Asymmetric Cut and Choose Games

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Fair Division

Assume there's a piece of cake, and we want Ann and Bob to fairly share it. An easy way to do so is to let Ann *cut* the cake in two, then let Bob *choose* his piece, and let Ann have the remaining piece.

This idea goes back to at least some hundred years BC. Modern mathematical investigation of *fair division* was initiated by Steinhaus, Banach and Knaster in the 1940ies. Galvin, Mycielski and Ulam, in the 1960ies, proposed various *infinite cut and choose games*.

An infinite cut and choose game

We have two players, going by the names of *Cut* and *Choose*, an infinite set X over which our game takes place, and a limit ordinal γ, which denotes the length of our game.

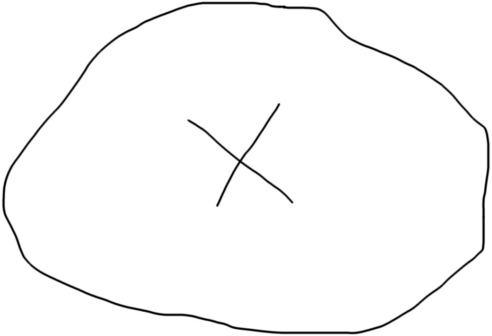
- In their first move, *Cut* partitions X into two (disjoint) pieces, and *Choose* picks one of them, call it X_0.
- Now *Cut* partitions X_0 into two pieces, and *Choose* picks one of them, call it X_1.
- Now *Cut* partitions X_1 ...
- At limit stages, intersections are taken, and then partitioned...
- If *Choose* ever picks a singleton or $\emptyset$, they immediately lose.
- Otherwise, this goes on for γ-many steps.

In the end, *Choose* wins if the intersection of all of their choices is nonempty. Otherwise, *Cut* wins.

Let us denote the above game as $\mathcal{U}(X, \gamma)$.

Note: If *Choose* were allowed to pick singletons, they could fix some $y \in X$ in advance, always pick the part that contains y as an element, and thus win.

...in pictures



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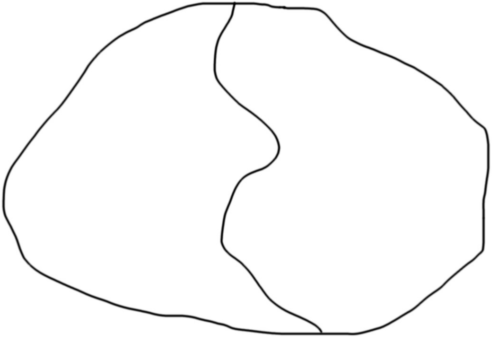

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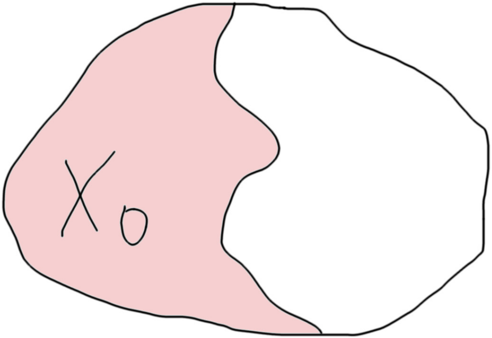

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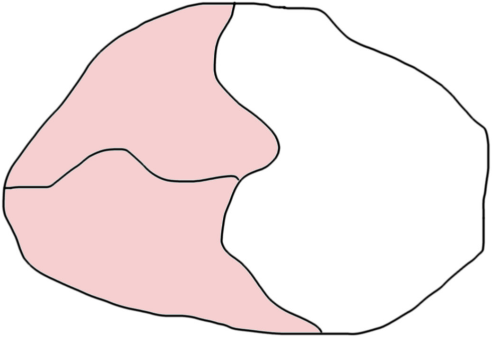

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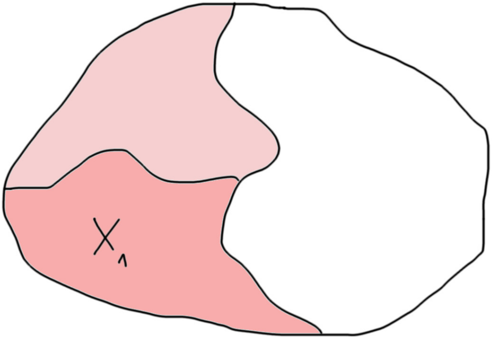

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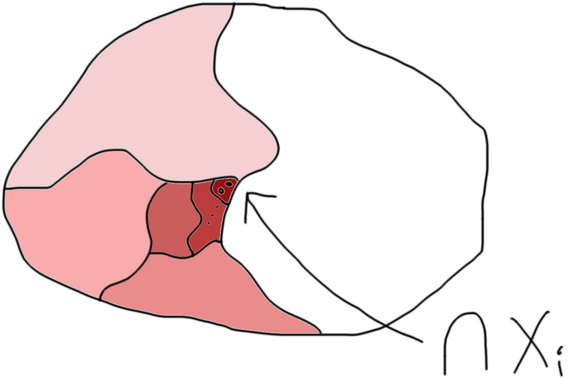


A simplification

For many practical purposes, rather than just cutting the intersection of all choices so far at any stage of our games, it is easier to think of *Cut* repeatedly cutting the starting set X into pieces, and *Choose* picking one of them. This is easily seen to be essentially the same game, for if we are only ever interested in intersections of choices in order to evaluate who wins, the cutting and choosing that happens outside of these intersections is clearly irrelevant. Let us this *redefine* our basic cut and choose game $\mathcal{U}(X, \gamma)$ as follows:

- In each of their moves, *Cut* partitions X into two (disjoint) pieces, and *Choose* picks one of them.
- *Choose* is not allowed to ever pick a singleton.
- This goes on for γ-many steps.

