LARGE CARDINAL CHARACTERIZATIONS VIA COMPACTNESS FOR LIST COLOURINGS

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Abstract. We investigate compactness properties with respect to list colouring, a certain form of graph colouring, with infinitely many colours. We introduce a new hierarchy of compactness cardinals, that also includes some well-established large cardinal notions, and use it to show that various types of large cardinals, including weakly compact, strongly compact, and δ-strongly compact cardinals, can be characterized in terms of compactness for list colouring. We also obtain lower bounds on the size of our newly introduced compactness cardinals.

1. Introduction

Given a set Ω, we say that a graph $\mathcal{G} = \langle G, E\rangle$ with vertex set G and edge relation E is *Ω-colourable* (or *chromatically Ω-colourable*) if there is a function $c\colon G \to \Omega$ such that $c(g) \neq c(h)$ whenever $E(g,h)$ holds. *Compactness for Ω-colouring* is the statement that for every graph $\mathcal{G}$, if every finite subgraph of $\mathcal{G}$ is Ω-colourable, then $\mathcal{G}$ itself is Ω-colourable. Classical results of De Bruijn and Erdős [6] and Läuchli [9] (see for example [5] or [8] for presentations of different arguments) show that compactness for 3-colouring is equivalent to the weak version of the axiom of choice that is the ultrafilter lemma, over the base system ZF. In this paper, we investigate variants of the above when allowing for infinitely many colours, over the base system ZFC. Those variants will turn out to be directly connected to certain types of large cardinals, which we will introduce next. Let δ be an infinite cardinal. $\mathcal{L}_{\delta,\omega}$ denotes the logic extending first-order logic by allowing for conjunctions and disjunctions of arbitrary size less than δ, however with formulas using only finitely many free variables. A theory (in any logic) is said to be *$<\kappa$-satisfiable* if every subset of this theory of size less than κ has a model.

Definition 1. Let $\delta \leq \kappa \leq \lambda$ be uncountable cardinals, with δ regular. Then, we say that κ is *(δ,λ)-compact* if every $<\kappa$-satisfiable $\mathcal{L}_{\delta,\omega}$-theory in a language of size at most λ is satisfiable. We say that κ is *$(\delta,<\lambda)$-compact* if κ is $(\delta,\bar\lambda)$-compact whenever $\bar\lambda < \lambda$, and we also allow for $\lambda = \mathrm{Ord}$ in this case.

Note that the above property of κ becomes stronger as δ or λ increase, and that if κ is (δ,λ)-compact, then every $\kappa^* \in [\kappa,\lambda]$ is (δ,λ)-compact as well. We will take a closer look at (δ,λ)-compactness in Section 6, and in particular show the following:

Theorem 2. *For any cardinals $\delta \leq \kappa$ with δ regular and uncountable, if κ is (δ,κ)-compact, then $\kappa \geq \aleph_\delta$.*

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In their [4], Boney and Unger introduce what they call (δ, λ)-strongly compact cardinals, and we will show in Section 7 that these are closely connected to our notion of (δ, λ)-compact cardinals.

However, the main point of introducing the above hierarchy of compactness properties is to unify the arguments that will follow, for the three well-known instances below – note that:

- κ is *weakly compact* if κ is (κ, κ)-compact.
- κ is *δ-strongly compact* if κ is $(\delta, {<}\mathrm{Ord})$-compact.
- κ is *strongly compact* if κ is $(\kappa, {<}\mathrm{Ord})$-compact.

The fact that compact cardinals yield compactness properties for graph colouring is well-known. The following proposition is verified as an example in [1], and we will obtain a more general result in Proposition 16.

Definition 3. Let $\delta < \kappa$ be cardinals with κ infinite. We say that *κ-compactness for δ-colouring* holds if whenever $\mathcal{G}$ is a graph with the property that every ${<}\kappa$-size subgraph $\mathcal{H}$ of $\mathcal{G}$ has a δ-colouring, also $\mathcal{G}$ itself has a δ-colouring.

Proposition 4 (Bagaria, Magidor)**.** *If a cardinal κ is ω_1-strongly compact, then κ-compactness for ω-colouring holds.*

However, compactness for graph colouring can consistently hold at small cardinals, at least for relatively small graphs. Shelah has shown [11, Theorem 5.2] that starting from a supercompact cardinal, using a model constructed by Ben-David and Magidor [3], it is consistent that $\aleph_\omega$-compactness holds for ω_n-colouring for graphs of size $\aleph_{\omega+1}$ whenever $0 < n < \omega$, while the GCH holds. In a note on his webpage [14], Unger adapts this to show that starting from a supercompact cardinal, it is also consistent that $\aleph_{\omega_1}$-compactness holds for $\omega_{\alpha+1}$-colouring for graphs of size $\aleph_{\omega_1+1}$ whenever $0 < \alpha < \omega_1$, while the GCH holds below $\aleph_{\omega_1}$.

What seems to be very difficult is to obtain any kind of reversals (i.e., implications of compactness properties for graph colouring) beyond Läuchli's above-mentioned classic result. A partial reversal by Shelah [12] is that if $\kappa > \omega_1$ is a regular cardinal with a non-reflecting stationary subset of E^κ_ω, and $\mu^\omega \leq \kappa$ whenever $\mu < \kappa$, then κ-compactness for ω-colouring fails for a graph of size κ.[1]

In this paper, we obtain reversals that allow us to retrieve optimal large cardinal strength from compactness properties with respect to *list colouring*, i.e., colouring with respect to a function (or *list*) L that assigns to each node of a graph a set of allowed colours. These reversals will also contrast the above-mentioned result of Shelah [11, Theorem 5.2], showing that compactness for graph colouring and compactness for list colouring behave fairly differently.

Definition 5. Given a graph $\mathcal{G} = \langle G, E \rangle$ and a function L with domain G, we say that a function c with domain G is an *L-colouring* of G if

- $\forall g \in G \ c(g) \in L(g)$ and
- $\forall g, h \in G \ [E(g, h) \rightarrow c(g) \neq c(h)]$.[2]

[1]Note that if κ is weakly compact or ω_1-strongly compact, well-known standard arguments show that every stationary subset of E^κ_ω reflects.

[2]Note that δ-colouring corresponds to L-colouring when L is constant with value δ.

In order to be able to talk about induced colourings of subgraphs more easily, we will also allow for the domains of L and of c to be (proper) supersets of G in the above. Corresponding compactness properties are given by the following:

Definition 6. Let δ and κ be cardinals, with κ infinite.

- We say that *κ-compactness for δ-list-colouring* holds if whenever $\mathcal{G}$ is a graph and $L\colon G \to \mathcal{P}(\delta)$ is such that every $<\kappa$-size subgraph $\mathcal{H}$ of $\mathcal{G}$ has an L-colouring, also $\mathcal{G}$ itself has an L-colouring.
- We say that *κ-compactness for $<\delta$-list-colouring* holds if whenever $\mathcal{G} = \langle G, E\rangle$ is a graph and $L\colon G \to [\delta]^{<\delta}$ is such that every $<\kappa$-size subgraph $\mathcal{H}$ of $\mathcal{G}$ has an L-colouring, then also $\mathcal{G}$ itself has an L-colouring.

Our main results will establish the following equivalences:

Theorem 7. *Assume that $\delta \leq \kappa \leq \lambda$ are infinite cardinals, with δ regular. Then:*

(1) If $\delta < \kappa$ and $\lambda^\delta = \lambda$, then κ is (δ^+, λ)-compact if and only if κ-compactness for δ-list-colouring for graphs of size at most λ holds.

(2) If $\delta > \omega$ and $\lambda^{<\delta} = \lambda$, then κ is (δ, λ)-compact if and only if κ-compactness for $<\delta$-list-colouring for graphs of size at most λ holds.

In particular, we obtain the following:

- *An uncountable cardinal κ is δ^+-strongly compact if and only if κ-compactness for δ-list-colouring holds.*
- *An uncountable cardinal κ is δ-strongly compact if and only if κ-compactness for $<\delta$-list-colouring holds.*
- *An uncountable cardinal κ is weakly compact if and only if $\kappa = \kappa^{<\kappa}$ and κ-compactness for $<\kappa$-list-colouring holds for graphs of size at most κ.*
- *An uncountable cardinal κ is strongly compact if and only if κ-compactness for $<\kappa$-list-colouring holds.*[3]

Together with Theorem 2, Theorem 7 yields some immediate consequences on compactness for list colouring: for example, under the GCH, κ^+-compactness (and hence also κ-compactness) for ω_n-list-colouring for graphs of size κ^+ fails whenever $n < \omega$ and $\kappa < \aleph_{\omega_{n+1}}$. This contrasts Shelah's result [11, Theorem 5.2].

Let us provide an overview of the contents of our paper. In Section 2, we introduce a family of filter extension properties, and show that they are closely related to the family of compactness properties for cardinals introduced in Definition 1, thus generalizing a classic result of Bell [2]. This relationship will be important for the verification of our main results. In Section 3, we introduce a family of homomorphism-compactness properties[4] generalizing the compactness for graph colouring and list colouring, and then provide the results that will yield the forward implications in Theorem 7. In Section 4, we construct certain families of infinite graphs that we will then make use of in the argument for the reverse implications in Theorem 7, which we provide in Section 5. We also provide a version of Theorem 7 with respect to homomorphism-compactness in that section. In Section 6, we verify

[3] Note that κ-compactness for κ-list-colouring is inconsistent, as is witnessed by the complete graph on κ^+ and the constant function with value κ.

[4] This paper was inspired by results that link the axiomatic strength of homomorphism-compactness for a finite target structure $\mathcal{A}$ over ZF to the computational complexity of the decision problem CSP($\mathcal{A}$); see [8, 10, 13]. Here, the latter problem—the *constraint satisfaction problem* of $\mathcal{A}$—consists of deciding whether a given finite input structure $\mathcal{X}$ homomorphically maps to $\mathcal{A}$.

