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The Computational Content of Arithmetical Proofs

Stefan Hetzl

Abstract For any extension T of $I\Sigma_1$ having a cut-elimination property extending that of $I\Sigma_1$, the number of different proofs that can be obtained by cut-elimination from a single T-proof cannot be bound by a function which is provably total in T.

1 Introduction

The notion of computational content of a proof is pervasive in proof theory. It can for example be found in the characterisation of the provably total functions of a theory (Kreisel [14]), in consistency proofs like Gödel's Dialectica interpretation [9] and Girard's system F [8] as well as in more recent applications in other mathematical areas (Kohlenbach [12]) or in proof complexity (Krajíček [13]). In this article we will concentrate on theories of classical first-order arithmetic. There are many different methods for extracting computations from arithmetical proofs, some of them like Gentzen's cut-elimination [7], the ε-substitution method of Ackermann [1] or term calculi such as Parigot [16] work directly in a classical system. Others like the Dialectica interpretation [9] or realisability (Kleene [11]) with Friedman's A-translation [5] typically require a translation to an intuitionistic system first, see Avigad [2] for a recent survey. Many of these methods extract a (program that implements a) *function* from a proof.

The possibility of extracting different programs from one and the same proof is well-known, see Ratiu and Trifonov [17] or Baaz et al. [4] for recent case studies and Urban and Bierman [18] for an interpretation of classical logic as non-deterministic computation. It is not clear however how far this non-canonicity goes. In Baaz and Hetzl [3] it has been shown that the number of (significantly different) cut-free proofs obtainable by cut-elimination in pure first-order logic can grow as fast the hyper-exponential function 2_n (where

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$2_0 = 1$ and $2_{i+1} = 2^{2_i}$) while the length of the original proof is polynomial in n. This function is exactly the growth rate of cut-elimination. In this paper we show an analogous result for arithmetical theories. To that aim we define the notion of *computational theory* – essentially – by the ability of computing witnesses from proofs of existential statements. We then show that for any computational theory T extending $I\Sigma_1$ the number of different cut-free proofs obtainable by cut-elimination from a single T-proof cannot be bound by a function in the size of the proof which is provably total in T.

2 Computational Theories

In this paper we will rely on several results of [3]. Let **LK** denote the sequent calculus (for first-order classical logic without equality) used there with the additional restriction that in the quantifier inferences

$$\frac{\Gamma \to \Delta, A[x\backslash t]}{\Gamma \to \Delta, \exists x\, A}\ \exists_{\mathrm{r}} \qquad \text{and} \qquad \frac{A[x\backslash t], \Gamma \to \Delta}{\forall x\, A, \Gamma \to \Delta}\ \forall_{\mathrm{l}}$$

the term t contains only such variables that appear free in the conclusion sequent of the inference. A proof not fulfilling this condition can easily be transformed into one that does by replacing the violating variables by a constant symbol. This condition has the technically convenient consequence that cut-free proofs of Σ_1-sentences are variable-free. We will work in the language of arithmetic $L = \{0, S, +, *, =, \leqslant\}$. For $n \in \mathbb{N}$ we write $\bar{n}$ for the term $S^n(0)$. When writing down concrete proofs we often omit structural inferences.

Definition 2.1. Let Seq denote the set of sequents in L, a k-ary *inference rule* is a subset of Seq^{k+1}. A *sequent calculus presentation* of an arithmetical theory T is a set of inference rules $\mathcal{R}$ s.t. $T \vdash A$ iff the sequent $\to A$ is provable in the calculus $\mathbf{LK} + \mathcal{R}$.

Q will denote the presentation of the theory of minimal arithmetic obtained from extending **LK** by the unary inference rules defined by

$$\frac{F, \Gamma \to \Delta}{\Gamma \to \Delta}\ F$$

for every sentence F in: reflexivity, symmetry, transitivity and compatibility of equality w.r.t. L as well as the universal closures of the axioms (Q1)-(Q8) in Hájek and Pudlák [10, Definition 1.1].

