Automata and Formal Language Theory

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Introduction

- Formal and natural languages

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- Formal and natural languages
- How to specify a formal language?
  - Automata
  - Grammars

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- Strong connections to:
  - Computability theory
  - Complexity theory

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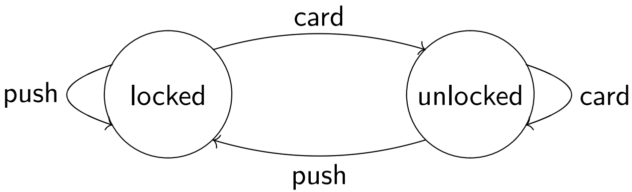
  - Automata
  - Grammars
- Strong connections to:
  - Computability theory
  - Complexity theory
- Applications in computer science:
  - Verification
  - Compiler construction
  - Data formats

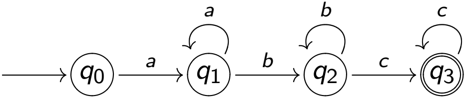
Outline

- Deterministic finite automata
- Nondeterministic finite automata
- Automata with ε-transitions
- The class of regular languages
- The pumping lemma for regular languages
- Context-free grammars and languages
- Right linear grammars
- Pushdown Automata
- The pumping lemma for context-free languages
- Grammars in computer science
- Further topics

Finite Automata – A First Example



Finite Automata – A More Abstract Example



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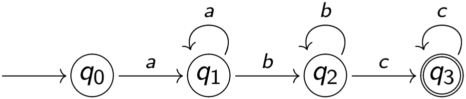

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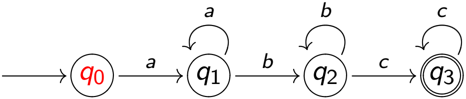

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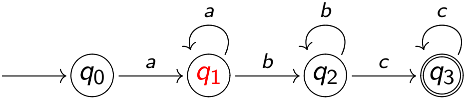

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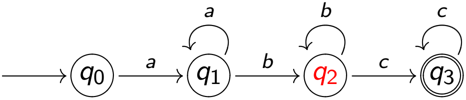

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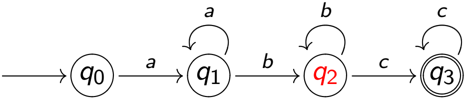

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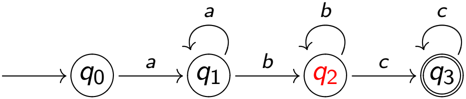

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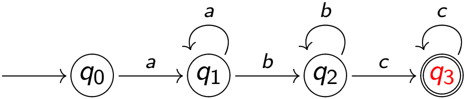

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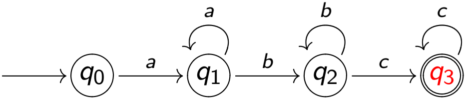

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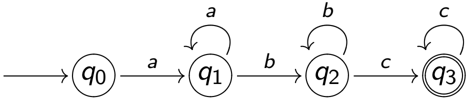

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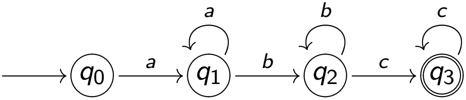

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aab

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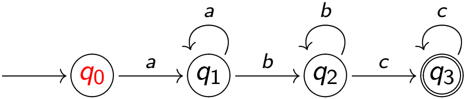

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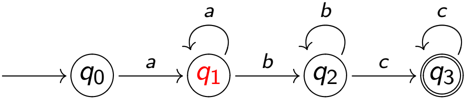

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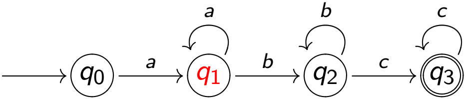

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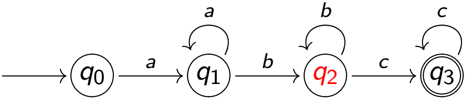

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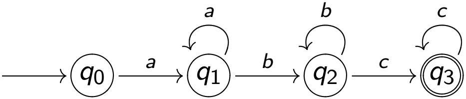

