21**] characterized moment bodies as the unique (non-trivial) homogeneous Minkowski valuations which intertwine volume preserving linear transformations. For $p > 1$, Ludwig [**21**] has introduced and characterized the two-parameter family

$$c_1 \cdot \mathrm{M}_p^+K +_p c_2 \cdot \mathrm{M}_p^-K, \qquad K \in \mathcal{K}_{\mathrm{o}}^n, \tag{4}$$

as all of the possible L_p analogs of moment bodies.

Our L_p Busemann–Petty centroid inequality for the entire class (4) of L_p moment bodies is:

Theorem 2. *Let $K \in \mathcal{K}_{\mathrm{o}}^n$ and $p > 1$. If Ψ_pK is the convex body defined by*

$$\Psi_pK = c_1 \cdot \mathrm{M}_p^+K +_p c_2 \cdot \mathrm{M}_p^-K,$$

where $c_1, c_2 \geq 0$ are not both zero, then

$$V(K)^{-n/p-1}V(\Psi_pK) \geq V(B)^{-n/p-1}V(\Psi_pB),$$

with equality if and only if K is an ellipsoid centered at the origin.

In fact, in Section 6 a stronger version of Theorem 2, valid for all star bodies, will be established.

For $K \in \mathcal{K}_{\mathrm{o}}^n$ and suitably normalized (say, e.g., $\Psi_pB = B$) L_p moment bodies (4), we will show that

$$V(\mathrm{M}_pK) \geq V(\Psi_pK) \geq V(\mathrm{M}_p^{\pm}K).$$

If K is not origin-symmetric and p is not an odd integer, these inequalities are strict unless $\Psi_p = \mathrm{M}_p$, or $\Psi_p = \mathrm{M}_p^{\pm}$, respectively. Consequently, each of the new inequalities established in Theorem 2 strengthens and implies inequality (3). The nonsymmetric operators $\mathrm{M}_p^{\pm}$ provide the strongest version of the L_p Busemann–Petty centroid inequality.

Recall that for $K \in \mathcal{K}_{\mathrm{o}}^n$, the Blaschke–Santaló inequality states

$$V(K)V(K^s) \leq V(B)^2,$$

with equality if and only if K is an ellipsoid. Here, $K^s = (K - s)^*$ is the polar body of K with respect to the Santaló point s of K, i.e., the unique point $s \in \operatorname{int} K$ which minimizes $V((K - x)^*)$ among all translates $K - x$, for $x \in \operatorname{int} K$. From Theorem 2, we obtain:

Corollary. *If Ψ_p is defined as in Theorem 2, then for $K \in \mathcal{K}_{\mathrm{o}}^n$,*

$$V(K)^{n/p+1}V(\Psi_p^sK) \leq V(B)^{n/p+1}V(\Psi_p^sB),$$

with equality if and only if K is an ellipsoid centered at the origin.

Here, the case $\Psi_p = \mathrm{M}_p$ was established by Lutwak and Zhang [**34**]. We remark that $\mathrm{M}_p^+ K$ converges to K as $p \to \infty$. Thus, as a limiting case we obtain for $\Psi_p = \mathrm{M}_p^+$ the classical Blaschke–Santaló inequality.

2. Background Material

In the following we state the necessary background material. For quick reference, we collect basic properties of L_p mixed and dual mixed volumes.

The setting for this article is Euclidean n-space $\mathbb{R}^n$ with $n \geq 3$. We will also assume throughout that $1 < p < \infty$. Thus, in the following we will omit these restrictions on n and p.

Associated with a convex body $K \in \mathcal{K}_{\mathrm{o}}^n$ is its surface area measure, $S(K, \cdot)$, on S^{n-1}. For a Borel set $\omega \subseteq S^{n-1}$, $S(K, \omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K for which there exists a normal vector of K belonging to ω. By Minkowski's uniqueness theorem (see e.g., [**42**, p. 397]), the convex body K is determined up to translation by the measure $S(K, \cdot)$.

We call a convex body $K \in \mathcal{K}_{\mathrm{o}}^n$ *smooth* if its boundary is C^2 with everywhere positive curvature. For a smooth convex body K, the surface area measure $S(K, \cdot)$ is absolutely continuous with respect to spherical Lebesgue measure:

$$dS(K, u) = f(K, u)\, du, \qquad u \in S^{n-1}.$$

The positive continuous function $f(K, \cdot)$ is called the *curvature function* of K. It is the reciprocal of the Gauss curvature as a function of the outer normals.

For $p \geq 1$, $K, L \in \mathcal{K}_{\mathrm{o}}^n$ and $\alpha, \beta \geq 0$ (not both zero), the L_p *Minkowski combination* $\alpha \cdot K +_p \beta \cdot L$ is the convex body defined by

$$h(\alpha \cdot K +_p \beta \cdot L, \cdot)^p = \alpha h(K, \cdot)^p + \beta h(L, \cdot)^p.$$

Introduced by Firey in the 1960's, this notion is the basis of what has become known as the L_p Brunn–Minkowski theory (or the Brunn–Minkowski–Firey theory). Obviously, the L_p Minkowski and the usual scalar multiplication are related by $\alpha \cdot K = \alpha^{1/p} K$.

For $K, L \in \mathcal{K}_{\mathrm{o}}^n$, the L_p mixed volume $V_p(K, L)$ was defined in [**27**] by

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

Clearly, for $K \in \mathcal{K}_{\mathrm{o}}^n$,

$$V_p(K, K) = V(K). \tag{5}$$

It was shown in [**27**] that corresponding to each convex body $K \in \mathcal{K}_{\mathrm{o}}^n$, there is a positive Borel measure on S^{n-1}, the L_p *surface area measure* $S_p(K, \cdot)$ of K, such that for every $L \in \mathcal{K}_{\mathrm{o}}^n$,

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p dS_p(K, u). \tag{6}$$

The measure $S_1(K,\cdot)$ is just the surface area measure of K. Moreover, the L_p surface area measure is absolutely continuous with respect to $S(K,\cdot)$:

$$dS_p(K,u) = h(K,u)^{1-p}\, dS(K,u), \qquad u \in S^{n-1}. \tag{7}$$

It was shown in [**27**] that, if $K, L \in \mathcal{K}^n_{\mathrm{o}}$ and $p \neq n$, then

$$S_p(K,\cdot) = S_p(L,\cdot) \implies K = L$$

and, if $p = n$, then

$$S_n(K,\cdot) = S_n(L,\cdot) \implies K = \lambda L, \quad \lambda > 0.$$

These uniqueness properties of L_p surface area measures are direct consequences of the L_p Minkowski inequality [**27**]: If $K, L \in \mathcal{K}^n_{\mathrm{o}}$, then

$$V_p(K,L)^n \geq V(K)^{n-p} V(L)^p, \tag{8}$$

with equality if and only if K and L are dilates.

Firey's L_p Brunn–Minkowski inequality states: If $K, L \in \mathcal{K}^n_{\mathrm{o}}$, then

$$V(K +_p L)^{p/n} \geq V(K)^{p/n} + V(L)^{p/n}, \tag{9}$$

with equality if and only if K and L are dilates.

For a compact set L in $\mathbb{R}^n$ which is star-shaped with respect to the origin, we denote by $\rho(L,x) = \max\{\lambda \geq 0 : \lambda x \in L\}$, $x \in \mathbb{R}^n \backslash \{0\}$, the radial function of L. If $\rho(L,\cdot)$ is positive and continuous, we call L a star body. The set of star bodies is denoted by $\mathcal{S}^n$.

If $K \in \mathcal{K}^n_{\mathrm{o}}$ is a convex body, then it follows from the definitions of support functions and radial functions, and the definition of the polar body of K, that

$$\rho(K^*,\cdot) = h(K,\cdot)^{-1} \qquad \text{and} \qquad h(K^*,\cdot) = \rho(K,\cdot)^{-1}. \tag{10}$$

For $\alpha, \beta \geq 0$ (not both zero), the L_p *harmonic radial combination* $\alpha \cdot K \,\widetilde{+}_p\, \beta \cdot L$ of $K, L \in \mathcal{S}^n$ is the star body defined by

$$\rho(\alpha \cdot K \,\widetilde{+}_p\, \beta \cdot L, \cdot)^{-p} = \alpha \rho(K,\cdot)^{-p} + \beta \rho(L,\cdot)^{-p}.$$

Although our notation does not reflect the obvious difference between L_p and dual L_p scalar multiplication, there should be no possibility of confusion. Clearly, the L_p harmonic radial and the usual scalar multiplication are related by $\alpha \cdot K = \alpha^{-1/p} K$.