Definition 2.2. Let $\mathcal{R}$ be a sequent calculus presentation of an arithmetical theory. An *$\mathcal{R}$-reduction rule* is a set $C \in \Pi \times \Pi$ where Π is the set of $\mathbf{LK} + \mathcal{R}$-proofs and $(\pi, \pi') \in C$ implies that the end-sequent of π' equals that of π. For a set $\mathcal{C}$ of reduction rules write $\to^{\mathcal{C}}$ for its reflexive, transitive and compatible (w.r.t. the inference rules $\mathbf{LK} + \mathcal{R}$) closure. A *normal form* of $\mathcal{C}$ is a proof π s.t. $\pi \to^{\mathcal{C}} \pi'$ implies $\pi = \pi'$.

A pair $(\mathcal{R}, \mathcal{C})$ is called *computational theory* if i) for every proof π in $\mathbf{LK} + \mathcal{R}$ of a Σ_1-sentence there is a cut-free Q-proof π' with $\pi \to^{\mathcal{C}} \pi'$ and ii) cut-free Q-proofs are normal forms.

A computational theory thus allows to compute a witness for a Σ_1-sentence from a given proof (by obtaining a cut-free Q-proof from $\mathcal{C}$ and then evaluating the matrix of the Σ_1-sentence for all witnesses of the existential quantifier present in that proof). This extends to proofs of Π_2-sentences in the straightforward way by applying the Σ_1-procedure to instances of the Π_2-proof.

$I\Sigma_1$ will denote the computational theory whose inference rules extend those of Q defined above by the unary inference rule

$$\frac{F(\alpha), \Gamma \to \Delta, F(S(\alpha))}{F(0), \Gamma \to \Delta, F(t)}\ \text{ind}$$

for F being a (not necessarily prenex) Σ_1-formula. The reduction rules of $I\Sigma_1$ consist of i) the local reduction rules of pure first-order logic as listed in Appendix A of [3] (which are those of Gentzen [6] adapted to the version of **LK** used here), ii) the permutation of cut upwards over any of the newly introduced rules F of Q or ind of $I\Sigma_1$ provided the cut formula is not active in that inference and iii) the reduction of

$$\frac{\begin{matrix}(\pi)\\ F(\alpha), \Gamma \to \Delta, F(S(\alpha))\end{matrix}}{F(0), \Gamma \to \Delta, F(t)}\ \text{ind}$$

to

$$\frac{\dfrac{\begin{matrix}(\pi[\alpha\backslash 0])\\ F(0), \Gamma \to \Delta, F(S(0))\end{matrix} \quad \begin{matrix}(\pi[\alpha\backslash S(0)])\\ F(S(0)), \Gamma \to \Delta, F(S(S(0)))\end{matrix}}{\begin{matrix}F(0), \Gamma \to \Delta, F(S(S(0)))\\ \vdots \\ F(0), \Gamma \to \Delta, F(\bar{n})\end{matrix}}\ \text{cut} \qquad \begin{matrix}(\omega(\bar{n}, t))\\ F(\bar{n}) \to F(t)\end{matrix}}{F(0), \Gamma \to \Delta, F(t)}\ \text{cut}$$

where π is any proof, t is a variable-free term whose value is n and $\omega(\bar{n}, t)$ denotes the straightforward proof of $F(\bar{n}) \to F(t)$ in Q.

Definition 2.3. Let $(\mathcal{R}, \mathcal{C})$ be a computational theory, another computational theory $(\mathcal{R}', \mathcal{C}')$ is called *computational extension* of $(\mathcal{R}, \mathcal{C})$ if $\mathcal{R} \subseteq \mathcal{R}'$ and $\mathcal{C} \subseteq \mathcal{C}'$.

These notions are very general as we do not require decidability, neither of the inference rules nor of the reduction rules. Even the set of true sentences qualifies as computational theory in the above sense by adding all true sentences as axioms (nullary inference rules) and relying on the Σ_1-completeness of Q for defining the reduction rules.

3 Translation to Arithmetic

From now on, and for the rest of this paper, let $T = (\mathcal{R}, \mathcal{C})$ be a computational extension of $I\Sigma_1$ and let Σ be any first-order language, disjoint from the language L of arithmetic, and containing at least one constant and one function symbol. The work of [3] has been carried out in the language $L \cup \{d, 2\} \cup \Sigma$ where d is a unary function symbol whose intended interpretation is the depth of a Σ-term and 2 is a unary function symbol whose intended interpretation is the exponential function with base 2. The function symbol 2 will not be used here so it is enough to treat the language $L \cup \{d\} \cup \Sigma$. We will now briefly describe how to translate formulas, proofs and reduction sequences from $L \cup \{d\} \cup \Sigma$ to L.