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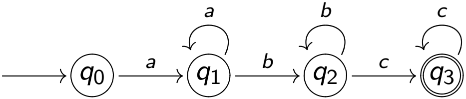

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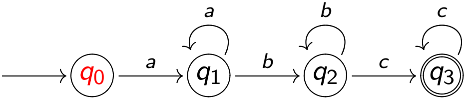

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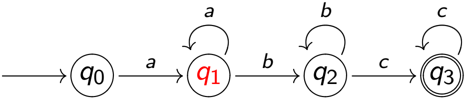

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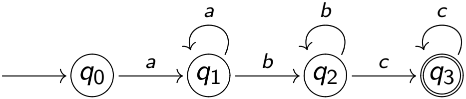

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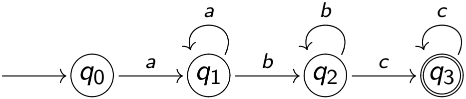

$abbbcc$ ✓
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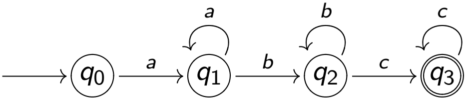
The language *accepted* by this automaton is

$$L = \{a^k b^n c^m \mid k, n, m \geq 1\}$$

The Error State



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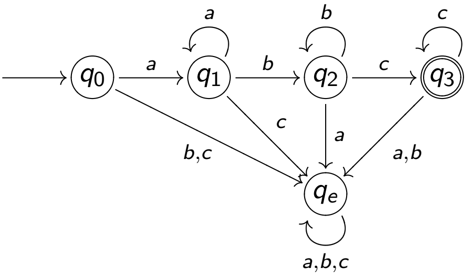


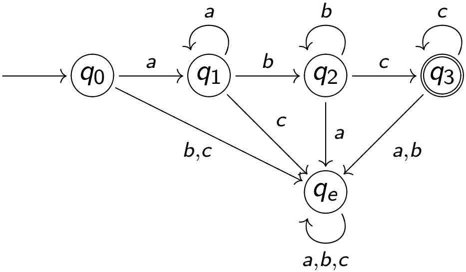
Deterministic Finite Automata – Definition

Definition

A *deterministic finite automaton (DFA)* is a tuple $A = \langle Q, \Sigma, \delta, q_0, F \rangle$ where:

1. Q is a finite set (the *states*).
2. Σ is a finite set (the *input symbols*).
3. $\delta : Q \times \Sigma \to Q$ (the *transition function*).
4. $q_0 \in Q$ (the *starting state*)
5. $F \subseteq Q$ (the *final states*).

DFA – Example



as tuple: BB1

The Language of a DFA

Definition

Extend $\delta : Q \times \Sigma \to Q$ to $\hat{\delta} : Q \times \Sigma^* \to Q$ as follows.

$$\hat{\delta}(q, w) = \begin{cases} q & \text{if } w = \varepsilon \\ \delta(\hat{\delta}(q, v), x) & \text{if } w = vx \text{ for } v \in \Sigma^*, x \in \Sigma \end{cases}$$

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Example

$\hat{\delta}(abc, q_0) = q_3, \ \hat{\delta}(aba, q_0) = q_e$

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Example

$\hat{\delta}(abc, q_0) = q_3,\ \hat{\delta}(aba, q_0) = q_e$

Definition

Let $A = \langle Q, \Sigma, \delta, q_0, F \rangle$ be a DFA. The language accepted by A is

$$\mathrm{L}(A) = \{w \in \Sigma^* \mid \hat{\delta}(q_0, w) \in F\}.$$

Designing an DFA

$$L = \{w \in \{a, b\}^* \mid w \text{ contains an even number of } a\text{'s and an even number of } b\text{'s}\}$$

BB2

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Nondeterministic Finite Automata – A Motivating Example

Automaton for accepting $L = \{wab \mid w \in \{a, b\}^*\}$?