Theorem 2. In Section 7, we investigate the relationship between (δ,λ)-compact and (δ,λ)-strongly compact cardinals, and provide a further characterization of the former. Finally, we argue that list colouring is a natural generalization of graph colouring (with finitely many colours) to the case of infinitely many colours in Appendix A.

Most of our paper should be readable with only very basic knowledge of set theory. However, Section 7 (which is not overly relevant to the other parts of our paper) is slightly more aimed at a set theoretic audience, and some basic knowledge about the interplay between ultrafilters and elementary embeddings will be helpful in order to follow the arguments in this part of our paper.

2. Filter Extension Properties

Definition 8. Let X be a set and $F \subseteq E \subseteq \mathcal{P}(X)$, with E closed under complements in X, and $X \in E$. Let κ be an infinite cardinal. We say that F is a *$<\kappa$-complete filter (on E)*[5] if

- $X \in F$, $\emptyset \notin F$,
- $\forall Y, Z \in E\ [Y \supseteq Z \in F \to Y \in F]$, and
- whenever $(Y_i)_{i<\lambda}$ is a family of elements of F with $\lambda < \kappa$, we have
$$\bigcap_{i<\lambda} Y_i \neq \emptyset.^6$$

We call F a *filter* if it is a $<\omega$-complete filter. We say that a filter F *measures* E if either $Y \in F$ or $X \setminus Y \in F$ whenever $Y \in E$.

Definition 9. Let $\delta \leq \kappa \leq \lambda$ be infinite cardinals, with δ regular.

- We say that κ has the *(δ,λ)-filter extension property* if whenever X is a set, $E \subseteq \mathcal{P}(X)$ is of size at most λ and closed under complements (in X) with $X \in E$, and $F \subseteq E$ is a $<\kappa$-complete filter, then F can be extended to a $<\delta$-complete filter that measures E.
- The *$(\delta,<\lambda)$-filter extension property* is the above property for E of size less than λ.

Our next proposition extends a result of Bell [2, Main Theorem]: we extend his result to include singular cardinals κ,[7] and also provide a stratification of his result with respect to the parameter λ that provides either the size of the languages for which we have certain forms of compactness, or the size of the families of sets that we would like to be able to measure. This requires some nontrivial extra care, in particular in the proof of the second part of the following proposition:

Proposition 10. *Let $\delta \leq \kappa \leq \lambda$ be uncountable cardinals, with δ regular.*

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[5] Note that $X = \bigcup E$.

[6] More commonly, the final requirement on F would be that $\bigcap_{i<\lambda} Y_i \in F$. However, for the case of singular κ, this seems to be an overly narrow restriction for many applications. Another common variant would be to require that $\bigcap_{i<\lambda} Y_i$ has the same cardinality as X. The definition that we chose is the most useful one for our present purposes.

[7] This may be folklore knowledge, but we could not find a written account for the case of singular cardinals, and moreover the set theoretic literature seems to be unspecific about what notion of $<\kappa$-completeness of a filter should be used in this case, while seemingly, only the less standard notion that we introduced above seems to make the argument work in the case of singular cardinals κ.

- *If κ is (δ,λ)-compact, then κ has the (δ,λ)-filter extension property.*
- *If κ has the $(\delta,\lambda^{<\delta})$-filter extension property, then κ is (δ,λ)-compact.*

Proof. First, assume that κ is (δ,λ)-compact and let $F\subseteq E\subseteq\mathcal{P}(X)$, with F a $<\kappa$-complete filter, and E of size at most λ, closed under complements and with $X\in E$. We ought to extend F to a $<\delta$-complete filter U that measures E. Let Σ be the $\mathcal{L}_{\delta,\omega}$-theory of the structure

$$\mathfrak{A}=(X,(P)_{P\in E})$$

with a unary relation P corresponding to every $P\in E$, together with the set of sentences $\{P(c)\mid P\in F\}$, where c is a new constant symbol. The language that we use here has size at most λ. As F is $<\kappa$-complete, Σ is $<\kappa$-satisfiable, as we may witness any collection $\Sigma'\subseteq\Sigma$ of $<\kappa$-many sentences by picking c to be an element of the (nonempty) intersection of the $<\kappa$-many elements P of F for which $P(c)$ is in Σ'. This means that by our assumption, Σ is satisfiable, so let

$$\mathfrak{B}=(Y,(Q_P)_{P\in E},d)\models\Sigma,$$

with d the interpretation of c in $\mathfrak{B}$. Let $U=\{P\in E\mid\mathfrak{B}\models Q_P(d)\}$. If $P\in U$ and $R\supseteq P$ with $R\in E$, then $Q_P(d)$ implies $Q_R(d)$, and hence $R\in U$. If $P\in E$ and $R=X\setminus P$, then $Q_R=Y\setminus Q_P$, hence either $Q_P(d)$ or $Q_R(d)$ holds, so either P or R is in U. If $P\in F$, then $Q_P(d)$ holds, so $P\in U$. Finally, whenever $(P_i)_{i<\bar\delta}$ is a family of elements of U for some $\bar\delta<\delta$, $\mathfrak{B}$ satisfies the $\mathcal{L}_{\delta,\omega}$-formula $\exists x\bigwedge_{i<\bar\delta}Q_{P_i}(x)$ as it is witnessed by d, hence $\bigcap_{i<\bar\delta}P_i$ is non-empty, yielding that U is a $<\delta$-complete filter that measures E.

For the second statement, assume that κ has the $(\delta,\lambda^{<\delta})$-filter extension property. Let Σ be a $<\kappa$-satisfiable $\mathcal{L}_{\delta,\omega}$-theory, in a language $\mathcal{L}$ of size at most λ. By the regularity of δ, it follows that Σ has size at most $\lambda^{<\delta}$. We want to show that Σ is satisfiable. For every $\Delta\subseteq\Sigma$ of size less than κ, let $\mathfrak{A}_\Delta\models\Delta$. Let $\mathfrak{A}=\prod_{\Delta\in[\Sigma]^{<\kappa}}\mathfrak{A}_\Delta$, and let A denote the domain of $\mathfrak{A}$. Using a Löwenheim-Skolem type argument, we may construct a substructure $\mathfrak{B}$ of $\mathfrak{A}$ of size at most $\lambda^{<\delta}$, with domain B, and with the additional property that whenever $\varphi(x_1,\ldots,x_{n+1})\in\mathcal{L}_{\delta,\omega}$ and $a=(a_1,\ldots,a_n)\in B^n$ for some $n<\omega$, there is $b\in B$ such that for every $\Delta\in[\Sigma]^{<\kappa}$, if $\mathfrak{A}_\Delta\models\exists x\,\varphi(x,a(\Delta))$, then $\mathfrak{A}_\Delta\models\varphi(b(\Delta),a(\Delta))$, where $a(\Delta):=(a_1(\Delta),\ldots,a_n(\Delta))$. This is possible for the number of $\mathcal{L}_{\delta,\omega}$-formulas is at most $\lambda^{<\delta}$. For $\varphi\in\Sigma$, let $\hat\varphi=\{\Delta\in[\Sigma]^{<\kappa}\mid\varphi\in\Delta\}$. Obviously, the family $\{\hat\varphi\mid\varphi\in\Sigma\}$ is the base of a $<\kappa$-complete filter we shall call F. For every $\mathcal{L}_{\delta,\omega}$-formula $\varphi(x_1,\ldots,x_n)$ and any $a\in A^n$, let

$$J_{\varphi,a}=\{\Delta\in[\Sigma]^{<\kappa}\mid\mathfrak{A}_\Delta\models\varphi(a(\Delta))\}.$$

Using our assumption, let $U\supseteq F$ be[8] a $<\delta$-complete filter on $[\Sigma]^{<\kappa}$ that measures all sets $J_{\varphi,a}$ for $\mathcal{L}_{\delta,\omega}$-formulas $\varphi(x_1,\ldots,x_n)$ and $a\in B^n$, of which there are $\lambda^{<\delta}$ many. We define, for $a,b\in B$, that

$$a\sim b\iff\{\Delta\in[\Sigma]^{<\kappa}\mid a(\Delta)=b(\Delta)\}\in U,$$

and note that a straightforward argument using that U is a filter that measures any relevant set (for all of them correspond to certain $J_{\varphi,a}$ for $a\in B^n$) shows that for all $n<\omega$, and all $a,b\in B^n$ with $a_i\sim b_i$ whenever $1\le i\le n$, it holds that $f(a)\sim f(b)$ whenever f is an n-ary function of $\mathfrak{B}$.

[8]Note that $U\supseteq F$ just means that U is a *fine* filter (see also Section 7).

For any $a = (a_1, \dots, a_n) \in B^n$, let $[a]_\sim = ([a_1]_\sim, \dots, [a_n]_\sim)$. We let $\mathfrak{D} = \mathfrak{B}/_\sim$ be the quotient structure of $\mathfrak{B}$ by $\sim$, with domain $D = B/_\sim$, and the following operations are well-defined by the above: If f is an n-ary function symbol and $a \in B^n$, $f^{\mathfrak{D}}([a]_\sim) = [f^{\mathfrak{A}}(a)]_\sim$. If R is an n-ary relation symbol, $R^{\mathfrak{D}}([a]_\sim)$ if and only if $\{\Delta \in [\Sigma]^{<\kappa} \mid \mathfrak{A}_\Delta \models R(a(\Delta))\} \in U$.