Using standard coding techniques, see e.g. [10], we arithmetise Σ-terms and write $\#t$ for the natural number representing the Σ-term t. We obtain Σ_1-formulas defining the set of Σ-terms, the depth of a Σ-term and the relation of one term being at the i-th position of another term which allows to translate any atom in $L \cup \Sigma \cup \{d\}$ to a formula in L. If π is an **LK**-proof and σ a substitution replacing each k-ary atom by a formula with k free variables, then $\pi\sigma$ is an **LK**-proof too. Furthermore, the reduction rules of first-order logic have the property that $\pi \to^{\mathcal{C}} \pi'$ implies $\pi\sigma \to^{\mathcal{C}} \pi'\sigma$. Therefore this translation of formulas extends to a translation of proofs and of reduction sequences.

Let $\mathcal{A}$ denote the translation of the (finite) set of axioms of [3] to L. The axioms containing d and symbols of Σ are

$$d(c) = 0$$

for every constant symbol c in Σ and

$$\begin{aligned} T_f^j \quad \equiv \quad & \forall x \forall y_1 \ldots \forall y_r (\ d(y_1) \leqslant x \supset \ldots \supset d(y_{j-1}) \leqslant x \supset d(y_j) = x \\ & \supset d(y_{j+1}) \leqslant x \supset \ldots \supset d(y_r) \leqslant x \supset d(f(y_1, \ldots, y_r)) = S(x)\ ) \end{aligned}$$

for every function symbol f of arity r in Σ and every $j \in \{1, \ldots, r\}$. Along the lines of [10] it is easy to check that the translations of these axioms are provable in $I\Sigma_1$. All other axioms of [3] that are used here are L-sentences and a quick check shows that they are also provable in $I\Sigma_1$. This will later allow to obtain a T-proof of F from a T-proof of $\mathcal{A} \to F$ by appending a cut on $\bigwedge_{A \in \mathcal{A}} A$.

4 Non-Confluence

The central idea for the construction of a proof with many normal forms is to modify a proof of the existence of a large number s.t. i) it proves the existence of a deep Σ-term instead and ii) it does so in a way that permits reduction to *any* Σ-term of that depth. Denote with $E(u)$ the translation of $\exists x\, d(x) = u$ to arithmetic, with $L(u)$ the translation of $\exists x\, d(x) \leqslant u$ and with $F(u)$ the formula $L(u) \wedge E(u)$. The central construction will be an induction on F using non-confluent constructors of Σ-terms for the induction base and step. We use F here in order to allow for reduction to any term of the desired depth which necessitates the $\leqslant$-part of the induction hypothesis. The slightly simpler proof using induction on E instead would allow reduction to any term of the desired depth all of whose branches are of equal depth.

Let τ_0 be the translation of the proof of $\mathcal{A} \to F(0)$ defined in Section 5.2. of [3]. As shown there, this proof possesses for any constant symbol of Σ and for both of the existential quantifiers in $L(0)$ and $E(0)$ respectively a normal form having this constant symbol as witness of that quantifier. This property carries over to the present setting as described in the previous section. Let $\tau_s'(u)$ be the translation of the proof of $\mathcal{A}, F(u) \to F(s(u))$ defined in Section 5.3. of [3]. This proof has the analogous property for function symbols, i.e. it allows the reduction to any top-level symbol as witness. Let $\psi(u)$ be

$$\dfrac{\dfrac{\begin{array}{c}(\tau_0)\\ \mathcal{A} \to F(0)\end{array} \quad \dfrac{\begin{array}{c}(\tau'_s(\gamma))\\ \mathcal{A}, F(\gamma) \to F(s(\gamma))\end{array}}{\mathcal{A}, F(0) \to F(u)}\,\text{ind}}{\mathcal{A} \to F(u)}\,\text{cut} \qquad \dfrac{E(u) \to E(u)}{F(u) \to E(u)}\,\wedge_{l_1}}{\mathcal{A} \to E(u)}\,\text{cut}$$

which is a proof in $I\Sigma_1$ as F is a Σ_1-formula.