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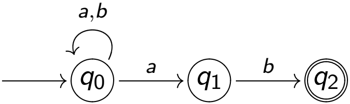

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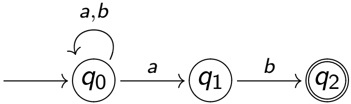

Nondeterminism $\implies$ consider *all* possible runs

Definition

A *nondeterministic finite automaton (NFA)* is a tuple $A = \langle Q, \Sigma, \Delta, q_0, F \rangle$ where:

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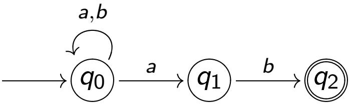
NFA – Example



as tuple: BB3

The Language of an NFA

Definition

Extend $\Delta \subseteq Q \times \Sigma \times Q$ to $\hat{\Delta} : Q \times \Sigma^* \times Q$ as follows.

$$
\begin{aligned}
(q, w, q) \in \hat{\Delta} &\quad \text{if } w = \varepsilon \\
(q, w, q^*) \in \hat{\Delta} &\quad \text{if } w = vx \text{ for } v \in \Sigma^*, x \in \Sigma, \\
&\qquad \text{and } (q, v, q') \in \hat{\Delta}, \text{and } (q', x, q) \in \Delta
\end{aligned}
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Example

$(q_0, ab, q_0), (q_0, ab, q_2) \in \hat{\Delta},\ (q_0, bb, q_0) \in \hat{\Delta}$

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Example

$(q_0, ab, q_0), (q_0, ab, q_2) \in \hat{\Delta},\ (q_0, bb, q_0) \in \hat{\Delta}$

Definition

Let $A = \langle Q, \Sigma, \Delta, q_0, F\rangle$ be a NFA. The language accepted by A is

$$\mathrm{L}(A) = \{w \in \Sigma^* \mid \exists q \in F \text{ s.t. } (q_0, w, q) \in \hat{\Delta}\}.$$

Theorem

Let $L \subseteq \Sigma^*$, then there is a DFA D with $\mathrm{L}(D) = L$ iff there is a NFA N with $\mathrm{L}(N) = L$.

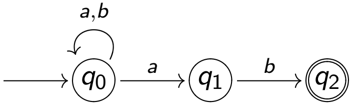
Proof (BB4).

1. Converting D to N: easy.
2. Converting N to D: subset construction.

$\square$

The Subset Construction – Example

Automaton for accepting $L = \{wab \mid w \in \{a, b\}^*\}$:



Conversion to DFA: BB5

In practice: only construct *reachable* states

Theorem

There are $L_n \subseteq \Sigma^*, n \geq 1$ and NFA N_n with $n + 1$ states with $\mathsf{L}(N_n) = L_n$ s.t. all DFA D_n with $\mathsf{L}(D_n) = L_n$ have at least 2^n states.

Proof.

Let $\Sigma = \{a, b\}$ and for $n \geq 1$ define

$$L_n = \{wav \mid w, v \in \Sigma^*, |v| = n - 1\}$$

BB6 $\square$

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Epsilon-Transitions – A Motivating Example

Automaton for accepting decimal representations of integers:

$$\Sigma = \{-, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$L = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}^{+} \\ \cup \{-w \mid w \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}^{+}\}$$

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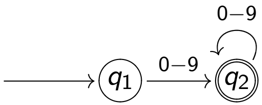

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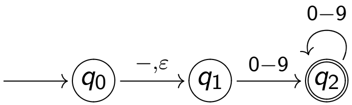

Definition

A *nondeterministic finite automaton with ε-transitions (ε-NFA)* is a tuple $A = \langle Q, \Sigma, \Delta, q_0, F \rangle$ where:

1. Q is a finite set (the *states*).
2. Σ is a finite set (the *input symbols*).
3. $\Delta \subseteq Q \times (\Sigma \cup \{\varepsilon\}) \times Q$ (the *transition relation*).
4. $q_0 \in Q$ (the *starting state*)
5. $F \subseteq Q$ (the *final states*).

ε-NFA – Properties

Definition

Transition relation $\hat{\Delta}$: includes ε-transitions.

Definition

Let $A = \langle Q, \Sigma, \Delta, q_0, F \rangle$ be a ε-NFA. The language accepted by A is

$$\mathrm{L}(A) = \{w \in \Sigma^* \mid \exists q \in F \text{ s.t. } (q_0, w, q) \in \hat{\Delta}\}.$$

Theorem

Let $L \subseteq \Sigma^*$, then there is a DFA D with $\mathrm{L}(D) = L$ iff there is a ε-NFA N with $\mathrm{L}(N) = L$.