For convex bodies, Firey started the investigations of harmonic L_p combinations which were continued by Lutwak leading to a dual L_p Brunn–Minkowski theory. The dual L_p mixed volume $\widetilde{V}_{-p}(K,L)$ of $K, L \in \mathcal{S}^n$ was defined in [**28**] by

$$-\frac{n}{p} \widetilde{V}_{-p}(K,L) = \lim_{\varepsilon \to 0^+} \frac{V(K \,\widetilde{+}_p\, \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

Clearly, for $L \in \mathcal{S}^n$,

$$\widetilde{V}_{-p}(L,L) = V(L). \tag{11}$$

The polar coordinate formula for volume leads to the following integral representation of the dual L_p mixed volume $\widetilde{V}_{-p}(K,L)$: For $K,L\in\mathcal{S}^n$,

$$\widetilde{V}_{-p}(K,L)=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n+p}\rho(L,u)^{-p}\,du. \tag{12}$$

Here, integration is with respect to spherical Lebesgue measure. An application of Hölder's integral inequality to (12) yields the dual L_p Minkowski inequality [**28**]: If $K,L\in\mathcal{S}^n$, then

$$\widetilde{V}_{-p}(K,L)^n\geq V(K)^{n+p}V(L)^{-p}, \tag{13}$$

with equality if and only if K and L are dilates.

The dual L_p Brunn–Minkowski inequality [**28**] is: If $K,L\in\mathcal{S}^n$, then

$$V(K\,\widetilde{+}_p\,L)^{-p/n}\geq V(K)^{-p/n}+V(L)^{-p/n}, \tag{14}$$

with equality if and only if K and L are dilates.

3. Nonsymmetric L_p Projection and Moment Bodies

In this section we define nonsymmetric L_p projection bodies Π_p^+K as well as nonsymmetric L_p moment bodies M_p^+K and discuss basic properties of the corresponding operators.

Recall that the volume of the Euclidean unit ball B is given by

$$\kappa_n=\pi^{n/2}/\Gamma(1+\tfrac{n}{2}).$$

We define $c_{n,p}$ by

$$c_{n,p}=\frac{\Gamma\left(\frac{n+p}{2}\right)}{\pi^{(n-1)/2}\Gamma\left(\frac{1+p}{2}\right)}.$$

For each finite Borel measure μ on S^{n-1}, we define a continuous function $\mathrm{C}_p^+\mu$ on S^{n-1}, the *nonsymmetric L_p cosine transform* of μ, by

$$(\mathrm{C}_p^+\mu)(u)=c_{n,p}\int_{S^{n-1}}(u\cdot v)_+^p\,d\mu(v),\qquad u\in S^{n-1},$$

where $(u\cdot v)_+=\max\{u\cdot v,0\}$. For $f\in C(S^{n-1})$, we define C_p^+f as the nonsymmetric L_p cosine transform of the absolutely continuous measure (with respect to spherical Lebesgue measure) with density f. The normalization above was chosen so that $\mathrm{C}_p^+1=1$.

The *nonsymmetric L_p projection body* Π_p^+K of $K\in\mathcal{K}_\mathrm{o}^n$ was first considered in [**28**]. It is the convex body defined by

$$h(\Pi_p^+K,\cdot)^p=\mathrm{C}_p^+S_p(K,\cdot).$$

For $L\in S^n$, define the *nonsymmetric L_p moment body* of L by

$$h(\mathrm{M}_p^+L,\cdot)^p=\mathrm{C}_p^+\rho(L,\cdot)^{n+p}.$$

Using polar coordinates, it is easy to verify that for $L\in\mathcal{S}^n$,

$$h(\mathrm{M}_p^+L,u)^p=c_{n,p}(n+p)\int_L(u\cdot x)_+^p\,dx,\qquad u\in S^{n-1}. \tag{15}$$

Note that the normalizations are chosen such that $\mathrm{M}_p^+ B = B$ and $\Pi_p^+ B = B$. For $K \in \mathcal{K}_{\mathrm{o}}^n$, we also define

$$\mathrm{M}_p^- K = \mathrm{M}_p^+(-K) \qquad \text{and} \qquad \Pi_p^- K = \Pi_p^+(-K).$$

For a finite measure μ on S^{n-1}, it is not hard to show that

$$\lim_{p \to 1^+} (\mathrm{C}_p^+ \mu)(u) = \frac{1}{2\kappa_{n-1}} \left\{ \int_{S^{n-1}} |u \cdot v| d\mu(v) + \int_{S^{n-1}} u \cdot v d\mu(v) \right\},$$

where the first integral is the spherical cosine transform $\mathrm{C}\mu$ of μ. Recall that pointwise convergence of support functions implies the convergence of the respective bodies in the Hausdorff metric (cf. [**42**, p. 54]). Thus, since $h(\Pi K, \cdot) = \frac{1}{2}\mathrm{C}S(K, \cdot)$ and since area measures have their center of mass at the origin, we obtain for every $K \in \mathcal{K}_{\mathrm{o}}^n$ as $p \to 1$,

$$\Pi_p^+ K \;\to\; \kappa_{n-1}^{-1} \Pi K \qquad \text{and} \qquad \mathrm{M}_p^+ K \;\to\; \frac{n+1}{2\kappa_{n-1}} \left(\mathrm{M}(K) + m(K) \right). \tag{16}$$

Here, $m(K)$ is up to volume normalization the centroid of K:

$$m(K) = \int_K x \, dx.$$

From representation (15), we obtain for $K \in \mathcal{K}_{\mathrm{o}}^n$ as $p \to \infty$,

$$\mathrm{M}_p^+ K \;\to\; K.$$

A map Φ defined on $\mathcal{K}^n$ and taking values in $\mathcal{K}^n$ is called $\mathrm{SL}(n)$ *covariant*, if for all $K \in \mathcal{K}^n$ and $\phi \in \mathrm{SL}(n)$,

$$\Phi(\phi K) = \phi \Phi K.$$

It is said to be $\mathrm{SL}(n)$ *contravariant*, if for all $K \in \mathcal{K}^n$ and $\phi \in \mathrm{SL}(n)$,

$$\Phi(\phi K) = \phi^{-\mathrm{T}} \Phi K,$$

where $\phi^{-\mathrm{T}}$ denotes the inverse of the transpose of ϕ.

As usual, Φ is called *homogeneous of degree* r, for $r \in \mathbb{R}$, if $\Phi(\lambda K) = \lambda^r \Phi(K)$ for all $K \in \mathcal{K}^n$ and $\lambda > 0$. We say Φ *is linearly associating* if Φ is $\mathrm{SL}(n)$ co- or contravariant and homogeneous of degree r for some $r \in \mathbb{R}$.

It was shown in [**21**] that $\Pi_p^\pm$ is an $n/p - 1$ homogeneous and $\mathrm{SL}(n)$ contravariant map, while $\mathrm{M}_p^\pm$ is $\mathrm{SL}(n)$ covariant and homogeneous of degree $n/p + 1$, i.e., for every $\phi \in \mathrm{SL}(n)$ and $\lambda > 0$,

$$\Pi_p^\pm(\phi K) = \phi^{-\mathrm{T}} \Pi_p^\pm K \qquad \text{and} \qquad \Pi_p^\pm(\lambda K) = \lambda^{n/p-1} \Pi_p^\pm K$$

for every $K \in \mathcal{K}_{\mathrm{o}}^n$ and

$$\mathrm{M}_p^\pm(\phi K) = \phi \mathrm{M}_p^\pm K \qquad \text{and} \qquad \mathrm{M}_p^\pm(\lambda K) = \lambda^{n/p+1} \mathrm{M}_p^\pm K.$$

A map $\Phi : \mathcal{K}_{\mathrm{o}}^n \to \mathcal{K}_{\mathrm{o}}^n$ is called an L_p *Minkowski valuation* if

$$\Phi(K \cup L) +_p \Phi(K \cap L) = \Phi K +_p \Phi L,$$

whenever $K, L, K \cup L \in \mathcal{K}_o^n$. The *trivial* L_p Minkowski valuations are L_p Minkowski combinations of the identity and central reflection. In [**21**] Ludwig has shown that L_p combinations of $\Pi_p^{\pm}$ and $\mathrm{M}_p^{\pm}$ are the (essentially) uniquely determined linearly associating L_p Minkowski valuations. In order to state her result, let $\mathcal{P}_{co}^n$ ($\mathcal{P}_o^n$) denote the set of polytopes in $\mathbb{R}^n$ which contain the origin (in their interior). For $n \geq 3$, Ludwig [**21**] proved the following:

Theorem 3.1. *If $\Phi_p : \mathcal{P}_{co}^n \to \mathcal{K}^n$ is a non-trivial L_p Minkowski valuation which is linearly associating, then there exist constants $c_0 \in \mathbb{R}$ and $c_1, c_2 \geq 0$ such that for every $K \in \mathcal{P}_o^n$,*

$$\Phi_p K = \begin{cases} c_1 \Pi K & \text{if } p = 1 \\ c_1 \cdot \Pi_p^+ K +_p c_2 \cdot \Pi_p^- K & \text{if } p > 1 \end{cases}$$

or

$$\Phi_p K = \begin{cases} c_0 m(K) + c_1 \mathrm{M} K & \text{if } p = 1 \\ c_1 \cdot \mathrm{M}_p^+ K +_p c_2 \cdot \mathrm{M}_p^- K & \text{if } p > 1. \end{cases}$$

Theorem 3.1 and (16) show that the L_p combinations of $\Pi_p^{\pm}$ and $\mathrm{M}_p^{\pm}$ are all L_p extensions of projection and moment bodies.