By induction on formula complexity, using existential quantification as the only quantifier step,[9] we show that for every $\mathcal{L}_{\delta,\omega}$-formula $\varphi(x_1, \dots, x_n)$ and $a \in B^n$,

$$\mathfrak{D} \models \varphi([a]_\sim) \iff J_{\varphi,a} \in U :$$

This is true for atomic formulas by definition. For the case of conjunctions of size less than δ, we use the $<\delta$-completeness of U in a straightforward way. The case of negation is straightforward. Let us argue for the step of existential quantification in detail. Assume first that $\mathfrak{D} \models \exists y \varphi(y, [a]_\sim)$. This means, by our inductive hypothesis for φ, that there is $b \in B$ such that the set

$$\{\Delta \in [\Sigma]^{<\kappa} \mid \mathfrak{A}_\Delta \models \varphi(b(\Delta), a(\Delta))\} \in U.$$

But this set is obviously contained in $J_{\exists y\varphi,a}$, which is thus also an element of U, as desired. If on the other hand, $J_{\exists y\varphi,a} \in U$, by the properties of $\mathfrak{B}$, we may pick $b \in B$ such that $\mathfrak{A}_\Delta \models \varphi(b(\Delta), a(\Delta))$ whenever such $b(\Delta)$ exists, and we know this is the case on a set in U. This means that $J_{\varphi,b^\frown a} \in U$. By our inductive hypothesis for φ, this implies that $\mathfrak{D} \models \exists y \varphi(y, [a]_\sim)$, as desired.

We now claim that $\mathfrak{D} \models \Sigma$. Therefore, pick $\sigma \in \Sigma$. Note that for every $\Delta \in [\Sigma]^{<\kappa}$, $\sigma \in \Delta$ implies $\mathfrak{A}_\Delta \models \sigma$, therefore $J_{\sigma,\emptyset} \supseteq \hat{\sigma} \in U$. Thus, by the equivalence that we have verified above, this means that $\mathfrak{D} \models \sigma$, as desired. $\square$

The above obviously yields an equivalence in case $\lambda^{<\delta} = \lambda$. The following proposition provides the forward direction of Theorem 7, (2).

Proposition 11. *Whenever $\delta \le \kappa \le \lambda$ are infinite cardinals, and κ has the (δ, λ)-filter extension property (or the $(\delta, <\lambda)$-filter extension property), then κ-compactness for $<\delta$-list-colouring holds for graphs of size at most λ (or of size $<\lambda$ respectively). In particular, by Proposition 10, this is the case if κ is (δ, λ)-compact.*

Proof. Assume that κ has the (δ, λ)-filter extension property, let $\mathcal{G} = \langle G, E\rangle$ be a graph of size at most λ, and let $L \colon G \to [\delta]^{<\delta}$. Assume that every $<\kappa$-size subgraph of $\mathcal{G}$ has an L-colouring. On the set δ^G, we define F' to be the collection of sets

$$A_H = \{f \in \delta^G \colon f \restriction H \text{ is an } L\text{-colouring of } \mathcal{H} = \langle H, E \restriction H^2\rangle\},$$

where H ranges over all $\le$2-element subsets of G. Let I consist of all sets in F' together with $B^g_i = \{f \in \delta^G \colon f(g) = i\}$ for $g \in G$ and $i \in L(g)$, as well as the complements of these sets. Let F be the closure under supersets of F' within I. Note that if $(H_i)_{i<\bar{\kappa}}$ for some $\bar{\kappa} < \kappa$ is a family of $\le$ 2-element subsets of G, with $H := \bigcup_{i<\bar{\kappa}} H_i$, then $\bigcap_{i<\bar{\kappa}} A_{H_i} \supseteq A_H \neq \emptyset$ by our assumption on $\mathcal{G}$. This means that F is a $<\kappa$-complete filter. By our assumption on κ, and since I is of size at most λ, we obtain a $<\delta$-complete filter $U \supseteq F$ which measures I. For every $g \in G$, there is a unique $i \in L(g)$ so that $B^g_i \in U$ (this uses the $<\delta$-completeness of U, that $A_{\{g\}} \in F'$, and that $|L(g)| < \delta$), and we denote this unique element i by i_g. Observe that by our choice of F', the assignment $h \colon g \mapsto i_g$ is an L-colouring of $\mathcal{G}$.

[9]That is, we replace subformulas of the form $\forall x \varphi$ by $\neg\exists x \neg\varphi$.

The statement regarding the $(\delta, {<}\lambda)$-filter extension property is verified by exactly the same argument. $\square$

3. Homomorphism-Compactness

Definition 12. *Homomorphism-compactness* for a relational structure $\mathcal{B}$ is the statement that, given a structure $\mathcal{A}$ in the same language, if there is a homomorphism $h\colon \mathcal{X} \to \mathcal{B}$ for every finite substructure $\mathcal{X}$ of $\mathcal{A}$, then there exists a homomorphism from $\mathcal{A}$ to $\mathcal{B}$.

Note that compactness for 3-colouring is the same as homomorphism-compactness for the complete graph K_3. It is well-known and not difficult to see that the classical results of De Bruijn, Erdős and Läuchli mentioned in the introduction of our paper can be adapted to yield the following (this is implicit for example in [8]):

Observation 13. *Over* ZF*, the ultrafilter lemma is equivalent to the statement that whenever $\mathcal{A}$ is a finite relational structure, homomorphism-compactness for $\mathcal{A}$ holds.*

We generalize Definition 12 as follows:

Definition 14. Let $\kappa \leq \lambda$ be infinite cardinals.

- (κ, λ)-*homomorphism-compactness* for a relational structure $\mathcal{B}$ is the statement that, given a structure $\mathcal{A}$ of size at most λ, in the same language as $\mathcal{B}$, if there is a homomorphism $h\colon \mathcal{X} \to \mathcal{B}$ for every ${<}\kappa$-size substructure $\mathcal{X}$ of $\mathcal{A}$, then there exists a homomorphism from $\mathcal{A}$ to $\mathcal{B}$.
- $(\kappa, {<}\lambda)$-*homomorphism-compactness* is the same statement for structures $\mathcal{A}$ of size less than λ, and we also allow for $\lambda = \mathrm{Ord}$ in this case.
- κ-*homomorphism-compactness* is $(\kappa, {<}\mathrm{Ord})$-homomorphism-compactness.

Note that κ-compactness for δ-list-colouring is a particular instance of the above:

Observation 15. *There is a relational structure $\mathcal{B}$ with domain δ such that κ-compactness for δ-list-colouring is equivalent to κ-homomorphism-compactness for $\mathcal{B}$.*

Proof. Let $\mathcal{B}$ be the complete graph K_δ with domain δ and with unary predicates c_x for every $x \subseteq \delta$, such that $c_x^{\mathcal{B}}(\gamma)$ holds if and only if $\gamma \in x$. If we consider a graph $\mathcal{G}$ and $L\colon G \to \mathcal{P}(\delta)$, we expand $\mathcal{G}$ to a structure $\mathcal{A}$ in the same language as $\mathcal{B}$, by letting $c_x^{\mathcal{A}}(g)$ hold if and only if $L(g) = x$. Now, a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ corresponds exactly to an L-colouring of $\mathcal{G}$. On the other hand, any structure $\mathcal{A} = \langle G, E, (c_x)_{x \subseteq \delta} \rangle$, where $\mathcal{G} = \langle G, E \rangle$ may be assumed to be a graph and the c_x are unary predicates, naturally yields a map $L\colon G \to \mathcal{P}(\delta)$, letting $L(g) = \bigcap \{x \mid c_x^{\mathcal{A}}(g)\}$, for every $g \in G$. Then, an L-colouring of G corresponds exactly to a homomorphism from $\mathcal{A}$ to $\mathcal{B}$. $\square$

We can now provide our promised generalization of Proposition 4. The argument is similar to the proof of Proposition 11. This proposition will also provide the forward directions in Theorem 7, (1) and Theorem 21.

Proposition 16. *If $\delta \leq \kappa \leq \lambda$ are infinite cardinals and κ has the (δ, λ)-filter extension property (or the $(\delta, {<}\lambda)$-filter extension property), then whenever $\mathcal{B}$ is a relational structure with domain of size less than δ, (κ, λ)-homomorphism-compactness (or $(\kappa, {<}\lambda)$-homomorphism-compactness) holds for $\mathcal{B}$. In particular, by Proposition 10, this is the case if κ is (δ, λ)-compact.*

Proof. Assume that κ has the (δ, λ)-filter extension property, and, without loss of generality, let $\mathcal{B}$ be a relational structure with domain $\bar{\delta} < \delta$. Let $\mathcal{A}$ be a structure with domain A of size at most λ, and in the same language as $\mathcal{B}$. Assume that for every $<\kappa$-size substructure $\mathcal{X}$ of $\mathcal{A}$, there is a homomorphism from $\mathcal{X}$ to $\mathcal{B}$. On the set $\bar{\delta}^A$, we define F' to be the collection of sets

$$A_H = \{f \in \bar{\delta}^A : f \restriction H \text{ is a homomorphism to } \mathcal{B}\},$$

where H ranges over all finite subsets of A. Let E consist of all sets in F' together with $B_i^x = \{f \in \bar{\delta}^A : f(x) = i\}$ for $x \in A$ and $i < \bar{\delta}$, as well as the complements of these sets. Note that E has size at most λ. Let F be the closure under supersets of F' within E. Note that if $(H_i)_{i<\bar{\kappa}}$ for some $\bar{\kappa} < \kappa$ is a family of finite subsets of A, with $H := \bigcup_{i<\bar{\kappa}} H_i$, then $\bigcap_{i<\bar{\kappa}} A_{H_i} \supseteq A_H \neq \emptyset$ by our assumption on $\mathcal{A}$, since $|H| < \kappa$. This means that F is a $<\kappa$-complete filter. By our assumption on κ, and since E is of size at most λ, we obtain a $<\delta$-complete filter $U \supseteq F$ which measures E. For every $x \in A$, there is a unique $i < \bar{\delta}$ so that $B_i^x \in U$ (this uses the $<\delta$-completeness of U), and we denote this unique element i by i_x. Observe that the assignment $h \colon x \mapsto i_x$ is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$: if some relation R holds for the finite tuple $a = (a_1, \ldots, a_n)$ in $\mathcal{A}$, then

$$\bigcap_{1 \leq i \leq n} B_{h(a_i)}^{a_i} \cap A_a \neq \emptyset,$$

which readily implies $R^{\mathcal{B}}(h(\vec{a}))$.

The statement regarding the $(\delta, <\lambda)$-filter extension property is verified by exactly the same argument. $\square$

4. Some Graph Constructions

Before we can prove our main results, we need to construct a couple of graphs (and lists of allowed colours) that we will then make use of in our arguments. Impatient readers may skip to Section 5, and refer back to the present section when necessary. The graph $\mathcal{R}_3$ below already appears in [5].