Lemma 4.1. *Let $n \in \mathbb{N}$ and t be any variable-free L-term with value n. Then for every Σ-term s of depth n there is a proof ψ_s s.t. $\psi(t) \to^{\mathcal{C}} \psi_s$ and the only witness of the existential quantifier in $E(t)$ in the end-sequent of ψ_s is an L-term with value $\#s$.*

Proof By reduction of induction and shifting the cut on $F(0)$ upwards using the reduction rules of pure logic we obtain $\psi(t) \to^{\mathcal{C}}$

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}(\tau_0)\\ \mathcal{A} \to F(0)\end{array} \quad \begin{array}{c}(\tau'_s(0))\\ \mathcal{A}, F(0) \to F(S(0))\end{array}}{\mathcal{A} \to F(S(0))}\,\text{cut}\\ \vdots \\ \mathcal{A} \to F(\bar{n})\end{array} \qquad \begin{array}{c}(\omega)\\ F(\bar{n}) \to F(t)\end{array}}{\mathcal{A} \to F(t)}\,\text{cut} \qquad \dfrac{E(t) \to E(t)}{F(t) \to E(t)}\,\wedge_{l_1}}{\mathcal{A} \to E(t)}\,\text{cut}$$

which is a proof whose form is slightly simpler than that of an F-*chain* from [3]. Therefore the proof of Lemma 10 from there readily adapts to this situation, in brief: use a bottom-up reduction of the $\tau'_s(\bar{i})$ making the right choices for obtaining s at each level and duplicating the proof of the assumption, thereby transforming the linear structure of the above proof to the tree structure of s. Finish the construction of s by appropriate reduction of the copies of τ_0 and finally reduce the two cuts at the bottom observing that they do not change the witness. □

Theorem 4.2. *Let $g : \mathbb{N} \to \mathbb{N}$ be a function provably total in T and let $G(x, y)$ be its definition. Then there is a T-proof $\chi(u)$ of $\exists y\,(G(u, y) \wedge E(y))$ s.t. for every $n \in \mathbb{N}$ and every Σ-term s of depth $g(n)$ there is a normal form χ_s of $\chi(\bar{n})$ s.t. every witness r of the existential quantifier in some $E(t)$ where t is an L-term with value $g(n)$ has the value $\#s$.*

Before proving this theorem, a remark on its formulation is appropriate: a cut-free Q-proof of $\exists y\,(G(\bar{n}, y) \wedge E(y))$ must contain some term t with value $g(n)$ as instance of $\exists y$ and hence it also contains $E(t)$. However, it might also contain other (irrelevant) instances of $\exists y$ with the same or other numerical values. In principle, it would be possible to rule those out by imposing an (intuitionistic) restriction on $\mathcal{C}$. As this option would render the reduction relation somewhat artificial we have opted for the more natural definition (and the more cumbersome theorem).

Proof Let ξ be any T-proof of $\rightarrow \forall x \exists y\, G(x,y)$, let

$$\xi(u) \quad = \quad \cfrac{\begin{matrix}(\xi)\\ \rightarrow \forall x \exists y\, G(x,y)\end{matrix} \qquad \cfrac{\exists y\, G(u,y) \rightarrow \exists y\, G(u,y)}{\forall x \exists y\, G(x,y) \rightarrow \exists y\, G(u,y)}\ \forall_{\mathrm{l}}}{\rightarrow \exists y\, G(u,y)}\ \mathrm{cut} \quad ,$$

$$\chi_0(u) \quad = \quad \cfrac{\cfrac{G(u,\beta) \rightarrow G(u,\beta) \qquad \begin{matrix}(\psi(\beta))\\ \mathcal{A} \rightarrow E(\beta)\end{matrix}}{\mathcal{A}, G(u,\beta) \rightarrow G(u,\beta) \wedge E(\beta)}\ \wedge_{\mathrm{r}}}{\mathcal{A}, \exists y\, G(u,y) \rightarrow \exists y\, (G(u,y) \wedge E(y))}\ \exists_{\mathrm{l}}, \exists_{\mathrm{r}}$$

and $\chi(u) =$

$$\cfrac{\begin{matrix}(\pi)\\ \rightarrow \bigwedge_{A \in \mathcal{A}} A\end{matrix} \qquad \cfrac{\cfrac{\begin{matrix}(\xi(u))\\ \rightarrow \exists y\, G(u,y)\end{matrix} \qquad \begin{matrix}(\chi_0(u))\\ \mathcal{A}, \exists y\, G(u,y) \rightarrow \exists y\, (G(u,y) \wedge E(y))\end{matrix}}{\mathcal{A} \rightarrow \exists y\, (G(u,y) \wedge E(y))}\ \mathrm{cut}}{\bigwedge_{A \in \mathcal{A}} A \rightarrow \exists y\, (G(u,y) \wedge E(y))}\ \wedge_{\mathrm{l}}^{*}}{\rightarrow \exists y\, (G(u,y) \wedge E(y))}\ \mathrm{cut} \quad .$$