Ludwig's classification results in [**21**] were formulated with a different parametrization of the families $c_1 \cdot \Pi_p^+ +_p c_2 \cdot \Pi_p^-$ and $c_1 \cdot \mathrm{M}_p^+ +_p c_2 \cdot \mathrm{M}_p^-$. These alternative representations will be very useful for us as well: For $\tau \in [-1, 1]$, define the function $\varphi_\tau : \mathbb{R} \to [0, \infty)$ by

$$\varphi_\tau(t) = |t| + \tau t.$$

For $K \in \mathcal{K}_o^n$, let $\Pi_p^\tau K \in \mathcal{K}_o^n$ be the convex body with support function

$$h(\Pi_p^\tau K, u)^p = c_{n,p}(\tau) \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p \, dS_p(K, v), \qquad u \in S^{n-1}, \tag{17}$$

where

$$c_{n,p}(\tau) = \frac{c_{n,p}}{(1+\tau)^p + (1-\tau)^p}.$$

The normalization is chosen such that $\Pi_p^\tau B = B$ for every $\tau \in [-1, 1]$. From the definition of $\Pi_p^{\pm}$ it is easy to verify that

$$\Pi_p^\tau K = \frac{(1+\tau)^p}{(1+\tau)^p + (1-\tau)^p} \cdot \Pi_p^+ K +_p \frac{(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p} \cdot \Pi_p^- K. \tag{18}$$

In particular, if $K \in \mathcal{K}_o^n$ is origin-symmetric, then for any $\tau, \sigma \in [-1, 1]$, we have $\Pi_p^\tau K = \Pi_p^\sigma K$.

By (18), the one-parameter family Π_p^τ constitutes a bridge between the L_p projection body operator Π_p ($\tau = 0$) as introduced by Lutwak, Yang, and Zhang and their non-symmetric analogs $\Pi_p^{\pm}$ ($\tau = \pm 1$)). From (18), it also follows that for every pair $c_1, c_2 \geq 0$ (not both zero) there exist a $\tau \in [-1, 1]$ and a constant $c > 0$ such that

$$c_1 \cdot \Pi_p^+ K +_p c_2 \cdot \Pi_p^- K = c \, \Pi_p^\tau K. \tag{19}$$

Thus, instead of working with the L_p combinations of the operators $\Pi_p^{\pm}$ we can consider multiples of the operators Π_p^{τ}, $\tau \in [-1,1]$.

For $L \in \mathcal{S}^n$, let $\mathrm{M}_p^{\tau} L \in \mathcal{K}_{\mathrm{o}}^n$ be the convex body defined by

$$h(\mathrm{M}_p^{\tau} L, u)^p = c_{n,p}(\tau) \int_{S^{n-1}} \varphi_{\tau}(u \cdot v)^p \rho(L,v)^{n+p}\, dv, \qquad u \in S^{n-1}. \tag{20}$$

Then $\mathrm{M}_p^{\tau} B = B$ for every $\tau \in [-1,1]$ and

$$\mathrm{M}_p^{\tau} L = \frac{(1+\tau)^p}{(1+\tau)^p + (1-\tau)^p} \cdot \mathrm{M}_p^{+} L +_p \frac{(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p} \cdot \mathrm{M}_p^{-} L. \tag{21}$$

In particular, if $L \in \mathcal{S}^n$ is origin-symmetric, then for any $\tau, \sigma \in [-1,1]$, we have $\mathrm{M}_p^{\tau} L = \mathrm{M}_p^{\sigma} L$.

The family M_p^{τ} forms a link between L_p moment bodies ($\tau = 0$) as introduced by Lutwak and Zhang and their non-symmetric analogs ($\tau = \pm 1$). From (21), it follows that for every pair $c_1, c_2 \geq 0$ (not both zero) there exists a $\tau \in [-1,1]$ and a constant $c > 0$ such that

$$c_1 \cdot \mathrm{M}_p^{+} K +_p c_2 \cdot \mathrm{M}_p^{-} K = c\, \mathrm{M}_p^{\tau} K. \tag{22}$$

Thus, instead of considering L_p combinations of $\mathrm{M}_p^{\pm}$ we can work with multiples of M_p^{τ}, $\tau \in [-1,1]$.

The following simple lemma will be crucial. Here and in the following, $\Pi_p^{\tau,*} K$ denotes the polar body of $\Pi_p^{\tau} K$.

Lemma 3.2. *If $K \in \mathcal{K}_{\mathrm{o}}^n$ and $L \in \mathcal{S}^n$, then*

$$V_p(K, \mathrm{M}_p^{\tau} L) = \widetilde{V}_{-p}(L, \Pi_p^{\tau,*} K).$$

Proof. If $K \in \mathcal{K}_{\mathrm{o}}^n$ and $L \in \mathcal{S}^n$, then, by (6) and definition (20),

$$V_p(K, \mathrm{M}_p^{\tau} L) = \frac{c_{n,p}(\tau)}{n} \int_{S^{n-1}} \int_{S^{n-1}} \varphi_{\tau}(u \cdot v) \rho(L,v)^{n+p}\, dv\, dS_p(K,u).$$

Thus, by Fubini's theorem, (10) and definition (17),

$$V_p(K, \mathrm{M}_p^{\tau} L) = \frac{1}{n} \int_{S^{n-1}} \rho(L,v)^{n+p} \rho(\Pi_p^{\tau,*} K, v)^{-p}\, dv = \widetilde{V}_{-p}(L, \Pi_p^{\tau,*} K).$$

□

In the following we discuss injectivity properties of the operators Π_p^{+} and M_p^{+}. To this end, we first collect some basic facts about spherical harmonics (see e.g., Schneider [**42**, Appendix]).

Let $\mathcal{H}_k^n$ denote the finite dimensional vector space of spherical harmonics of dimension n and order k. We use $N(n,k)$ to denote the dimension of $\mathcal{H}_k^n$. Let $L_2(S^{n-1})$ denote the Hilbert space of square integrable functions on S^{n-1} with its usual inner product $(\cdot,\cdot)$. The spaces $\mathcal{H}_k^n$ are pairwise orthogonal with respect to this inner product. In each space $\mathcal{H}_k^n$ we choose an orthonormal basis $\{Y_{k1}, \ldots, Y_{kN(n,k)}\}$. Then $\{Y_{k1}, \ldots, Y_{kN(n,k)} : k \in$

$\mathbb{N}\}$ forms a complete orthogonal system in $L_2(S^{n-1})$, i.e., for every $f \in L_2(S^{n-1})$, the Fourier series

$$f \sim \sum_{k=0}^{\infty} \pi_k f$$

converges in quadratic mean to f. Here, $\pi_k f$ denotes the orthogonal projection of f onto $\mathcal{H}_k^n$:

$$\pi_k f = \sum_{i=1}^{N(n,k)} (f, Y_{ki})\, Y_{ki}.$$

In particular, for $f \in C(S^{n-1})$,

$$\pi_k f = 0 \quad \text{for all } k \in \mathbb{N} \qquad \Longrightarrow \qquad f = 0. \tag{23}$$

Thus, $f \in C(S^{n-1})$ is uniquely determined by its series expansion.