Lemma 17. *For every $2 \leq \alpha < \omega$ there exists a finite graph $\mathcal{R}_\alpha$ with distinguished vertices $x_0, \ldots, x_{\alpha-1}$ and t such that*

- *(i) if c is a 3-colouring of $\mathcal{R}_\alpha$ such that all of $x_0, \ldots, x_{\alpha-1}$ receive the same colour, then also t has to receive that colour;*
- *(ii) for any $a \in \{0,1\}^\alpha$ and $b \in \{a(i) \mid i < \alpha\}$ there is a 3-colouring c of $\mathcal{R}_\alpha$ such that $c(x_i) = a(i)$ for $i < \alpha$ and $c(t) = b$.*

Proof. We construct $\mathcal{R}_\alpha = \langle R_\alpha, E\rangle$ by induction over α. We let $\mathcal{R}_2$ be the graph with vertex set $\{x_0, x_1, a, b, t\}$ such that the restriction to $\{a, b, t\}$ is complete and with edges between x_0 and a and between x_1 and b, see Figure 1. One easily verifies (i) and (ii).

For $\alpha \geq 2$ we construct $\mathcal{R}_{\alpha+1}$ as follows. We take one copy of $\mathcal{R}_\alpha$, whose distinguished vertices we call $x_0, \ldots, x_{\alpha-2}, x'_{\alpha-1}$ and t, and one copy of $\mathcal{R}_2$, whose distinguished vertices we call $x_{\alpha-1}, x_\alpha$ and t', and let $\mathcal{R}_{\alpha+1}$ be the quotient graph obtained by identifying $x'_{\alpha-1}$ with t'. The construction is depicted in Figure 2. Using that $\mathcal{R}_\alpha$ and $\mathcal{R}_2$ satisfy properties (i) and (ii) inductively, one easily concludes that $\mathcal{R}_{\alpha+1}$ does so as well. $\square$

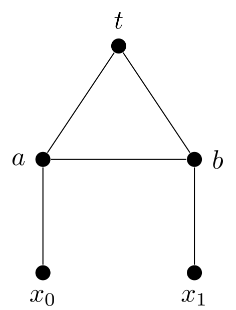

FIGURE 1. The graph $\mathcal{R}_2$.

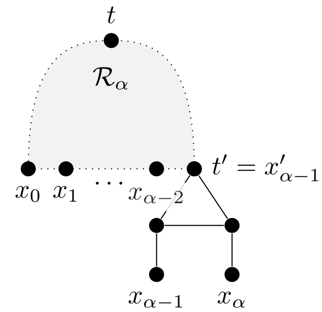

FIGURE 2. The graph $\mathcal{R}_{\alpha+1}$.

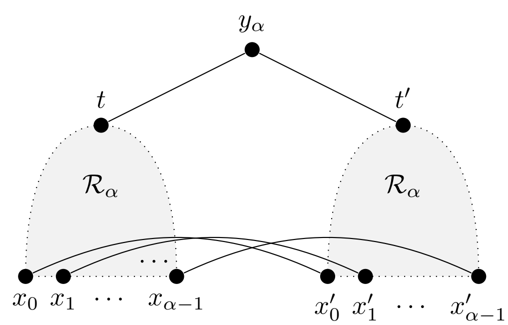

FIGURE 3. The graph $\mathcal{S}_\alpha$ for finite $\alpha \geq 2$.

Lemma 18. *Let $\alpha \geq 2$ be an ordinal. There exists a graph $\mathcal{S}_\alpha = \langle S_\alpha, E\rangle$, which is of size $|\alpha|$ for infinite α, and finite otherwise, with distinguished vertices y_α and x_i for $i < \alpha$, and there exists a function $L_\alpha\colon S_\alpha \to \mathcal{P}(\alpha+1)$ such that*

(i) if c is an L_α-colouring of $\mathcal{S}_\alpha$ such that $\{x_i \mid i < \alpha\}$ is monochromatic, then $c(y_\alpha) = \alpha$ and furthermore, there are L_α-colourings extending both monochromatic colourings of $\{x_i \mid i < \alpha\}$;

(ii) for any non-constant $a \in \{0,1\}^\alpha$ and any $b \in \{0,1,\alpha\}$, there is an L_α-colouring c of $\mathcal{S}_\alpha$ such that $c(x_i) = a(i)$ for $i < \alpha$ and $c(y_\alpha) = b$.

Proof. We construct $\mathcal{S}_\alpha$ and L_α by induction over all ordinals $\alpha \geq 2$. For finite $\alpha \geq 2$, we construct $\mathcal{S}_\alpha$ as follows. We take two disjoint copies of $\mathcal{R}_\alpha$, as obtained by Lemma 17, and whose distinguished vertices we denote by x_i for $i < \alpha$ and t, as well as x'_i for $i < \alpha$ and t' respectively. For every $i < \alpha$, we add an edge between the vertices x_i and x'_i, and furthermore, we connect both the vertices t and t' with a newly added vertex y_α. We define $L_\alpha(x_i) = L_\alpha(x'_i) = \{0,1\}$ for $i < \alpha$ and let L_α have value $\{0,1,\alpha\}$ on the remaining vertices of $\mathcal{S}_\alpha$; see Figure 3 for a graphical representation. Using the properties of $\mathcal{R}_\alpha$, it is easy to verify that $\mathcal{S}_\alpha$ satisfies (i) and (ii).

Next, we deal with the case when α is not a cardinal. We have already defined $\mathcal{S}_{|\alpha|}$, with distinguished vertices $y_{|\alpha|}$ and x_i for $i < |\alpha|$, as well as the function $L_{|\alpha|}$. Pick a bijection f_α from $|\alpha|$ to α. We let $\mathcal{S}_\alpha$ be the same graph as $\mathcal{S}_{|\alpha|}$. We let y_α be the distinguished vertex $y_{|\alpha|}$ of $\mathcal{S}_{|\alpha|}$, and we replace the distinguished vertex x_i

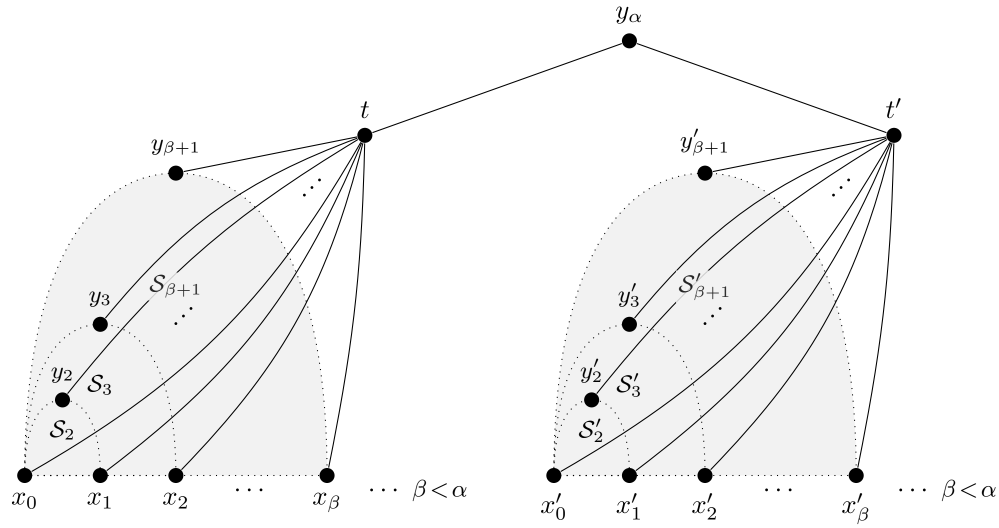

FIGURE 4. The graph $\mathcal{S}_\alpha$ for an infinite cardinal α. For better readability the edges between x_i and x'_i are omitted.

of $\mathcal{S}_{|\alpha|}$ by $x_{f_\alpha(i)}$ for every $i < |\alpha|$. For $v \in S_\alpha$, we let $L_\alpha(v) \subseteq \alpha + 1$ be the same set as $L_{|\alpha|}(v)$, except that we replace $|\alpha|$ by α whenever $|\alpha| \in L_{|\alpha|}(v)$. The desired properties of $\mathcal{S}_\alpha$ and L_α follow immediately from the properties of $\mathcal{S}_{|\alpha|}$ and $L_{|\alpha|}$.

Finally, let α be an infinite cardinal. For every $2 \leq \beta < \alpha$, we have a graph $\mathcal{S}_\beta$, whose distinguished vertices we shall call y^β_β and x^β_i (rather than just y_β and x_i) for $i < \beta$, and a function L_β, meeting conditions (i) and (ii). We construct $\mathcal{S}_\alpha$ as follows. Let $\mathcal{G}$ be a graph with vertices t, t', y_α, and x_i, x'_i for $i < \alpha$, as well as y_i, y'_i for $2 \leq i < \alpha$, and edges from y_α to t and to t', as well as edges from x_i to x'_i, from t to x_i and y_i and from t' to both x'_i and y'_i for all possible $i < \alpha$. For every $2 \leq \beta < \alpha$, we glue two copies of the graph $\mathcal{S}_\beta$ into $\mathcal{G}$ as follows. For the first copy, to which we will simply refer as $\mathcal{S}_\beta$ in the following, we identify the vertex y_β of $\mathcal{G}$ with the vertex y^β_β of $\mathcal{S}_\beta$, and for all $i < \beta$, the vertex x_i of $\mathcal{G}$ with the vertex x^β_i of $\mathcal{S}_\beta$. Similarly, for the second copy, to which we will refer as $\mathcal{S}'_\beta$ in the following, we identify the vertex y'_β of $\mathcal{G}$ with the vertex y^β_β of $\mathcal{S}'_\beta$, and for all $i < \beta$, the vertex x'_i of $\mathcal{G}$ with the vertex x^β_i of $\mathcal{S}'_\beta$. We thus obtain the graph $\mathcal{S}_\alpha$ visualized in Figure 4.