As T is a computational extension of $I\Sigma_1$, there is a cut-free Q-proof ξ' of $\rightarrow \exists y\, G(\bar{n}, y)$ with $\xi(\bar{n}) \rightarrow^{\mathcal{C}} \xi'$ having terms $t_1, \ldots, t_k$ as witnesses of $\exists y$. Using reductions rules from pure logic we obtain a proof ξ^* with $\chi(\bar{n}) \rightarrow^{\mathcal{C}} \xi^*$ from ξ' by successively replacing

$$\cfrac{\begin{matrix}(\pi_i)\\ \Gamma_i \rightarrow \Delta_i, G(\bar{n}, t_i)\end{matrix}}{\Gamma_i \rightarrow \Delta_i, \exists y\, G(\bar{n}, y)}\ \exists_{\mathrm{r}}$$

by

$$\cfrac{\cfrac{\begin{matrix}(\pi_i)\\ \Gamma_i \rightarrow \Delta_i, G(\bar{n}, t_i)\end{matrix} \qquad \begin{matrix}(\psi(t_i))\\ \mathcal{A} \rightarrow E(t_i)\end{matrix}}{\mathcal{A}, \Gamma_i \rightarrow \Delta_i, G(\bar{n}, t_i) \wedge E(t_i)}\ \wedge_{\mathrm{r}}}{\mathcal{A}, \Gamma_i \rightarrow \Delta_i, \exists y\, (G(\bar{n}, y) \wedge E(y))}\ \exists_{\mathrm{r}}$$

for $i \in \{1, \ldots, k\}$. For all t_i with value $g(n)$ we apply Lemma 4.1 to obtain a cut-free ψ_i with $\psi(t_i) \rightarrow^{\mathcal{C}} \psi_i$ having an L-term with value $\#s$ as only witness. For all t_i whose value is not $g(n)$ we reduce to an arbitrary cut-free proof. Finally, the reduction of the cut on $\bigwedge_{A \in \mathcal{A}} A$ does not change the witnesses and finishes with a cut-free Q-proof because variable-freeness of the t_i ensures that we can reduce the inductions coming from π. This cut-free Q-proof is χ_s and has the desired property. □

Corollary 4.3. *The number of normal forms of a proof in a computational extension T of $I\Sigma_1$ cannot be bound by a function in the size of the proof which is provably total in T.*

5 Conclusion

It should be emphasised that apart from the theory-specific part (which is arbitrary) the above reduction sequences consist exclusively of the natural standard reductions of a sequent calculus for $I\Sigma_1$. Furthermore, the proofs with cut are completely symmetric w.r.t. their normal forms in the sense that there is no reason for preferring one normal form over another.

The central technical insight is that the non-determinism of classical logic can be isolated in a manner that permits the cut-elimination process to distribute it throughout a large proof it generates. Consequently an analogous result should be expected for every calculus containing even the slightest non-determinism.

The contribution of this work to the discussion of the computational content of classical logic is a new demonstration that, in a strikingly strong sense, the computational content of an arithmetical proof is *not a function*. As useful it is, from both a theoretical and a practical point of view, to extract a function from a proof, such extraction methods in general fall short of doing justice to the notion of computational content, as they cannot satisfy the unambiguity suggested by the term *content*.

The above results and remarks refer to *formal* proofs. As pointed out e.g. in [2] and Kreisel [15] there is another, more fundamental, reason for the ambiguity of the computational content of a *mathematical* proof, which is that a given mathematical proof allows many different formalisations which in turn may induce different computations.

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Laboratoire Preuves, Programmes et Systèmes (PPS)
Université Paris Diderot – Paris 7
175 Rue du Chevaleret, 75013 Paris, France
stefan.hetzl@pps.jussieu.fr
http://www.pps.jussieu.fr/~hetzl/