For a finite Borel measure μ on S^{n-1}, we define

$$\pi_k \mu = \sum_{i=1}^{N(n,k)} \int_{S^{n-1}} Y_{ki}(u)\, d\mu(u)\, Y_{ki}.$$

If $f \in C(S^{n-1})$, then

$$(f, \pi_k \mu) = \int_{S^{n-1}} (\pi_k f)(u)\, d\mu(u).$$

Thus, by (23), the measure μ is uniquely determined by its (formal) series expansion:

$$\pi_k \mu = 0 \quad \text{for all } k \in \mathbb{N} \qquad \Longrightarrow \qquad \mu = 0. \tag{24}$$

Of particular importance for us is the Funk–Hecke theorem: Let ϕ be a continuous function on $[-1, 1]$. If T_ϕ is the transformation on the set of finite Borel measures on S^{n-1} defined by

$$(\mathrm{T}_\phi \mu)(u) = \int_{S^{n-1}} \phi(u \cdot v)\, d\mu(v),$$

then there are real numbers $a_k[\mathrm{T}_\phi]$, the *multipliers* of T_ϕ, such that

$$\mathrm{T}_\phi Y_k = a_k[\mathrm{T}_\phi]\, Y_k$$

for every $Y_k \in \mathcal{H}_k^n$. In particular, by Fubini's theorem,

$$\pi_k (\mathrm{T}_\phi \mu) = a_k[\mathrm{T}_\phi] \pi_k \mu. \tag{25}$$

A transformation T defined on the space of finite Borel measures on S^{n-1} and satisfying (25) is called a *multiplier transformation*. Using (24) and (25), it follows that a multiplier transformation T_ϕ is injective if and only if all multipliers $a_k[\mathrm{T}_\phi]$ are non-zero.

By the Funk–Hecke theorem, the nonsymmetric L_p cosine transform C_p^+ is a multiplier transformation. The numbers $a_k[\mathrm{C}_p^+]$ have been calculated in [**39**], see also [**15**]: If p is not an integer, then

$$a_k[\mathrm{C}_p^+] \neq 0, \tag{26}$$

and if $p \in \mathbb{N}$, then

$$a_k[\mathrm{C}_p^+] = 0 \text{ if and only if } k = 2+p, 4+p, 6+p, \ldots \tag{27}$$

Consequently, C_p^+ is injective if and only if p is not an integer.

Since a convex body $K \in \mathcal{K}_\mathrm{o}^n$ is uniquely determined by its support function, by its radial function and by its L_p surface area measure, we conclude that the operators Π_p^+ and M_p^+ are injective if and only if p is not an integer. It is easy to verify that

$$\Pi_p^- K = \Pi_p^+(-K) = -\Pi_p^+ K \qquad \text{and} \qquad \mathrm{M}_p^- K = \mathrm{M}_p^+(-K) = -\mathrm{M}_p^+ K.$$

Thus, injectivity properties of Π_p^+ and M_p^+ carry over to Π_p^- and M_p^-.

We will frequently use the following consequence of (26) and (27).

Lemma 3.3. *If $K \in \mathcal{K}_\mathrm{o}^n$, $L \in \mathcal{S}^n$ and p is not an odd integer, then*

$$\Pi_p^+ K = \Pi_p^- K \qquad \textit{or} \qquad \mathrm{M}_p^+ L = \mathrm{M}_p^- L$$

holds if and only if K, respectively L, is origin-symmetric.

Proof. From the definition of Π_p^- and M_p^-, it follows directly that $\Pi_p^+ K = \Pi_p^- K$ and $\mathrm{M}_p^+ L = \mathrm{M}_p^- L$ for origin-symmetric bodies K and L.

Conversely, assume that $\Pi_p^+ K = \Pi_p^- K$. Then the convex body $\Pi_p^+ K$ is origin-symmetric, i.e., $h(\Pi_p^+ K, \cdot)^p$ is even. Note that $f \in C(S^{n-1})$ (or a measure μ on S^{n-1}) is even if and only if $\pi_k f = 0$ (or $\pi_k \mu = 0$, respectively) for every odd $k \in \mathbb{N}$.

Since C_p^+ is a multiplier transformation, we obtain from (26) and (27) that $S_p(K, \cdot)$ is even. Thus, by the uniqueness property of $S_p(K\cdot)$, the body K must be origin-symmetric.

The case $\mathrm{M}_p^+ L = \mathrm{M}_p^- L$ is similar, using $\rho(L, \cdot)^{n+p}$ instead of $S_p(K, \cdot)$. □

4. Class Reduction

A standard method for establishing geometric inequalities is to prove them first for a dense class of bodies (e.g, polytopes or smooth bodies) and then, by taking the limit, the inequality is obtained for all bodies. This approach has the major disadvantage that the critical equality conditions are usually lost for the limiting case. In order to prove affine isoperimetric inequalities *along* with their equality conditions for all convex bodies, it is often sufficient to establish the inequalities only for a very small class of bodies, e.g., the class of L_p moment bodies. This class reduction technique was introduced by Lutwak [**25**] and further applied in [**30**] and [**34**].

The crucial result in this section, Lemma 4.2, shows that in order to establish Theorem 1, we need only prove it for the class of smooth convex bodies (in fact the much smaller class of L_p moment bodies will suffice). The tools to derive this fact are provided by Lemma 3.2 and the following lemma.

Lemma 4.1. *If $K \in \mathcal{K}_o^n$, then the convex body $\mathrm{M}_p^\tau K$ is smooth.*

Proof. In order to show that $\mathrm{M}_p^\tau K$ is smooth, we need to prove that its support function $h := h(\mathrm{M}_p^\tau K, \cdot)$ is of class C^2 and that the convex body $\mathrm{M}_p^\tau K$ has everywhere positive radii of curvature (see [**42**, p. 111]). To this end, we first assume that $\tau = 1$, i.e., $h = h(\mathrm{M}_p^+ K, \cdot)$. Let f be a continuous function on $\mathbb{R}^n$ and let $u \in \mathbb{R}^n \backslash \{0\}$. A simple calculation shows that

$$\frac{\partial}{\partial u_i} \int_K (u \cdot x)_+^p f(x)\, dx = p \int_K (u \cdot x)_+^{p-1} x_i f(x)\, dx. \tag{28}$$

Thus, the function h is of class C^2 if $\tau = 1$. Let $(h_{ij})_{i,j=1}^{n-1}$ denote the Hessian matrix of h at u with respect to an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ with $e_n = u$. By [**42**, Corollary 2.5.3], the convex body $\mathrm{M}_p^\tau K$ has everywhere positive radii of curvature if and only if

$$\det(h_{ij}(u))_{i,j=1}^{n-1} > 0.$$

Using (28), we obtain for $h_{ij}(u)$ up to some positive constant

$$\int_K (x \cdot u)_+^p\, dx \int_K (x \cdot u)_+^{p-2} (x \cdot b_i)(x \cdot b_j)\, dx \\ - \int_K (x \cdot u)_+^{p-1} (x \cdot b_i)\, dx \int_K (x \cdot u)_+^{p-1} (x \cdot b_j)\, dx.$$

An application of Hölder's inequality shows that $(h_{ij})_{i,j=1}^{n-1}$ is positive definite and thus, in particular, $\det(h_{ij}(u))_{i,j=1}^{n-1} > 0$. Hence, $\mathrm{M}_p^+ K$ is smooth and, since $\mathrm{M}_p^- K = -\mathrm{M}_p^+ K$, we also obtain that $\mathrm{M}_p^- K$ is smooth. For $\tau \in (-1, 1)$, the assertion follows from a similar (but more tedious) calculation, by using (21) and (28). $\square$

The crucial result of this section is contained in the following lemma which reduces the proof of Theorem 1 to the class of smooth convex bodies.

Lemma 4.2. *In order to prove Theorem 1, it is sufficient to verify the following assertion: If $K \in \mathcal{K}_o^n$ is smooth, then for every $\tau \in [-1, 1]$,*

$$V(K)^{n/p-1} V(\Pi_p^{\tau,*} K) \leq V(B)^{n/p},$$

with equality if and only if K is an ellipsoid centered at the origin.

Proof. For $K \in \mathcal{K}_o^n$, let $\Phi_p K = c_1 \cdot \Pi_p^+ K +_p c_2 \cdot \Pi_p^- K$, where $c_1, c_2 \geq 0$ are not both zero. By (19), there exist a $\tau \in [-1, 1]$ and a constant $c > 0$ such that $\Phi_p K = c\, \Pi_p^\tau K$. Since $\Pi_p^\tau B = B$, we conclude, that the assertion of Theorem 1 is equivalent to the following statement: If $K \in \mathcal{K}_{\mathrm{o}}^n$, then for every $\tau \in [-1,1]$,

$$V(K)^{n/p-1}V(\Pi_p^{\tau,*}K) \leq V(B)^{n/p}, \tag{29}$$

with equality if and only if K is an ellipsoid centered at the origin.