The function $L_\alpha \colon S_\alpha \to \mathcal{P}(\alpha + 1)$ is defined as follows. If $x \in S_\alpha$ is an element of a glued-in copy of $\mathcal{S}_\beta$ for some $2 \leq \beta < \alpha$, we let $L_\alpha(x) = L_\beta(x)$. Note that this assignment is well-defined, as x being contained in multiple glued-in copies of $\mathcal{S}_\beta$ implies that it equals x_i or x'_i for some $i < \alpha$, in which case $L_\beta(x) = \{0, 1\}$ for all $\beta > i$. Furthermore, we let $L_\alpha(t) = L_\alpha(t') = \alpha$ and $L_\alpha(y_\alpha) = \{0, 1, \alpha\}$.

$\mathcal{S}_\alpha$ and L_α satisfy (i) and (ii): To see that (i) holds, suppose that c is an L_α-colouring of $\mathcal{S}_\alpha$ such that $\{x_i \mid i < \alpha\}$ is monochromatic, say $c(x_i) = 0$ for all $i < \alpha$. Since for every $2 \leq \beta < \alpha$, $L_\alpha \restriction \mathcal{S}_\beta = L_\beta$, we conclude that $c(y_\beta) = \beta$ by property (i) of $\mathcal{S}_\beta$ and L_β. The edges from x_i to x'_i for all $i < \alpha$ imply that $c(x'_i) = 1$ for all $i < \alpha$, so the analogous argument shows that $c(y'_\beta) = \beta$ for all $2 \leq \beta < \alpha$. Since t has edges to x_0 as well as to y_β for all $\beta < \alpha$, we have $c(t) = 1$

and similarly $c(t') = 0$. Thus, $c(y_\alpha) = \alpha$, as desired. The above also yields that both monochromatic colourings of $\{x_i \mid i < \alpha\}$ extend to L_α-colourings of $\mathcal{S}_\alpha$.

To see that (ii) holds, let $a \in \{0,1\}^\alpha$ be non-constant, and let $b \in \{0,1,\alpha\}$. Let $a' \in \{0,1\}^\alpha$ be such that $a(i) = 0 \iff a'(i) = 1$, for all $i < \alpha$. Then, since α is a limit ordinal and a is non-constant, there is some $\beta < \alpha$ such that $a \restriction \beta$ (and hence also $a' \restriction \beta$) is non-constant. Using our inductive assumption, we find, for every $2 \leq \gamma < \alpha$, L_γ-colourings c_γ of $\mathcal{S}_\gamma$ and c'_γ of $\mathcal{S}'_\gamma$ such that for all $i < \gamma < \alpha$, it holds that $c_\gamma(x_i) = a(i)$, $c'_\gamma(x'_i) = a'(i)$ and $c_\gamma(y_\gamma), c'_\gamma(y'_\gamma) < \beta$. We let c be the extension of $\bigcup_{\gamma<\alpha}(c_\gamma \cup c'_\gamma)$ to domain S_α with $c(t) = c(t') = \beta$ and $c(y_\alpha) = b$. This is an L_α-colouring of $\mathcal{S}_\alpha$ with the desired properties, concluding our induction. □

Lemma 19. *Let $\delta \geq 3$ be a cardinal. There exists a graph $\mathcal{T}_\delta = \langle T_\delta, E\rangle$, which is of size δ for infinite δ, and finite otherwise, with distinguished vertices y and x_i for $i < \delta$, and a map $L' \colon T_\delta \to \mathcal{P}(\delta)$ with $L'(x_i) = L'(y) = \{0,1\}$ for all $i < \delta$ such that the following holds.*

(i) If c is an L'-colouring of $\mathcal{T}_\delta$ such that $c(x_i) = c(x_j)$ for all $i,j < \delta$, then $c(y) = c(x_0)$.

(ii) For every $a \in \{0,1\}^\delta$ and $b \in \{a(i) \mid i < \delta\}$ there is an L'-colouring $c \colon T_\delta \to \delta$ such that $c(y) = b$ and $c(x_i) = a(i)$ for all $i < \delta$.

Proof. For finite $\delta \geq 3$ this follows immediately from Lemma 17, so we assume that δ is infinite. We let $\mathcal{G}$ be the graph with vertices y, t, x_α for $\alpha < \delta$, and y_α for $2 \leq \alpha < \delta$, with edges between y and t, between x_0 and t, and between y_α and t for $2 \leq \alpha < \delta$. Similar to the argument of the previous lemma, for every $2 \leq \alpha < \delta$, we glue a copy of $\mathcal{S}_\alpha$, whose distinguished vertices we shall call Y_α and X_i^α for $i < \alpha$, into $\mathcal{G}$, by identifying the vertex Y_α with y_α and X_i^α with x_i for $i < \alpha$. We obtain the graph $\mathcal{T}_\delta = \langle T_\delta, E\rangle$ depicted in Figure 5. We define $L' \colon T_\delta \to \mathcal{P}(\delta)$ as follows. We require that on every copy of $\mathcal{S}_\alpha$ for $2 \leq \alpha < \delta$, L' restricts to the function L_α from Lemma 18. Furthermore, we define $L'(t) = \delta$ and $L'(y) = \{0,1\}$. Note that this specifies a well-defined map $T_\delta \to \mathcal{P}(\delta)$, and one easily verifies properties (i) and (ii). □

5. Retrieving large cardinals

In this section, we provide reversals to Propositions 11 and 16, which, together with Proposition 10, establish Theorem 7. Some of the basic ideas for this argument stem from the presentation of Läuchli's theorem in [5].

Theorem 20. *Assume that $\delta \leq \kappa \leq \lambda$ are infinite cardinals, with δ regular. Then:*

- *If $\delta < \kappa$, then κ-compactness for δ-list-colouring for graphs of size at most λ^δ implies that κ has the (δ^+, λ)-filter extension property.*
- *If $\delta \leq \kappa$, then κ-compactness for $<\delta$-list-colouring for graphs of size at most $\lambda^{<\delta}$ implies that κ has the (δ, λ)-filter extension property.*

Proof. Assume first that κ-compactness for δ-list-colouring holds for graphs of size at most λ^δ. We want to show that κ has the (δ^+, λ)-filter extension property. Given a set J, and $I \subseteq \mathcal{P}(J)$ of size at most λ, we may enlarge I to size at most λ^δ, so that I is closed under δ-intersections and complements, and $J \in I$. Let $F \subseteq I$ be a $<\kappa$-complete filter. We need to extend F to a $<\delta^+$-complete filter $F' \supseteq F$ that measures I. We construct a graph $\mathcal{G} = \langle G, E\rangle$ with $G \supseteq I$, and a map $L \colon G \to \mathcal{P}(\delta)$,

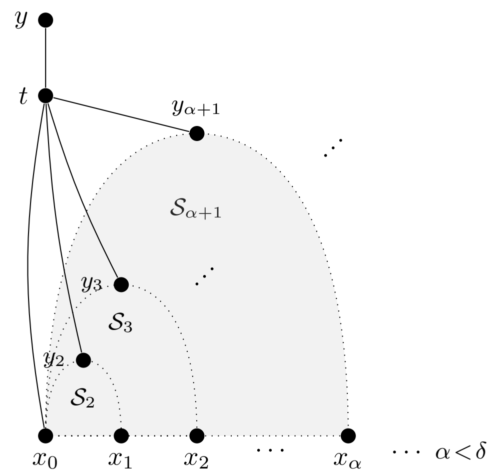

FIGURE 5. The graph $\mathcal{T}_\delta$ for a cardinal δ.

with the property that every L-colouring c of $\mathcal{G}$ corresponds to a $<\delta^+$-complete filter that extends F and measures I, and which also has the property that L-colourings of subgraphs $\mathcal{H} = \langle H, E\cap H^2\rangle$ of $\mathcal{G}$ correspond to $<\delta^+$-complete filters extending F and measuring all elements of I in H. Let $\mathcal{G}' = \langle I, E'\rangle$ be the graph with edges $E' = \{(X, J\setminus X) \mid X \in I\}$. We construct $\mathcal{G} = \langle G, E\rangle$ with $G \supseteq I$ and $E \supseteq E'$ from $\mathcal{G}'$ by gluing in a copy of $\mathcal{T}_\delta$ for every family $(X_i)_{i<\delta}$ of elements of I, identifying the vertex x_i of $\mathcal{T}_\delta$ with the vertex X_i of $\mathcal{G}'$ for all $i < \delta$, and identifying y with $\bigcap_{i<\delta} X_i \in I$. Note that G has size at most λ^δ. We define the function $L\colon G \to \mathcal{P}(\delta)$ as follows. For $X \in F$, we set $L(X) = \{1\}$, for $X \in I \setminus F$ we set $L(X) = \{0,1\}$ and for $g \in G \setminus I$ we set $L(g) = L'(g)$, where L' refers to the map $T_\delta \to \mathcal{P}(\delta)$ defined for the respective copy of $\mathcal{T}_\delta$ from Lemma 19 that g belongs to. Observe that for every L-colouring c of $\mathcal{G}$, the set

$$F' = \{X \in I \mid c(X) = 1\}$$

is a $<\delta^+$-complete filter that extends F and measures I: F' extends F by our demand that $L(X) = \{1\}$ whenever $X \in F$, and for every $X \in I$, precisely one of X and $J \setminus X$ is an element of F', for they are connected by an edge in $\mathcal{G}$. Furthermore, if $(X_i)_{i<\delta}$ is a family of elements of F', i.e., $c(X_i) = 1$ for all $i < \delta$, then Property (i) from Lemma 19 for the suitable copy of $\mathcal{T}_\delta$ contained in $\mathcal{G}$ yields $c(\bigcap_{i<\delta} X_i) = 1$, so $\bigcap_{i<\delta} X_i \in F'$. Finally, if $X \subseteq Y$ are both in I and $X \in F'$, i.e., $c(X) = 1$, then if $c(Y) = 0$, it follows by the above that $c(J\setminus Y) = 1$, and thus again by the above that $c(X \cap (J \setminus Y)) = c(\emptyset) = 1$, and hence that $c(J) = 0$, contradicting that $J \in F$.