Proof.

Modified subset construction (ε-closed subsets). $\square$

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The Class of Regular Languages

Corollary

Let $L \subseteq \Sigma^*$. *The following are equivalent:*

- *There is a DFA* D *with* $\mathrm{L}(D) = L$.
- *There is a NFA* N *with* $\mathrm{L}(N) = L$.
- *There is a* ε*-NFA* N' *with* $\mathrm{L}(N') = L$.

Definition

$L \subseteq \Sigma^*$ is called *regular language* if there is a finite automaton A with $\mathrm{L}(A) = L$.

Closure Properties of Regular Languages

Theorem

If $L_1, L_2 \subseteq \Sigma^*$ are regular, then $L_1 \cup L_2$ is regular.

Proof.

BB7 $\square$

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If $L \subseteq \Sigma^*$ is regular, then $L^{\mathrm{c}} = \Sigma^* \setminus L$ is regular.

Proof.

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Theorem

If $L_1, L_2 \subseteq \Sigma^*$ are regular, then $L_1 \cap L_2$ is regular.

Proof.

$L_1 \cap L_2 = (L_1^{\mathrm{c}} \cup L_2^{\mathrm{c}})^{\mathrm{c}}$. $\square$

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The Pumping Lemma for Regular Languages

Lemma (Pumping Lemma)

Let L be a regular language. Then there is an $n \in \mathbb{N}$ s.t. for every $w \in L$ with $|w| \geq n$ we have $w = v_1 v_2 v_3$ with

1. $v_2 \neq \varepsilon$,
2. $|v_1 v_2| \leq n$, *and*
3. *for all* $k \geq 0$ *also* $v_1 v_2^k v_3 \in L$.

Proof.

BB9 $\square$

Lemma (Pumping Lemma)

Let L be a regular language. Then there is an $n \in \mathbb{N}$ s.t. for every $w \in L$ with $|w| \geq n$ we have $w = v_1 v_2 v_3$ with

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3. *for all* $k \geq 0$ *also* $v_1 v_2^k v_3 \in L$.

Example

$L = \{a^m b^m \mid m \geq 1\}$ is not regular (BB10).

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Context-Free Grammars – A First Example

How can we specify the set of all arithmetical expressions?
E.g. $12, 30 + 21 \cdot 6, (123 + 7) \cdot 15 + 88, \ldots$

$$\begin{aligned}
E &\to N \mid E + E \mid E \cdot E \mid (E) \\
N &\to D \mid DN \\
D &\to 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9
\end{aligned}$$

Context-Free Grammars – Definition

Definition

A *context-free grammar (CFG)* is a tuple $G = \langle N, T, P, S \rangle$ where

1. N is a finite set of symbols (the *nonterminals*),
2. T is a finite set of symbols (the *terminals*),
3. P is a finite set of *production rules* of the form:

$$A \rightarrow w \text{ where } A \in N \text{ and } w \in (N \cup T)^*$$

4. $S \in N$ (the *start symbol*).

$$
\begin{aligned}
G &= \langle NT, T, P, S\rangle \\
NT &= \{N, D, E\} \\
T &= \{+, \cdot, (,), 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \\
P &= \{E \to N, E \to E + E, E \to E \cdot E, E \to (E), \\
&\qquad N \to D, N \to DN, \\
&\qquad D \to 0, D \to 1, D \to 2, D \to 3, D \to 4, \\
&\qquad D \to 5, D \to 6, D \to 7, D \to 8, D \to 9\} \\
S &= E
\end{aligned}
$$

The Language of a Grammar

Let $G = \langle N, T, P, S \rangle$ be a CFG.

Definition

For every $A \to w \in P$ and every $uAv \in (N \cup T)^*$ define

$$uAv \Rightarrow_G uwv.$$

The *derivation relation* $\Rightarrow_G^*$ is the reflexive and transitive closure of $\Rightarrow_G$.

Definition

The language of G is $\mathrm{L}(G) = \{w \in T^* \mid S \Rightarrow_G^* w\}$.

Definition

$L \subseteq \Sigma^*$ is called *context-free* if there is a context-free grammar G with $\mathrm{L}(G) = L$.