It remains to show that inequality (29) along with its equality conditions holds if and only if it holds for smooth bodies. To this end, we will prove that, for $K \in \mathcal{K}_{\mathrm{o}}^n$,

$$V(K)^{n/p-1}V(\Pi_p^{\tau,*}K) \leq V(\mathrm{M}_p^{\tau}\Pi_p^{\tau,*}K)^{n/p-1}V(\Pi_p^{\tau,*}\mathrm{M}_p^{\tau}\Pi_p^{\tau,*}K), \tag{30}$$

with equality if and only if K and $\mathrm{M}_p^{\tau}\Pi_p^{\tau,*}K$ are dilates. Thus, by Lemma 4.1, any convex body at which $V(K)^{n/p-1}V(\Pi_p^{\tau,*}K)$ attains a maximum must be smooth.

In order to see (30), take $L = \Pi_p^{\tau,*}K$ in Lemma 3.2 and use (11) to conclude

$$V(\Pi_p^{\tau,*}K) = V_p(K, \mathrm{M}_p^{\tau}\Pi_p^{\tau,*}K).$$

Thus, by the L_p Minkowski inequality (8), we obtain

$$V(\Pi_p^{\tau,*}K)^n \geq V(K)^{n-p}V(\mathrm{M}_p^{\tau}\Pi_p^{\tau,*}K)^p, \tag{31}$$

with equality if and only if K and $\mathrm{M}_p^{\tau}\Pi_p^{\tau,*}K$ are dilates. Conversely, replace K by $\mathrm{M}_p^{\tau}L$, for some star body L, in Lemma 3.2 and use (5) to obtain

$$V(\mathrm{M}_p^{\tau}L) = \widetilde{V}_{-p}(L, \Pi_p^{\tau,*}\mathrm{M}_p^{\tau}L).$$

Thus, the dual L_p Minkowski inequality (13) yields

$$V(\mathrm{M}_p^{\tau}L)^n \geq V(L)^{n+p}V(\Pi_p^{\tau,*}\mathrm{M}_p^{\tau}L)^{-p}, \tag{32}$$

with equality if and only if L and $\Pi_p^{\tau,*}\mathrm{M}_p^{\tau}L$ are dilates.

Now take $L = \Pi_p^{\tau,*}K$ in (32) to get

$$V(\mathrm{M}_p^{\tau}\Pi_p^{\tau,*}K)^n \geq V(\Pi_p^{\tau,*}K)^{n+p}V(\Pi_p^{\tau,*}\mathrm{M}_p^{\tau}\Pi_p^{\tau,*}K)^{-p}, \tag{33}$$

with equality if and only if $\Pi_p^{\tau,*}K$ and $\Pi_p^{\tau,*}\mathrm{M}_p^{\tau}\Pi_p^{\tau,*}K$ are dilates.

A combination of inequalities (31) and (33) finally yields (30) and finishes the proof. □

By (16), the case $p = 1$ of inequality (29) reduces to the classical Petty projection inequality. Since we do not wish to reprove this inequality, we note again that we restrict our attention to the case $1 < p < \infty$.

In Section 6, we will again use the class reduction technique to show that Theorem 2 follows from Theorem 1.

5. Steiner Symmetrization and $\Pi_p^{\tau,*}$

In this section we establish the important fact that Steiner symmetrization intertwines with the operator $\Pi_p^{\tau,*}$ for every $\tau \in [-1,1]$. This was proved in [**30**] for the case $\tau = 0$. For arbitrary $\tau \in [-1,1]$, the proof is similar but certain modifications are needed to settle the equality conditions in Theorem 1.

In the following let $\{e_1,\ldots,e_n\}$ be an orthonormal basis of $\mathbb{R}^n$. We will frequently use the decomposition $\mathbb{R}^n = \mathbb{R}^{n-1}\times\mathbb{R}$, where we assume that $e_n^{\perp} = \mathbb{R}^{n-1}$. Clearly, for every convex body $K \in \mathcal{K}_{\mathrm{o}}^n$ there exist functions $\underline{z}, \overline{z} : K|e_n^{\perp} \to \mathbb{R}$ such that K can be represented in the form

$$K = \{(x,t) \in \mathbb{R}^{n-1}\times\mathbb{R} : \underline{z}(x) \le t \le \overline{z}(x), x \in K|e_n^{\perp}\}. \tag{34}$$

Note that the number $\overline{z}-\underline{z}$ is the length of the chord of K through x parallel to e_n. It is easy to verify that $\underline{z}$ is convex and that $\overline{z}$ is a concave function. Thus, $\underline{z}$ and $\overline{z}$ are continuous on $K_{\mathrm{o}} := \operatorname{relint} K|e_n^{\perp}$. If K is smooth, then $\underline{z}$ and $\overline{z}$ are C^1 functions on K_{o}.

Let $D \subseteq \mathbb{R}^{n-1}$ be an open convex set which contains the origin in its interior. For a C^1 function $z : D \to \mathbb{R}$ define

$$\langle z\rangle(x) = z(x) - x\cdot\nabla z(x), \qquad x \in D.$$

Note that the operator $\langle\cdot\rangle$ is linear. Moreover, the kernel of $\langle\cdot\rangle$ consists only of linear functions:

$$\langle z\rangle(x) = 0 \ \text{ for all } x \in D \qquad \Longrightarrow \qquad z \text{ is linear on } D. \tag{35}$$

The following auxiliary result can be found in [**30**, Lemma 11].

Lemma 5.1. *If $K \in \mathcal{K}_{\mathrm{o}}^n$ is a smooth convex body given by*

$$K = \{(x,t) \in \mathbb{R}^{n-1}\times\mathbb{R} : \underline{z}(x) \le t \le \overline{z}(x),\ x \in K|e_n^{\perp}\},$$

then for every $x \in \operatorname{relint} K|e_n^{\perp}$,

$$h(K,(\nabla\underline{z}(x),-1)) = \langle -\underline{z}\rangle(x) \qquad \textit{and} \qquad h(K,(-\nabla\overline{z}(x),1)) = \langle\overline{z}\rangle(x).$$

Recall that for smooth $K \in \mathcal{K}_{\mathrm{o}}^n$, the surface area measure $S(K,\cdot)$ and thus, by (7), also the L_p surface area measure $S_p(K,\cdot)$ are absolutely continuous with respect to spherical Lebesgue measure:

$$dS_p(K,u) = h(K,u)^{1-p} f(K,u)\,du, \qquad u \in S^{n-1}. \tag{36}$$

Here $f(K,\cdot)$ is the curvature function of the smooth convex body K.

For smooth $K \in \mathcal{K}_{\mathrm{o}}^n$, the *spherical image map* $\nu : \operatorname{bd} K \to S^{n-1}$ is defined by letting $\nu(x)$, for $x \in \operatorname{bd} K$, be the unique outer unit normal vector of K at x. By [**42**, p. 112], for any integrable function g on S^{n-1} we have

$$\int_{S^{n-1}} g(u) f(K,u)\,du = \int_{\operatorname{bd} K} g(\nu(x))\,d\mathcal{H}^{n-1}(x),$$

where $\mathcal{H}^{n-1}$ denotes $(n-1)$-dimensional Hausdorff measure. Thus, (17) and (36) yield the following representation of $\Pi_p^\tau K$, $\tau \in [-1,1]$:

$$h(\Pi_p^\tau K, u)^p = c_{n,p}(\tau) \int_{\operatorname{bd} K} \varphi_\tau(u \cdot \nu(x))^p h(K, \nu(x))^{1-p}\, d\mathcal{H}^{n-1}(x). \tag{37}$$

If the smooth convex body K is given by (34), then for any continuous function h on S^{n-1},

$$\begin{aligned}&\int_{\operatorname{bd} K} h(\nu(x))\, d\mathcal{H}^{n-1}(x)\\ &= \int_{K_{\mathrm{o}}} h(\nu(x, \underline{z}(x)))\sqrt{1 + \|\nabla \underline{z}(x)\|^2} + h(\nu(x, \overline{z}(x)))\sqrt{1 + \|\nabla \overline{z}(x)\|^2}\, dx.\end{aligned}$$

Recall that $K_{\mathrm{o}} = \operatorname{relint} K|e_n^\perp$. Since for any $x \in K_{\mathrm{o}}$,

$$\nu(x, \underline{z}(x)) = \frac{(\nabla \underline{z}(x), -1)}{\sqrt{1 + \|\nabla \underline{z}(x)\|^2}} \quad \text{and} \quad \nu(x, \overline{z}(x)) = \frac{(-\nabla \overline{z}(x), 1)}{\sqrt{1 + \|\nabla \overline{z}(x)\|^2}},$$

we obtain from Lemma 5.1, (37), and the homogeneity of $h(K, \cdot)$ and φ_τ:

Lemma 5.2. *If $K \in \mathcal{K}_{\mathrm{o}}^n$ is a smooth convex body given by*

$$K = \{(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R} : \underline{z}(x) \le t \le \overline{z}(x),\ x \in K|e_n^\perp\},$$

then for every $(y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$,

$$\begin{aligned}&c_{n,p}^{-1}(\tau) h(\Pi_p^\tau K, (y, t))^p\\ &= \int_{K_{\mathrm{o}}} \varphi_\tau(t - y \cdot \nabla \overline{z}(x))^p \langle \overline{z} \rangle(x)^{1-p} + \varphi_\tau(y \cdot \nabla \underline{z}(x) - t)^p \langle -\underline{z} \rangle(x)^{1-p}\, dx.\end{aligned}$$

For $K \in \mathcal{K}_{\mathrm{o}}^n$ and $u \in S^{n-1}$, we denote by $S_u K$ the Steiner symmetral of K with respect to the hyperplane $u^\perp$, c.f. [**9**, p. 30]. If K is given by (34), then

$$S_{e_n} K = \{(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R} : \tfrac{1}{2}(\underline{z} - \overline{z})(x) \le t \le \tfrac{1}{2}(\overline{z} - \underline{z})(x), x \in K|e_n^\perp\}.$$

Our next result forms the critical part of the proof of Theorem 1:

Lemma 5.3. *If $K \in \mathcal{K}_{\mathrm{o}}^n$ is smooth, then for every $u \in S^{n-1}$,*

$$S_u \Pi_p^{\tau,*} K \subseteq \Pi_p^{\tau,*} S_u K.$$

If equality holds in the above inclusion, there exists an $r \in [0,1]$ such that the points which divide the (directed) chords of K parallel to u in the proportion $r : 1 - r$ are coplanar.

Proof. Since $\Pi_p^{\tau,*}$ is linearly associating, we can assume without loss of generality that $u = e_n$. Let the convex body K be given by

$$K = \{(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R} : \underline{z}(x) \le t \le \overline{z}(x),\ x \in K|e_n^\perp\}.$$

The definition of Steiner symmetrization and (10) show that

$$S_{e_n} \Pi_p^{\tau,*} K \subseteq \Pi_p^{\tau,*} S_{e_n} K \tag{38}$$

holds if and only if

$$h(\Pi_p^\tau K,(y,s)) = h(\Pi_p^\tau K,(y,t)) = 1 \quad \text{with } s \neq t$$

implies

$$h(\Pi_p^\tau S_{e_n} K,(y,\tfrac{1}{2}s - \tfrac{1}{2}t)) \leq 1.$$

Let $(y,s),(y,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ with $s \neq t$ and suppose that

$$h(\Pi_p^\tau K,(y,s)) = h(\Pi_p^\tau K,(y,t)) = 1.$$

Note that the Steiner symmetral of a smooth convex body is again smooth. Since the triangle inequality implies

$$\varphi_\tau(\tfrac{1}{2}(s-t) - y \cdot \tfrac{1}{2}\nabla(\overline{z} - \underline{z})(x)) \leq \frac{1}{2}\Big(\varphi_\tau(s - y \cdot \nabla\overline{z}(x)) + \varphi_\tau(y \cdot \nabla\underline{z}(x) - t)\Big)$$

and

$$\varphi_\tau(y \cdot \tfrac{1}{2}\nabla(\underline{z} - \overline{z})(x) - \tfrac{1}{2}(s-t)) \leq \frac{1}{2}\Big(\varphi_\tau(y \cdot \nabla\underline{z}(x) - s) + \varphi_\tau(t - y \cdot \nabla\overline{z}(x))\Big),$$

we obtain from Lemma 5.2 and the linearity of the operator $\langle\cdot\rangle$,

$$\begin{aligned}
&c_{n,p}^{-1}(\tau)\, h(\Pi_p^\tau S_{e_n} K,(y,\tfrac{1}{2}s - \tfrac{1}{2}t))^p \\
&= \int_{K_\circ} \varphi_\tau(\tfrac{1}{2}(s-t) - y \cdot \tfrac{1}{2}\nabla(\overline{z} - \underline{z})(x))^p \left\langle \tfrac{1}{2}(\overline{z} - \underline{z})\right\rangle (x)^{1-p}\, dx \\
&\quad + \int_{K_\circ} \varphi_\tau(y \cdot \tfrac{1}{2}\nabla(\underline{z} - \overline{z})(x) - \tfrac{1}{2}(s-t))^p \left\langle \tfrac{1}{2}(\overline{z} - \underline{z})\right\rangle (x)^{1-p}\, dx \\
&\leq \frac{1}{2}\int_{K_\circ} \Big(\varphi_\tau(s - y \cdot \nabla\overline{z}(x)) + \varphi_\tau(y \cdot \nabla\underline{z}(x) - t)\Big)^p \langle \overline{z} - \underline{z}\rangle (x)^{1-p}\, dx \\
&\quad + \frac{1}{2}\int_{K_\circ} \Big(\varphi_\tau(y \cdot \nabla\underline{z}(x) - s) + \varphi_\tau(t - y \cdot \nabla\overline{z}(x))\Big)^p \langle \overline{z} - \underline{z}\rangle (x)^{1-p}\, dx.
\end{aligned}$$

By the convexity of the function $t \mapsto t^p$, it follows that for real numbers $a,b \geq 0$ and $c,d > 0$,

$$(a+b)^p(c+d)^{1-p} \leq a^p c^{1-p} + b^p d^{1-p}, \tag{39}$$

with equality if and only if $ad = bc$, see [**30**, Lemma 8]. Since $K \in \mathcal{K}_\circ^n$, Lemma 5.1 implies that $\langle\overline{z}\rangle(x), \langle -\underline{z}\rangle(x) > 0$ for every $x \in K_\circ$. Thus, we obtain the desired inequality

$$\begin{aligned}
&2\, c_{n,p}^{-1}(\tau) h(\Pi_p^\tau S_{e_n} K,(y,\tfrac{1}{2}s - \tfrac{1}{2}t))^p \\
&\leq \int_{K_\circ} \varphi_\tau(s - y \cdot \nabla\overline{z}(x))^p \langle\overline{z}\rangle(x)^{1-p} + \varphi_\tau(y \cdot \nabla\underline{z}(x) - s)\langle -\underline{z}\rangle(x)^{1-p}\, dx \\
&\quad + \int_{K_\circ} \varphi_\tau(t - y \cdot \nabla\overline{z}(x))^p \langle\overline{z}\rangle(x)^{1-p} + \varphi_\tau(y \cdot \nabla\underline{z}(x) - t)^p \langle -\underline{z}\rangle(x)^{1-p}\, dx \\
&= c_{n,p}^{-1}(\tau) h(\Pi_p^\tau K,(y,s))^p + c_{n,p}^{-1}(\tau) h(\Pi_p^\tau K,(y,t))^p = 2\, c_{n,p}^{-1}(\tau).
\end{aligned}$$

If there is equality in (38), then $h(\Pi_p^\tau K,(y,s)) = h(\Pi_p^\tau K,(y,t)) = 1$ with $s \neq t$ implies $h(\Pi_p^\tau S_{e_n} K,(y,\frac{1}{2}s - \frac{1}{2}t)) = 1$. Consequently, equality must hold in the above chain of inequalities. The equality conditions of (39) now yield for every $x \in K_{\mathrm{o}}$,

$$\begin{aligned}\varphi_\tau(s - y \cdot \nabla \overline{z}(x))\langle -\underline{z}\rangle(x) &= \varphi_\tau(y \cdot \nabla \underline{z}(x) - t)\langle \overline{z}\rangle(x),\\ \varphi_\tau(y \cdot \nabla \underline{z}(x) - s)\langle \overline{z}\rangle(x) &= \varphi_\tau(t - y \cdot \nabla \overline{z}(x))\langle -\underline{z}\rangle(x).\end{aligned}$$

Hence, for $y = 0$, we conclude that

$$\begin{aligned}(|s| + \tau s)\langle -\underline{z}\rangle(x) &= (|t| - \tau t)\langle \overline{z}\rangle(x),\\ (|s| - \tau s)\langle \overline{z}\rangle(x) &= (|t| + \tau t)\langle -\underline{z}\rangle(x).\end{aligned}$$

Since $(0,s),(0,t) \in \operatorname{bd}\Pi_p^{\tau,*}K$, there must exist a constant $c > 0$ such that $\langle \underline{z} + c\overline{z}\rangle = 0$. Thus, by (35), $\underline{z} + c\overline{z}$ has to be linear. Define $r := c/(c+1)$. Since $\underline{z} + c\overline{z}$ is linear, the points which divide the (directed) chords of K parallel to e_n in the proportion $r : r-1$ are coplanar. □

6. Proof of the main theorems

We are now in the position to establish our main results. We first complete the proof of Theorem 1. Then we show that the nonsymmetric L_p projection bodies lead to the strongest affine isoperimetric inequality among the family of inequalities in Theorem 1. The corresponding result for nonsymmetric L_p moment bodies will be given after the proof of Theorem 2. We emphasize again that we are assuming throughout that $n \geq 3$ and $1 < p < \infty$.