Thus, it remains to show that there exists an L-colouring of $\mathcal{G}$, which by the κ-compactness for δ-list-colouring for graphs of size at most λ^δ is equivalent to showing that every subgraph $\mathcal{H} \subseteq \mathcal{G}$ of size $<\kappa$ has an L-colouring. Let $\mathcal{H} = \langle H, D\rangle$ be a subgraph of $\mathcal{G}$ of size less than κ. We may assume that whenever $h \in H$ is an element of some copy of $\mathcal{T}_\delta$ in $\mathcal{G}$ with $h \notin I$, this copy of $\mathcal{T}_\delta$ (including its nodes from I) is completely contained in $\mathcal{H}$. The $<\kappa$-completeness of F implies that the set $\bigcap(H \cap F) \subseteq J$ is non-empty, so it contains some $a \in J$ as an element. We define an L-colouring $c\colon H \to \delta$ as follows. For $X \in H \cap I$, we set $c(X) = 1$ if $a \in X$, and $c(X) = 0$ otherwise. The remaining vertices of H are contained in (unique) copies of $\mathcal{T}_\delta$, and we define the function c on those vertices for each copy of $\mathcal{T}_\delta$ as follows. Let us consider a particular copy of $\mathcal{T}_\delta$ in $\mathcal{H}$. Observe that $c(y)$ and $c(x_i)$ for $i < \delta$ have been defined already, and that $c(y) \in \{c(x_i) \mid i < \delta\}$. Property (ii) from Lemma 19 shows that we find an L-colouring of the whole copy of $\mathcal{T}_\delta$ that extends c as already defined on $H \cap I$. Since different copies of $\mathcal{T}_\delta$ in $\mathcal{H}$ are not connected on nodes outside of I, c is an L-colouring of $\mathcal{H}$.

Now let us consider the case when κ-compactness for $<\delta$-list-colouring holds for graphs of size at most $\lambda^{<\delta}$. We now want to show that κ has the (δ, λ)-filter extension property. The argument is basically the same as in the previous case, but now we enlarge a given set $I \subseteq \mathcal{P}(J)$ of size at most λ to size at most $\lambda^{<\delta}$, with $J \in I$, and I closed under $<\delta$-intersections and complements. We then construct a graph $\mathcal{G}$ such that every L-colouring of $\mathcal{G}$ corresponds to a $<\delta$-complete filter on J that extends a given $<\kappa$-complete filter $F \subseteq I$ on J that measures I. The graph is constructed as before, but we now consider families $(X_i)_{i<\bar\delta}$ of elements of I for cardinals $3 \leq \bar\delta < \delta$, and glue a copy of $\mathcal{T}_{\bar\delta}$ into $\mathcal{G}$ for every such family. We also define L as before, and note that $L\colon G \to [\delta]^{<\delta}$, since this is the case for every map L' that is associated to $\mathcal{T}_{\bar\delta}$ for $3 \leq \bar\delta < \delta$. We then define F' as in the previous case, and the same argument, using the $\mathcal{T}_{\bar\delta}$'s rather than $\mathcal{T}_\delta$, shows that $F' \supseteq F$ is a $<\delta$-complete filter that measures I. Finally, the argument that $\mathcal{G}$ has an L-colouring proceeds exactly as in the previous case. □

We also obtain an analogous characterization of certain compact cardinals in terms of homomorphism-compactness:

Theorem 21. *Assume that $\delta < \kappa \leq \lambda$ are infinite cardinals, with δ regular. Then:*

- *If $\lambda^\delta = \lambda$, then κ is (δ^+, λ)-compact if and only if (κ, λ)-homomorphism-compactness holds for target structures of size at most δ.*
- *If $\bar\lambda^\delta < \lambda$ whenever $\bar\lambda < \lambda$, then κ is $(\delta^+, <\lambda)$-compact if and only if $(\kappa, <\lambda)$-homomorphism-compactness holds for target structures of size at most δ.*
- *κ is δ^+-strongly compact if and only if κ-homomorphism-compactness holds for target structures of size at most δ.*

Proof. Immediate by Propositions 16, 10 and Theorem 20. □

6. On the largeness of compact cardinals

The goal of this section is to verify Theorem 2, which states that for any cardinals $\delta \leq \kappa$ with δ regular and uncountable, if κ is (δ, κ)-compact, then $\kappa \geq \aleph_\delta$.

Proof of Theorem 2. Assume for a contradiction that κ is a cardinal in $[\delta, \aleph_\delta)$ that is (δ, κ)-compact. Let $\mathcal{L} = \{\alpha_i \mid i \leq \kappa\} \cup \{E, \in, f\}$, with constant symbols α_i, a unary predicate E and binary relations $\in$ and f. For every limit cardinal $\lambda \leq \kappa$, we fix a sequence of cardinals $\langle \lambda_i \mid i < \mathrm{cof}(\lambda)\rangle$ below λ that is cofinal in λ. Let T be the theory consisting of the following:

1. for every $i \leq \kappa$, $E(\alpha_i)$,
2. *E is transitive:* $\forall x, y\ [(x \in y \wedge E(y)) \to E(x)]$,
3. *$\in$ is a strict linear ordering of all elements satisfying E*, that is:
   - $\forall x, y, z\ E(z) \to [(x \in y \wedge y \in z) \to x \in z]$,

- $\forall x \neq y\ [(E(x) \wedge E(y)) \to (x \in y \vee y \in x)]$,
- $\forall x\ E(x) \to \neg(x \in x)$,

(4) for all $i < j \leq \kappa$, $\alpha_i \in \alpha_j$,

(5) for $m < \omega$,

$$\forall x \left(x \in \alpha_m \iff \bigvee_{k<m} x = \alpha_k \right),$$

(6) for every limit cardinal $\lambda \leq \kappa$,

$$\forall x \left(x \in \alpha_\lambda \iff \bigvee_{i<\mathrm{cof}(\lambda)} x \in \alpha_{\lambda_i} \right),$$

(7) for every cardinal $\lambda \leq \kappa$, the sentence that for all $x \in \alpha_\lambda$, x is a surjective image of $\alpha_{\bar\lambda}$ for some cardinal $\bar\lambda < \lambda$,[10] where z being a surjective image of $\alpha_{\bar\lambda}$ means that there is g consisting (via $\in$) of ordered pairs (x, y) with $x \in \alpha_{\bar\lambda}$ and $y \in z$, such that
- $\forall x \in \alpha_{\bar\lambda}\ \exists! y \in z\ (x, y) \in g$, and
- $\forall y\ [y \in z \to \exists x \in \alpha_{\bar\lambda}\ (x, y) \in g]$.

(8) f is a function, and its image of α_κ contains $\{\alpha_i \mid i \leq \kappa\}$, or more precisely:
- $\forall x \exists! y f(x, y)$,
- for every $i \leq \kappa$, $\exists x \in \alpha_\kappa\ f(x, \alpha_i)$.

We will write $f(x) = y$ rather than $f(x, y)$ in the following.

(9) f is weakly order-preserving with respect to $\in$, that is if $x \in y \in \alpha_\kappa$, then either $f(x) \in f(y)$ or $f(x) = f(y)$.

T is an $\mathcal{L}_{\delta,\omega}$-theory in a language of size κ.

Claim. *T is $<\kappa$-satisfiable.*

Proof. Let $\bar T \subseteq T$ be of size less than κ. Let R be the set (of size $<\kappa$) of all $i \leq \kappa$ for which α_i is mentioned in a sentence in $\bar T$. We let $\theta < \kappa$ be the order type of R and write $R = \{i_\xi \mid \xi \in \theta\}$, with the i_ξ strictly increasing. We obtain $M \models \bar T$ as follows. We let $\alpha_i = i$ for $i \leq \kappa$, and let E hold exactly for all α_i, $i \leq \kappa$. We pick $\in$ to be the actual elementhood relation, and also include suitable surjections (together with all ordered pairs (x, y), coded as $\{\{x\}, \{x, y\}\}$, contained in them, as well as the corresponding sets $\{x\}$ and $\{x, y\}$) into M in order to satisfy Item 7. Trivially, this model satisfies the axioms from items 1–7; to satisfy the axioms contained in $\bar T$ from the remaining items, we let f be the assignment $\xi \mapsto i_\xi$ for $\xi < \theta$ and $\xi \mapsto \kappa$, for $\theta \leq \xi < \kappa$. For the remaining $x \in M$ we set $f(x) = 0$. Then, f is weakly order-preserving with respect to $\in$ for elements of $\alpha_\kappa = \kappa$, as required by Item 9, and it is straightforward to observe that the axioms from Item 8 that are contained in $\bar T$ are satisfied. $\square$

We will now show that T is not satisfiable, thus yielding the statement of our theorem. Assume for a contradiction that $M \models T$.

Claim.
- *For all cardinals $\lambda \leq \kappa$, $|\{x \in M \mid M \models x \in \alpha_\lambda\}| = \lambda$.*
- *If $\lambda \leq \kappa$ is an infinite cardinal, then $\{\alpha_i \mid i < \lambda\}$ is unbounded in α_λ with respect to $\in$.*

[10] Note that since $\lambda < \aleph_\delta$, there are less than δ-many cardinals below λ, so this can be formulated as an $\mathcal{L}_{\delta,\omega}$-sentence.