Context-Free Grammars – Example Derivation

$$
\begin{aligned}
E &\rightarrow N \mid E + E \mid E \cdot E \mid (E) \\
N &\rightarrow D \mid DN \\
D &\rightarrow 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9
\end{aligned}
$$

Example derivation: BB11

Formalisms

	Automata	Grammars
Regular Languages	DFAs, NFAs	?
Context-Free Languages	?	CFG

Outline

- ✓ Deterministic finite automata
- ✓ Nondeterministic finite automata
- ✓ Automata with ε-transitions
- ✓ The class of regular languages
- ✓ The pumping lemma for regular languages
- ✓ Context-free grammars and languages
- $\Rightarrow$ Right linear grammars
- Pushdown Automata
- The pumping lemma for context-free languages
- Grammars in computer science
- Further topics

Right Linear Grammars

Definition

A grammar $G = \langle N, T, P, S \rangle$ is called *right linear* if all productions are of one of the following forms:

$$A \to xB \text{ where } x \in T, B \in N$$
$$A \to x \text{ where } x \in T$$
$$A \to \varepsilon$$

Theorem

Let $L \subseteq \Sigma^*$. Then L is regular iff L has a right linear grammar.

Proof (BB12).

1. From right linear grammar to ε-NFA.
2. From NFA to right linear grammar.

□

Remark: notion of left linear grammars with analogous result

Right Linear Grammars – Example

Let $L = \{a^m b^n \mid m, n \geq 0\}$ (BB13)

Right linear grammar

Automaton

Formalisms

	Automata	Grammars
Regular Languages	DFA, NFA	LLG, RLG
Context-Free Languages	?	CFG

Regular vs. Context-Free Languages

Theorem

Every regular language is context-free.

Proof.

Every regular language has a right linear grammar. Every right linear grammar is context-free. □

Theorem

There is a context-free language which is not regular.

Proof.

$L = \{a^n b^n \mid n \geq 1\}$ is not regular (pumping lemma), but $S \to ab \mid aSb$ is a context-free grammar for L. □

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Automata for Context-Free Languages ?

- Well-balanced strings of parentheses, e.g.
  $(), (()()), ((()())()) \in W$ but
  $((),)(() \notin W$
- Context-free grammar for W: $E \rightarrow EE \mid (E) \mid \varepsilon$
- W is not regular (by pumping lemma, BB14)
- Generating a language vs. accepting a language
- How to accept a well-balanced string? (BB15)

Automata for Context-Free Languages ?

- Well-balanced strings of parentheses, e.g.
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- W is not regular (by pumping lemma, BB14)
- Generating a language vs. accepting a language
- How to accept a well-balanced string? (BB15)
  $\Longrightarrow$ Need more than constant memory
  $\Longrightarrow$ For context-free languages: automaton with a stack

Pushdown Automata – Definition

Definition

A *pushdown automaton* (PDA) is a tuple $A = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F\rangle$ where:

1. Q is a finite set (the *states*).
2. Σ is a finite set (the *input symbols*).
3. Γ is a finite set (the *stack symbols*).
4. $\Delta \subseteq Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \times Q \times \Gamma^*$ (the *transition relation*). where for each $(q, x, z) \in Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma$ there are only finitely many $(q', w) \in Q \times \Gamma^*$ s.t. $(q, x, z, q', w) \in \Delta$.
5. $q_0 \in Q$ (the *starting state*).
6. $Z_0 \in \Gamma$ (the *starting stack symbol*).
7. $F \subseteq Q$ (the *final states*).

Computation of a PDA

Let $A = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle$ be a PDA.

Definition

A *configuration* of A is a triple $(q, w, v) \in Q \times \Sigma^* \times \Gamma^*$ where

- q is the *current state*,
- w is the *remaining input*, and
- v is the *current stack contents*.

Convention: top of the stack is on the left

Definition

Define the binary *step relation* $\vdash_A$ on configurations of A as:

$$(q, xw, yv) \vdash_A (p, w, uv) \;\text{ if } (q, x, y, p, u) \in \Delta$$

Define $\vdash_A^*$ as reflexive and transitive closure of $\vdash_A$.