In order to settle the equality conditions of Theorem 1, we will need a generalization of the Bertrand–Brunn theorem due to Gruber [**13**]:

Theorem 6.1. *Let $K \in \mathcal{K}_{\mathrm{o}}^n$ be a convex body. Suppose that for any family of parallel chords of K there exists an $r \in [0,1]$ such that the points which divide the (directed) chords of K in the proportion $r : r-1$ are coplanar. Then K is an ellipsoid.*

By Lemma 4.2, our next result completes the proof of Theorem 1:

Theorem 6.2. *If $K \in \mathcal{K}_{\mathrm{o}}^n$ is smooth, then for every $\tau \in [-1,1]$,*

$$V(K)^{n/p-1}V(\Pi_p^{\tau,*}K) \leq V(B)^{n/p},$$

with equality if and only if K is an ellipsoid centered at the origin.

Proof. Since Steiner symmetrization does not affect the volume, we deduce from Lemma 5.3 that for every direction u,

$$V(K)^{n/p-1}V(\Pi_p^{\tau,*}K) \leq V(S_uK)^{n/p-1}V(\Pi_p^{\tau,*}S_uK).$$

If equality holds, there exists an $r \in [0,1]$ such that the points which divide the chords of K parallel to u in the proportion $r : 1-r$ are coplanar.

We can now choose a sequence of Steiner symmetrals of the convex body K which converges to $(V(K)/\kappa_n)^{1/n}B$ (see e.g., [**14**, p. 172]). By the continuity and the homogeneity of $\Pi_p^{\tau,*}$, we obtain

$$V(K)^{n/p-1}V(\Pi_p^{\tau,*}K) \le V(B)^{n/p}.$$

If equality holds, then for every $u \in S^{n-1}$ there exists an $r \in [0,1]$ such that the points which divide the chords of K parallel to u in the proportion $r : 1-r$ are contained in a subspace of codimension 1 (by the proof of Lemma 5.3). Together with Theorem 6.1, this implies that K must be an ellipsoid centered at the origin. $\square$

If $K \in \mathcal{K}_{\mathrm{o}}^n$ is origin-symmetric, then for any $\tau, \sigma \in [-1,1]$, we have $\Pi_p^\tau K = \Pi_p^\sigma K$ and thus, Theorem 1 reduces to the L_p Petty projection inequality established in [**30**]. If K is not origin-symmetric, the following theorem shows that the nonsymmetric operators $\Pi_p^{\pm}$ provide the strongest inequalities:

Theorem 6.3. *For every $K \in \mathcal{K}_{\mathrm{o}}^n$,*

$$V(\Pi_p^{*}K) \le V(\Pi_p^{\tau,*}K) \le V(\Pi_p^{\pm,*}K).$$

If K is not origin-symmetric and p is not an odd integer, equality holds in the left inequality if and only if $\tau = 0$ and equality holds in the right inequality if and only if $\tau = \pm 1$.

Proof. We may assume that K is not origin-symmetric and that p is not an odd integer (otherwise the statement is trivial or follows by approximation). Let $-1 < \tau < 1$. From (10) and the definition of Π_p^τ, we obtain

$$\Pi_p^{\tau,*}K = \frac{(1+\tau)^p}{(1+\tau)^p+(1-\tau)^p} \cdot \Pi_p^{+,*}K \,\tilde{+}_p\, \frac{(1-\tau)^p}{(1+\tau)^p+(1-\tau)^p} \cdot \Pi_p^{-,*}K. \tag{40}$$

Here, multiplication is the dual L_p scalar multiplication, i.e., $\lambda \cdot K = \lambda^{-1/p}\,K$. From the dual L_p Brunn–Minkowski inequality (14), we obtain

$$V(\Pi_p^{\tau,*}K) \le V(\Pi_p^{\pm,*}K), \tag{41}$$

with equality if and only if $\Pi_p^{+,*}K$ and $\Pi_p^{-,*}K$ are dilates which is only possible if $\Pi_p^+K = \Pi_p^-K$. From Lemma 3.3, it follows that inequality (41) is strict for every $\tau \in (-1,1)$ which completes the proof of the right inequality.

In order to see the left inequality, note that the polar coordinate formula for volume yields

$$V(\Pi_p^{\tau,*}K) = \frac{1}{n}\int_{S^{n-1}} \rho(\Pi_p^{\tau,*}K, u)^n \, du.$$

Thus, using (40), the derivative of the function $\tau \mapsto V(\Pi_p^{\tau,*}K)$ is given by

$$f(\tau)\int_{S^{n-1}} \rho(\Pi_p^{\tau,*}K,u)^{n+p}\left(\rho(\Pi_p^{+,*}K,u)^{-p} - \rho(\Pi_p^{-,*}K,u)^{-p}\right) du,$$

where

$$f(\tau) = -\frac{2(1-\tau)^{p-1}(1+\tau)^{p-1}}{((1+\tau)^p + (1-\tau)^p)^2} < 0. \tag{42}$$

The continuous function $\tau \mapsto V(\Pi_p^{\tau,*}K)$ must attain a minimum on $[-1,1]$. By the first part of the proof, the points where this minimum is attained, are contained in $(-1,1)$. If $\bar{\tau}$ is such a point, then

$$\frac{\partial}{\partial \tau} V(\Pi_p^{\tau,*}K)\bigg|_{\tau=\bar{\tau}} = 0.$$

By the calculation above and definition (12), this is equivalent to

$$\widetilde{V}_{-p}(\Pi_p^{\bar{\tau},*}K, \Pi_p^{+,*}K) = \widetilde{V}_{-p}(\Pi_p^{\bar{\tau},*}K, \Pi_p^{-,*}K). \tag{43}$$

Since, for $Q, K, L \in \mathcal{S}^n$ and $\alpha, \beta > 0$,

$$\widetilde{V}_{-p}(Q, \alpha \cdot K \,\widetilde{+}\, \beta \cdot L) = \alpha \widetilde{V}_{-p}(Q, K) + \beta \widetilde{V}_{-p}(Q, L),$$

the representation (40) and the identity (43) imply

$$\widetilde{V}_{-p}(\Pi_p^{\bar{\tau},*}K, \Pi_p^{\bar{\tau},*}K) = \widetilde{V}_{-p}(\Pi_p^{\bar{\tau},*}K, \Pi_p^{\bar{\tau},*}(-K)).$$

By (11) and since $\Pi_p^{\bar{\tau},*}(-K) = -\Pi_p^{\bar{\tau},*}K$, we therefore obtain

$$V(\Pi_p^{\bar{\tau},*}K) = \widetilde{V}_{-p}(\Pi_p^{\bar{\tau},*}K, -\Pi_p^{\bar{\tau},*}K).$$

Using the dual L_p Minkowski inequality (13), we conclude that $\Pi_p^{\bar{\tau},*}K$ is origin-symmetric. By (40), this is equivalent to

$$((1+\bar{\tau})^p - (1-\bar{\tau})^p)\left(\rho(\Pi_p^{+,*}K, u)^{-p} - \rho(\Pi_p^{-,*}K, u)^{-p}\right) = 0$$

for every $u \in S^{n-1}$. As before, an application of Lemma 3.3, shows that $\Pi_p^{+,*}K \neq \Pi_p^{-,*}K$. Thus, we must have $\bar{\tau} = 0$ which proves the left inequality. $\square$

In view of (22), our next result is a stronger version of Theorem 2:

Theorem 6.4. *If $L \in \mathcal{S}^n$, then for every $\tau \in [-1,1]$,*

$$V(L)^{-n/p-1} V(\mathrm{M}_p^\tau L) \geq V(B)^{-n/p},$$

with equality if and only if L is an ellipsoid centered at the origin.

Proof. By definition, $\mathrm{M}_p^\tau L \in \mathcal{K}_o^n$. In Theorem 1, take $K = \mathrm{M}_p^\tau L$, to obtain

$$V(\Pi_p^{\tau,*}\mathrm{M}_p^\tau L)^{-p} \geq V(B)^{-n} V(\mathrm{M}_p^\tau L)^{n-p}, \tag{44}$$

with equality if and only if $\mathrm{M}_p^\tau L$ is an ellipsoid centered at the origin. Combine this with (32) and get

$$V(L)^{-n/p-1} V(\mathrm{M}_p^\tau L) \geq V(B)^{-n/p}.$$

If equality holds in this inequality, then equality must hold in (32) and (44). Consequently, L and $\Pi_p^{\tau,*}\mathrm{M}_p^\tau L$ are dilates and $\mathrm{M}_p^\tau L$ is an ellipsoid centered at the origin. Since $\Pi_p^{\tau,*}$ is linearly associating, this implies that L is an ellipsoid centered at the origin. $\square$

A combination of Theorem 2 with the Blaschke–Santaló inequality now yields the Corollary stated in the introduction.