Proof. By induction on $\lambda \leq \kappa$: The start of the induction (for all $\lambda < \omega$) is provided by Item 5. If $\lambda \leq \kappa$ is a limit cardinal, the claim is immediate by Item 6. Assume that the claimed statement holds for $\lambda < \kappa$. By Item 4, it holds that $|\{x \in M \mid M \models x \in \alpha_{\lambda^+}\}| \geq \lambda^+$. Note that our induction hypothesis implies that there cannot be $x \in M$ with $\forall i < \lambda^+ \; M \models \alpha_i \in x \in \alpha_{\lambda^+}$, since the existence of such x would clearly contradict Item 7. This means that the α_i for $i < \lambda^+$ are unbounded in α_{λ^+} with respect to $\in$ in M. This in turn means that

$$\{x \in M \mid M \models x \in \alpha_{\lambda^+}\} = \bigcup_{i<\lambda^+} \{x \in M \mid M \models x \in \alpha_i\}.$$

By Item 7 with respect to the α_i's, and by our inductive hypothesis, this set has cardinality at most λ^+. □

We will now obtain a contradiction by considering the interpretation of f in M. By Item 8, there has to be some $x \in \alpha_\kappa$ such that $f(x) = \alpha_\kappa$. But note that by Item 7 and the previous claim, $\{y \in M \mid M \models y \in x\}$ has cardinality less than κ. This means that there has to be $i < \kappa$ such that $\neg f(y, \alpha_i)$ whenever $M \models y \in x$. But whenever $M \models x \in y$, by Item 9, it follows that $\alpha_\kappa \in f(y)$ or $f(y) = \alpha_\kappa$. Summing up, α_i is not an element of $f(\alpha_\kappa)$, contradicting Item 8.

□

The following small observation, which we thought was nice enough to include, is an adaptation of one part of [7, Proposition 4.4]:

Observation 22. *If $\omega \leq \bar{\delta} < \delta \leq \kappa$, with δ regular, and κ is (δ, κ)-compact, then $2^{\bar{\delta}} < \kappa$.*

Proof. Assume that κ is (δ, κ)-compact. Assume for a contradiction that $2^{\bar{\delta}} \geq \kappa$ for some $\bar{\delta} < \delta$. Let c_α and d^i be constants for $\alpha < \bar{\delta}$ and $i < 2$. Let T be the theory that consists of the following:

- $\{c_\alpha = d^0 \vee c_\alpha = d^1 \mid \alpha < \bar{\delta}\}$.
- $\{\bigvee_{\alpha<\bar{\delta}} (c_\alpha \neq d^{f(\alpha)}) \mid f \in {}^{\bar{\delta}}2\}$.

Then, T is an $\mathcal{L}_{\delta,\omega}$-theory in a language of size $\bar{\delta} < \kappa$. Since $2^{\bar{\delta}} \geq \kappa$, T is $<\kappa$-satisfiable. However, T is not satisfiable, which is a contradiction. □

7. More on compact cardinals

Our next goal is to investigate the relationship between (δ, λ)-compact cardinals and (δ, λ)-strongly compact cardinals, as introduced by Boney and Unger [4]. Let κ be a regular cardinal, and let A be any set. We let $\mathcal{P}_\kappa(A)$ denote the collection of subsets of A of cardinality less than κ. If $F \subseteq \mathcal{P}(\mathcal{P}_\kappa(A))$ is a filter, we say that F is *fine* in case whenever $a \in A$, $\{x \in \mathcal{P}_\kappa(A) \mid a \in x\} \in F$. We say that F is an *ultrafilter* on a set X if it measures X. In the terminology of [4], for uncountable cardinals $\delta \leq \kappa \leq \lambda$, κ is (δ, λ)-*strongly compact* if there is a fine, $<\delta$-complete ultrafilter on $\mathcal{P}(\mathcal{P}_\kappa(\lambda))$.

Proposition 23. *If $\delta \leq \kappa \leq \lambda$ are uncountable cardinals with δ regular, and κ has the $(\delta, 2^{(\lambda^{<\kappa})})$-filter extension property (in particular, this is the case if κ is $(\delta, 2^{(\lambda^{<\kappa})})$-compact by Proposition 10), then there is a fine, $<\delta$-complete ultrafilter on $\mathcal{P}(\mathcal{P}_\kappa(\lambda))$ – that is, κ is (δ, λ)-strongly compact, or equivalently, there is an* *elementary embedding* $j\colon V \to W$ *for some transitive class* $W \subseteq V$*, with critical point* $\operatorname{crit} j \geq \delta$[11] *and* $D \in (\mathcal{P}_{j(\kappa)}(j(\lambda)))^W$ *with* $j[\lambda] \subseteq D$*.*

Proof. Let $F = \{\{x \in \mathcal{P}_\kappa(\lambda) \mid \gamma \in x\} \mid \gamma < \lambda\}$. Since F is $<\kappa$-complete, we may use the $(\delta, 2^{(\lambda^{<\kappa})})$-filter extension property to find $U \supseteq F$ which is $<\delta$-complete and measures $\mathcal{P}(\mathcal{P}_\kappa(\lambda))$. Note that for U to extend F just means that U is fine. The rest of the argument is very much standard, but will be provided for the benefit of our readers, and also since we will need to use a variation of this argument later on: We use U to define an equivalence relation

$$f \sim g \iff \{x \in \mathcal{P}_\kappa(\lambda) \mid f(x) = g(x)\} \in U$$

for $f, g\colon \mathcal{P}_\kappa(\lambda) \to V$, denote the equivalence class of $f\colon \mathcal{P}_\kappa(\lambda) \to V$ by $[f]_U$, and define the ultrapower

$$W := \operatorname{Ult}(V, U) = \{[f]_U \mid f\colon \mathcal{P}_\kappa(\lambda) \to V\}.$$

Repeated application of Łoś's theorem yields the following: By the $<\omega_1$-completeness of U, $\operatorname{Ult}(V, U)$ is well-founded, and we may identify it with its transitive collapse W. Let $j\colon V \to W$ be defined via $j(x) = [c_x]_U$, for $x \in V$, where c_x denotes the constant function with value x and domain $\mathcal{P}_\kappa(\lambda)$. Let $D := [\mathrm{ot}]_U \in W$. The embedding j is nontrivial, since $\forall \alpha < \kappa\ \alpha < D < j(\kappa)$. By the $<\delta$-completeness of U, $\operatorname{crit} j \geq \delta$. Moreover, $j[\lambda] \subseteq D$ by the fineness of U, and $|D| < j(\kappa)$ in W easily follows, as desired.

Given $j\colon V \to W$ with the above properties, let

$$U = \{A \in \mathcal{P}(\mathcal{P}_\kappa(\lambda)) \mid D \in j(A)\}.$$

Then, U is easily seen to be a $<\delta$-complete, fine ultrafilter on $\mathcal{P}(\mathcal{P}_\kappa(\lambda))$, using that $\operatorname{crit} j \geq \delta$, $j[\lambda] \subseteq D$, and that $D \in (\mathcal{P}_{j(\kappa)}(j(\lambda)))^W$. □

Proposition 24. *If* $\delta \leq \kappa \leq \lambda$ *are uncountable cardinals with* δ *regular and with* $\lambda^{<\delta} = \lambda$*, and* κ *is* (δ, λ)*-strongly compact then* κ *is* (δ, λ)*-compact.*

Proof. This follows from the second part of the proof of Proposition 10 – note that, using the notation of that proof, the set Σ has size at most $\lambda^{<\delta} = \lambda$ and hence we can identify $[\Sigma]^{<\kappa}$ with $\mathcal{P}_\kappa(\lambda)$. Then clearly, a fine, $<\delta$-complete ultrafilter on $\mathcal{P}_\kappa(\lambda)$ is sufficient to proceed with the argument from that proof. □

We now modify the above so that we obtain an actual characterization of (δ, λ)-compactness. Given a set M, we say that a filter $U \subseteq \mathcal{P}(\mathcal{P}_\kappa(\alpha)) \cap M$ is an *M-ultrafilter* on $\mathcal{P}(\mathcal{P}_\kappa(\alpha))$ if U measures $\mathcal{P}(\mathcal{P}_\kappa(\alpha)) \cap M$. We say that an M-ultrafilter U is $<\delta$-complete if U is closed under $<\delta$-intersections in M. We say that an M-ultrafilter U is *fine* if it contains the set $\{x \in \mathcal{P}_\kappa(\alpha) \mid \gamma \in x\}$ as an element whenever $\gamma \in \alpha \cap M$. Furthermore, if $M \prec H(\theta)$ for some regular cardinal θ, and $j\colon (M, \in) \to (N, \in_N)$ is an elementary embedding with N not necessarily transitive, the *critical point* $\operatorname{crit} j$ of j (which may or may not exist) is the least ordinal $\gamma \in M$ with the property that $\{\bar\gamma \in N \mid \bar\gamma \in_N j(\gamma)\} \supsetneq j[\gamma \cap M]$.

[11]The *critical point* of an elementary embedding between transitive sets or classes is the least ordinal α that is moved by j nontrivially, that is $j(\alpha) > \alpha$. Statements like $\operatorname{crit} j \geq \delta$ are meant to include the statement that $\operatorname{crit} j$ exists.

Proposition 25. *Let $\delta \leq \kappa \leq \lambda \leq \alpha < \theta$ be uncountable cardinals with δ and θ regular. Then, if κ has the (δ, λ)-filter extension property and $M \prec H(\theta)$ is of size λ with $\lambda + 1 \subseteq M$ and $\mathcal{P}_\kappa(\alpha) \in M$, then there is a $<\delta$-complete, fine M-ultrafilter U on $\mathcal{P}(\mathcal{P}_\kappa(\alpha))$.*

The existence of such U is equivalent to the existence of an elementary embedding $j \colon M \to N$ with $\operatorname{crit} j \geq \delta$ (but N not necessarily transitive), and a set $D \in (\mathcal{P}_{j(\kappa)}(j(\alpha)))^N$ with $j[\alpha \cap M] \subseteq D$.

Proof. We proceed similarly to the argument for Proposition 23. First, assume that κ has the (δ, λ)-filter extension property and that $M \prec H(\theta)$ is of size λ with $\lambda + 1 \subseteq M$ and $\mathcal{P}_\kappa(\alpha) \in M$. We let

$$F = \{\{x \in \mathcal{P}_\kappa(\alpha) \mid \gamma \in x\} \mid \gamma \in \alpha \cap M\}.$$

Since F is $<\kappa$-complete and $F \subseteq M$, we may use the (δ, λ)-filter extension property to find $U \supseteq F$ which is $<\delta$-complete and measures $\mathcal{P}(\mathcal{P}_\kappa(\alpha)) \cap M$, since M has size λ.