The Language of a PDA

Definition

Let $A = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle$ be a PDA, then the *language accepted by A by* **final state** is defined as:

$$\mathrm{L}(A) = \{w \in \Sigma^* \mid (q_0, w, Z_0) \vdash_A^* (q, \varepsilon, v) \text{ for some } q \in F \text{ and any } v\}$$

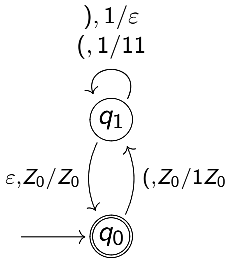
Definition

Let $A = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle$ be a PDA, then the *language accepted by A by* **empty stack** is defined as:

$$\mathrm{N}(A) = \{w \in \Sigma^* \mid (q_0, w, Z_0) \vdash_A^* (q, \varepsilon, \varepsilon) \text{ for any } q\}$$

Example

$\Sigma = \{(,)\}, \Gamma = \{Z_0, 1\}$



Derivation of (()()): BB16

Theorem

Let $L \subseteq \Sigma^*$, then the following are equivalent:

1. *There is a PDA A_N with $\mathrm{N}(A_N) = L$.*
2. *There is a PDA A_F with $\mathrm{L}(A_F) = L$.*

Proof (BB17).

$1 \Rightarrow 2$

$2 \Rightarrow 1$ $\square$

Theorem

A language L has a context-free grammar iff it has a PDA.

Proof (BB18).

1. Given grammar G, construct PDA A.
2. Given PDA A, construct grammar G.

$\square$

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Context-Free Languages	PDA	CFG

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- ▶ Further topics

Lemma (Pumping Lemma)

Let L be a context-free language. Then there is an $n \in \mathbb{N}$ s.t. for every $w \in L$ with $|w| \geq n$ we have $w = v_1 v_2 v_3 v_4 v_5$ with

1. $|v_2 v_3 v_4| \leq n$,
2. $v_2 v_4 \neq \varepsilon$, *and*
3. *for all* $k \geq 0$ *also* $v_1 v_2^k v_3 v_4^k v_5 \in L$.

Proof Sketch (BB19).

$\square$

Lemma (Pumping Lemma)

Let L be a context-free language. Then there is an $n \in \mathbb{N}$ s.t. for every $w \in L$ with $|w| \geq n$ we have $w = v_1 v_2 v_3 v_4 v_5$ with

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3. *for all* $k \geq 0$ *also* $v_1 v_2^k v_3 v_4^k v_5 \in L$.

Example

$L = \{a^m b^m c^m \mid m \geq 1\}$ is not context-free (BB20).

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HTML is a Context-free Language

Source of website

$$
\begin{aligned}
\text{Char} &\to \textbf{a} \mid \textbf{A} \mid \textbf{b} \mid \textbf{B} \mid \cdots \\
\text{String} &\to \varepsilon \mid \text{Char String} \\
\text{Element} &\to \text{Heading} \mid \text{Paragraph} \mid \text{Link} \mid \text{String} \mid <\textbf{br}> \mid \cdots \\
\text{Elements} &\to \varepsilon \mid \text{Element Elements} \\
\text{Heading} &\to <\textbf{h3}> \text{ String } </\textbf{h3}> \mid \cdots \\
\text{Paragraph} &\to <\textbf{p}> \text{ Elements } </\textbf{p}> \\
\text{Link} &\to <\textbf{a href} = \text{“String“}> \text{ String } </\textbf{a}> \\
&\vdots
\end{aligned}
$$

XML

Generalization: XML (Extensible Markup Language)

- DTD (Document Type Definition) is a grammar
- There are DTDs for:
  HTML, office formats, mathematical formulas, address data, vector graphics, cooking recipes, formal proofs, ...
- Very rich infrastructure available

Further Topics

- Context-sensitive languages: $uAv \to uwv$
- Regular expressions
- Decidability/complexity of, e.g., membership, emptyness, ...
- Parser generators
- ...

Introductory Textbook:
J. E. Hopcroft, R. Motwani, J. D. Ullman: *Introduction to Automata Theory, Languages, and Computation*, 2nd edition, 2001, Addison-Wesley.