Our final result shows that the strongest inequalities in Theorem 6.4 are provided by the nonsymmetric operators $\mathrm{M}_p^{\pm}$:

Theorem 6.5. *For every $L \in \mathcal{S}^n$,*

$$V(\mathrm{M}_p L) \geq V(\mathrm{M}_p^{\tau} L) \geq V(\mathrm{M}_p^{\pm} L),$$

If L is not origin-symmetric and p is not an odd integer, equality holds in the left inequality if and only if $\tau = 0$ and equality holds in the right inequality if and only if $\tau = \pm 1$.

Proof. We may again assume that L is not origin-symmetric and p is not an odd integer. Let $-1 < \tau < 1$. Using that for L_p scalar multiplication $\lambda \cdot K = \lambda^{1/p} K$, an application of the L_p Brunn–Minkowski inequality (9) to the representation (21) yields

$$V(\mathrm{M}_p^{\tau} L) \geq V(\mathrm{M}_p^{\pm} L), \tag{45}$$

with equality if and only if $\mathrm{M}_p^{+} L$ and $\mathrm{M}_p^{-} L$ are dilates which is only possible if $\mathrm{M}_p^{+} L = \mathrm{M}_p^{-} L$. From Lemma 3.3, it follows that inequality (45) is strict for every $\tau \in (-1, 1)$ which completes the proof of the right inequality.

In order to prove the left inequality, we have to calculate the derivative of the function $\tau \mapsto V(\mathrm{M}_p^{\tau} L)$ with respect to τ: For fixed $\bar{\tau} \in (-1, 1)$, note that

$$\frac{V(\mathrm{M}_p^{\tau} L) - V_1(\mathrm{M}_p^{\tau} L, \mathrm{M}_p^{\bar{\tau}} L)}{\tau - \bar{\tau}} = \frac{1}{n} \int_{S^{n-1}} \frac{h(\mathrm{M}_p^{\tau} L, u) - h(\mathrm{M}_p^{\bar{\tau}} L, u)}{\tau - \bar{\tau}} \, dS(\mathrm{M}_p^{\tau} L, u),$$

and

$$\frac{V_1(\mathrm{M}_p^{\bar{\tau}} L, \mathrm{M}_p^{\tau} L) - V(\mathrm{M}_p^{\bar{\tau}} L)}{\tau - \bar{\tau}} = \frac{1}{n} \int_{S^{n-1}} \frac{h(\mathrm{M}_p^{\tau} L, u) - h(\mathrm{M}_p^{\bar{\tau}} L, u)}{\tau - \bar{\tau}} \, dS(\mathrm{M}_p^{\bar{\tau}} L, u).$$

From the uniform convergence of support functions and the weak convergence of surface area measures, we deduce that the limits

$$\lim_{\tau \to \bar{\tau}} \frac{V(\mathrm{M}_p^{\tau} L) - V_1(\mathrm{M}_p^{\tau} L, \mathrm{M}_p^{\bar{\tau}} L)}{\tau - \bar{\tau}}, \quad \lim_{\tau \to \bar{\tau}} \frac{V_1(\mathrm{M}_p^{\bar{\tau}} L, \mathrm{M}_p^{\tau} L) - V(\mathrm{M}_p^{\bar{\tau}} L)}{\tau - \bar{\tau}} \tag{46}$$

exist and are both equal to

$$g(\bar{\tau}) := \frac{1}{n} \int_{S^{n-1}} \frac{\partial}{\partial \tau} h(\mathrm{M}_p^{\tau} L, u) \bigg|_{\bar{\tau}} \, dS(\mathrm{M}_p^{\bar{\tau}} L, u).$$

Using the L_p Minkowski inequality (8) for $p = 1$ in (46), shows that

$$g(\bar{\tau}) \leq V(\mathrm{M}_p^{\bar{\tau}} L)^{(n-1)/n} \liminf_{\tau \to \bar{\tau}} \frac{V(\mathrm{M}_p^{\tau} L)^{1/n} - V(\mathrm{M}_p^{\bar{\tau}} L)^{1/n}}{\tau - \bar{\tau}}$$

and

$$g(\bar{\tau}) \geq V(\mathrm{M}_p^{\bar{\tau}} L)^{(n-1)/n} \limsup_{\tau \to \bar{\tau}} \frac{V(\mathrm{M}_p^{\tau} L)^{1/n} - V(\mathrm{M}_p^{\bar{\tau}} L)^{1/n}}{\tau - \bar{\tau}}.$$

Thus, we obtain

$$g(\bar{\tau}) = V(\mathrm{M}_p^{\bar{\tau}} L)^{(n-1)/n} \lim_{\tau \to \bar{\tau}} \frac{V(\mathrm{M}_p^{\tau} L)^{1/n} - V(\mathrm{M}_p^{\bar{\tau}} L)^{1/n}}{\tau - \bar{\tau}}.$$

In particular, the function $\tau \to V(\mathrm{M}_p^{\tau} L)^{1/n}$ is differentiable at $\bar{\tau}$. The definition of $g(\bar{\tau})$ yields

$$\frac{\partial}{\partial \tau} V(\mathrm{M}_p^{\tau} L) = \int_{S^{n-1}} \frac{\partial}{\partial \tau} h(\mathrm{M}_p^{\tau} L, u)\, dS(\mathrm{M}_p^{\tau} L, u).$$

Using (21), we obtain for this derivative

$$-f(\tau) \int_{S^{n-1}} h(\mathrm{M}_p^{\tau} L, u)^{1-p} \left(h(\mathrm{M}_p^{+} L, u)^p - h(\mathrm{M}_p^{-} L, u)^p \right) dS(\mathrm{M}_p^{\tau} L, u),$$

where $f(\tau)$ is given by (42).

The continuous function $\tau \mapsto V(\mathrm{M}_p^{\tau} L)$ must attain a maximum on $[-1, 1]$. By the first part of the proof, the points where this maximum is attained, are contained in $(-1, 1)$. If $\bar{\tau}$ is such a point, then

$$\left. \frac{\partial}{\partial \tau} V(\mathrm{M}_p^{\tau} L) \right|_{\tau = \bar{\tau}} = 0.$$

By the calculation above and definition (6), this is equivalent to

$$V_p(\mathrm{M}_p^{\bar{\tau}} L, \mathrm{M}_p^{+} L) = V_p(\mathrm{M}_p^{\bar{\tau}} L, \mathrm{M}_p^{-} L). \tag{47}$$

Since, for $Q, K, L \in \mathcal{K}_{\mathrm{o}}^n$ and $\alpha, \beta > 0$,

$$V_p(Q, \alpha \cdot K +_p \beta \cdot L) = \alpha V_p(Q, K) + \beta V_p(Q, L),$$

the representation (21) and the identity (47) imply

$$V_p(\mathrm{M}_p^{\bar{\tau}} L, \mathrm{M}_p^{\bar{\tau}} L) = V_p(\mathrm{M}_p^{\bar{\tau}} L, \mathrm{M}_p^{\bar{\tau}}(-L)).$$

By (5) and since $\mathrm{M}_p^{\bar{\tau}}(-L) = -\mathrm{M}_p^{\bar{\tau}} L$, we therefore obtain

$$V(\mathrm{M}_p^{\bar{\tau}} L) = V_p(\mathrm{M}_p^{\bar{\tau}} L, -\mathrm{M}_p^{\bar{\tau}} L).$$

Using the L_p Minkowski inequality (8), we conclude that $\mathrm{M}_p^{\bar{\tau}} L$ is origin-symmetric. By (21), this is equivalent to

$$((1 + \bar{\tau})^p - (1 - \bar{\tau})^p) \left(h(\mathrm{M}_p^{+} L, u)^p - h(\mathrm{M}_p^{-} L, u)^p \right) = 0$$

for every $u \in S^{n-1}$. By Lemma 3.3, $\mathrm{M}_p^{+} L \neq \mathrm{M}_p^{-} L$. Thus, we must have $\bar{\tau} = 0$ which proves the left inequality. $\square$

Acknowledgement. The authors are very grateful for the help of Monika Ludwig, Erwin Lutwak, Deane Yang, and Gaoyong Zhang in the presentation of this article. This work was supported by the Austrian Science Fund (FWF), within the project P 18308, "Valuations on convex bodies".

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