Given such U, we define the ultrapower

$$N := \mathrm{Ult}(M, U) = \{[f]_U \mid f \text{ is a function with domain } \mathcal{P}_\kappa(\alpha) \text{ and } f \in M\}.$$

Let $j \colon M \to N$ be defined via $j(x) = [c_x]_U$, for $x \in M$. Let $D := [\mathrm{ot}]_U \in N$. Repeated application of Łoś's theorem yields the following: The critical point $\operatorname{crit} j$ of j exists, since $\forall \alpha < \kappa\ [c_\alpha] < D < j(\kappa)$. By the $<\delta$-completeness of U, $\operatorname{crit} j \geq \delta$. Moreover, $j[\alpha \cap M] \subseteq D$ by the fineness of U, and $|D| < j(\kappa)$ in N easily follows, as desired.

Given $j \colon M \to N$ and $D \in N$ as in the final statement of our proposition, let

$$U = \{A \in \mathcal{P}(\mathcal{P}_\kappa(\alpha)) \mid A \in M \wedge D \in j(A)\}.$$

Then, U is easily seen to be a $<\delta$-complete, fine M-ultrafilter on $\mathcal{P}(\mathcal{P}_\kappa(\alpha))$, using that $\operatorname{crit} j \geq \delta$, $j[\alpha \cap M] \subseteq D$, and that $D \in (\mathcal{P}_{j(\kappa)}(j(\alpha)))^N$. $\square$

Corollary 26. *Let $\delta \leq \kappa \leq \lambda$ be uncountable cardinals with δ regular and $\lambda^{<\delta} = \lambda$. Then, κ is (δ, λ)-compact if and only if for all (equivalently, for some) regular $\theta > \lambda^{<\kappa}$, whenever $M \prec H(\theta)$ is of size λ with $\lambda + 1 \subseteq M$, there is U which is a $<\delta$-complete, fine M-ultrafilter on $\mathcal{P}_\kappa(\lambda)$.*

The existence of such U is equivalent to the existence of an elementary embedding $j \colon M \to N$ with $\operatorname{crit} j \geq \delta$ (but N not necessarily transitive), and a set $D \in (\mathcal{P}_{j(\kappa)}(j(\lambda)))^N$ with $j[\lambda \cap M] \subseteq D$.

Proof. The forward direction and the equivalence with respect to elementary embeddings are immediate from Propositions 25 and 10 above. The reverse direction follows from the second part of the proof of Proposition 10 – note that, using the notation of that proof, the set Σ has size at most $\lambda^{<\delta} = \lambda$ and hence we can identify $[\Sigma]^{<\kappa}$ with $\mathcal{P}_\kappa(\lambda)$. Then, again observing that proof, a fine, $<\delta$-complete filter that measures a particular λ-size subset of $\mathcal{P}(\mathcal{P}_\kappa(\lambda))$ – namely all the relevant sets $J_{\varphi,a}$, of which there are $\lambda^{<\delta} = \lambda$ many, is sufficient to proceed with the argument from that proof, and whenever $\theta > \lambda^{<\kappa}$ is regular, $H(\theta)$ contains all subsets of $\mathcal{P}_\kappa(\lambda)$ as elements, so we may pick $M \prec H(\theta)$ to contain all the relevant sets $J_{\varphi,a}$. $\square$

Appendix A. On the naturalness of list colouring

The point of this section is to remark that for finitely many colours, colouring and list colouring are essentially the same, and that therefore, list colouring is just a way of generalizing graph colouring with finitely many colours to infinitely many colours.

Observation 27. *If $\mathcal{G} = \langle G, E\rangle$ is a graph and $L\colon G \to \mathcal{P}(\delta)$ with δ finite, then there is a graph $\mathcal{H}(\mathcal{G}) = \mathcal{H} = \langle H, F\rangle$ with $H = G \dot\cup K_\delta$ and $F \cap G^2 = E$, with the following properties:*

- *Every L-colouring of $\mathcal{G}$ can be extended to a δ-colouring of $\mathcal{H}$.*
- *For any δ-colouring d of $\mathcal{H}$, there is a permutation π of δ such that $(\pi\circ d)\restriction G$ is an L-colouring of $\mathcal{G}$.*
- *If $\bar{\mathcal{G}} \leq \mathcal{G}$, then $\mathcal{H}(\bar{\mathcal{G}})$ is the restriction of $\mathcal{H} = \mathcal{H}(\mathcal{G})$ to $\bar{G} \dot\cup K_\delta$.*

Proof. Given $\mathcal{G}$, extend it by adding a disjoint copy of K_δ, the complete graph with δ-many nodes, in order to obtain $\mathcal{H}$. Whenever $g \in G$ and $n \notin L(g)$, we also add an edge between g and n in $\mathcal{H}$. Now, given an L-colouring c of $\mathcal{G}$, we may extend it to a δ-colouring d of $\mathcal{H}$ by letting $d(n) = n$ for every $n \in K_\delta$. Since n is connected to $g \in G$ only if $n \notin L(g)$ it follows that $d(g) = c(g) \neq n = d(n)$, which implies that d is a δ-colouring of $\mathcal{H}$, as desired.

On the other hand, given a δ-colouring d of $\mathcal{H}$, its restriction to K_δ is injective, hence we may find a permutation π of δ such that $\pi\circ d\restriction K_\delta = \mathrm{id}_\delta$. Then obviously, also $\pi\circ d$ is a δ-colouring of $\mathcal{H}$, and its restriction to G is an L-colouring of $\mathcal{G}$, since whenever $g \in G$ and $n \notin L(g)$, then there is an edge between g and n in $\mathcal{H}$, and therefore $\pi \circ d(g) \neq n$.

The final item is immediate from our construction of $\mathcal{H}$. □

Corollary 28. *If δ is finite, then κ-compactness for δ-colouring and κ-compactness for δ-list-colouring are equivalent.*

Proof. Since every δ-colouring is an L-colouring for L the constant function with value δ, κ-compactness for δ-list-colouring implies κ-compactness for δ-colouring. Now assume that κ-compactness for δ-colouring holds. Let $\mathcal{G}$ be a graph and let $L\colon G \to \mathcal{P}(\delta)$ be a map such that every subgraph of $\mathcal{G}$ of size less than κ has an L-colouring. Let $\mathcal{H}$ be the graph witnessing Observation 27 with respect to $\mathcal{G}$. By Observation 27 applied to subgraphs $\bar{\mathcal{G}}$ of $\mathcal{G}$ of size less than κ, this means that every subgraph of $\mathcal{H}$ of size less than κ has a δ-colouring, and therefore by our assumption, $\mathcal{H}$ has a δ-colouring. Now, again by Observation 27, this implies that $\mathcal{G}$ has an L-colouring, as desired. □

Open Questions

The most obvious questions that naturally arise from the results of our paper seem to be those concerning the properties of our hierarchy of compact cardinals, and we ask what we think might be very interesting sample questions regarding (ω_1, κ)-compact cardinals κ below:

Question 29.

- Can $\kappa = \aleph_{\omega_1}$ be (ω_1, κ)-compact?
- Is an (ω_1, κ)-compact cardinal κ always a limit cardinal?
- Does an (ω_1, κ)-compact cardinal κ always have uncountable cofinality?
- Does the assumption of the existence of an (ω_1, κ)-compact cardinal κ have consistency strength beyond ZFC?

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References

1. Joan Bagaria and Menachem Magidor, *On ω_1-strongly compact cardinals*, The Journal of Symbolic Logic **79** (2014), no. 1, 266–278.
2. John Lane Bell, *On the relationship between weak compactness in $L_{\omega_1,\omega}$, L_{ω_1,ω_1}, and restricted second-order languages*, Archiv für mathematische Logik und Grundlagenforschung **15** (1972), no. 1, 74–78.
3. Shai Ben-David and Menachem Magidor, *The weak $\square^*$ is really weaker than the full $\square$*, The Journal of Symbolic Logic **51** (1986), no. 4, 1029–1033.
4. Will Boney and Spencer Unger, *Large cardinal axioms from tameness in AECs*, Proceedings of the AMS **145** (2017), 4517–4532.
5. Robert Cowen, *Two hypergraph theorems equivalent to BPI*, Notre Dame Journal of Formal Logic **31** (1990), 232–240.
6. Nicolaas Govert de Bruijn and Paul Erdős, *A colour problem for infinite graphs and a problem in the theory of relations*, Indagationes Mathematicae (Proceedings) **54** (1951), 371–373.
7. Akihiro Kanamori, *The higher infinite*, second ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, Large cardinals in set theory from their beginnings.
8. Tamás Kátay, László Márton Tóth, and Zoltán Vidnyánszky, *The CSP dichotomy, the axiom of choice, and cyclic polymorphisms*, Preprint (2024).
9. Hans Läuchli, *Coloring infinite graphs and the boolean prime ideal theorem*, Israel Journal of Mathematics **9** (1971), no. 4, 422–429.
10. Danny Rorabaugh, Claude Tardif, and David Wehlau, *Logical compactness and constraint satisfaction problems*, Logical Methods in Computer Science **Volume 13, Issue 1** (2017).
11. Saharon Shelah, *Incompactness for chromatic numbers of graphs*, A tribute to Paul Erdős, Cambridge Univ. Press, Cambridge, 1990, pp. 361–371.
12. ———, *On incompactness for chromatic number of graphs*, Acta Mathematica Hungarica **139** (2013), no. 4, 363–371.
13. Claude Tardif, *Constraint satisfaction problems, compactness and non-measurable sets*, Logical Methods in Computer Science **Volume 22, Issue 3** (2026).
14. Spencer Unger, *Compactness for the chromatic number at $\aleph_{\omega_1+1}$*, Unpublished manuscript, available at `https://www.math.toronto.edu/sunger/chromatic-compactness.pdf`, December 2015.

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