In the end, *Choose* wins if the intersection of all of their choices is nonempty. Otherwise, *Cut* wins.

Cut wins over ω

It is very easy to see that *Cut* wins $\mathcal{U}(\omega, \omega)$ – *Cut* wins by *removing* one natural number in each step:

- In their first move, they partition ω into $\{0\}$ and $[1, \omega)$.
- *Choose* will have to pick $[1, \omega)$.
- In their second move, they present the choices $\{1\}$ and $\omega \setminus \{1\}$.
- *Choose* will have to pick $\omega \setminus \{1\}$.
- Next options will be $\{2\}$ and $\omega \setminus \{2\}$, ...

Obviously, the intersection of all choices will be empty, hence *Cut* wins.

Note that this was only about ω being countable.

In the same way, for any γ, *Cut* wins $\mathcal{U}(\gamma, \gamma)$.

So for *Choose* to have a chance of winning, the starting set will have to be reasonably large.

Some fairly easy observations

It is not very interesting to investigate winning strategies for *Cut* in this game. Fairly easy arguments show the following:

Observation

Cut has a winning strategy for the game $\mathcal{U}(\kappa, \gamma)$ if and only if $\kappa \leq 2^{<\gamma}$.

Another fairly easy argument shows the following:

Observation

If γ is regular and $\kappa \leq 2^{\gamma}$, then *Choose* does not have a winning strategy for the game $\mathcal{U}(\kappa, \gamma)$.

In particular, this means that if $2^{<\gamma} < \kappa \leq 2^{\gamma}$, then $\mathcal{U}(\kappa, \gamma)$ is undetermined.

Choose wins at Measurable Cardinals

An uncountable cardinal κ is *measurable* if there is a $<\kappa$-complete, non-principal ultrafilter on κ (we call such a filter a *measurable ultrafilter*).

Observation

Choose wins $\mathcal{U}(\kappa, \gamma)$ whenever $\gamma < \kappa$ in case κ is a measurable cardinal.

Proof: Make choices according to some fixed measurable ultrafilter. □

In the above, *measurable* can be replaced by certain forms of *generically measurable*.

Generically measurable cardinals

Definition

A cardinal κ is *generically measurable as witnessed by the notion of forcing* P if in every P-generic extension, there is a uniform V-normal V-ultrafilter on κ that induces a well-founded (generic) ultrapower of V. Equivalently, in every P-generic extension $V[G]$, there is an elementary embedding $j\colon V \to M$ with critical point κ for some transitive $M \subseteq V[G]$.

Observation

Choose wins $\mathcal{U}(\kappa, \gamma)$ whenever κ is generically measurable as witnessed by $<\gamma^+$-closed forcing.

Proof: Let $\dot{U}$ be a P-name for a uniform V-normal V-ultrafilter on κ. In each step, *Choose* picks conditions p_i forcing their choices X_i to be in $\dot{U}$, so that the p_i's are decreasing in P. By our closure assumption, the p_i's have a lower bound in P, which forces the intersection of their choices to be in $\dot{U}$, and thus in particular to be nonempty. □

Some classical results

Choose winning has large cardinal strength:

Theorem (Silver-Solovay, 1970ies, for the case $\gamma = \omega$)

If $\gamma < \kappa$ are regular cardinals and Choose has a winning strategy in the game $\mathcal{U}(\kappa, \gamma)$, then there is a generically measurable cardinal $\leq \kappa$, as witnessed by $<\gamma$-closed forcing. In particular, this yields an inner model with a measurable cardinal.

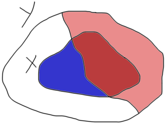
Choose can also win at successor cardinals:

Theorem (essentially Laver, 1970ies)

If λ is a regular uncountable cardinal, $\gamma < \lambda$ is regular, $\kappa > \lambda$ is a measurable cardinal, and we perform a Lévy collapse to turn κ into λ^+, then in the generic extension, Choose has a winning strategy in the game $\mathcal{U}(\lambda^+, \gamma)$.

Leastness

If *Choose* has a winning strategy in the game $\mathcal{U}(\kappa,\gamma)$ and $\kappa \leq \lambda$, then they have a winning strategy in $\mathcal{U}(\lambda,\gamma)$ as well – simply consider restrictions of possible choices to κ, and use the strategy for the game $\mathcal{U}(\kappa,\gamma)$.



So what is interesting is the *smallest* κ so that *Choose* can win $\mathcal{U}(\kappa,\gamma)$, if there is any at all. Starting over an inner model for a (single) measurable cardinal, the above results can easily be modified to show that either measurable cardinals or certain successor cardinals κ can be least so that *Choose* wins $\mathcal{U}(\kappa,\gamma)$.

Cut and choose games at small inaccessibles

We can also extend this range of possibilities to the following:

Theorem

If κ is a measurable cardinal, and $\gamma < \kappa$ is regular, then there is a forcing extension in which κ is an inaccessible cardinal that is not measurable (in fact, not even weakly compact), however is the least cardinal λ such that Choose has a winning strategy in the game $\mathcal{U}(\lambda, \gamma)$.

Proof-Sketch (for the experts): First force with a reverse Easton iteration, adding a Cohen subset to every inaccessible cardinal below κ. This will be our ground model. Adding a Cohen subset to κ will ensure that κ is measurable again in that further extension. Now we force to add a homogeneous κ-Suslin tree T that is closed under ascending γ-sequences. In that extension, κ is generically measurable, as witnessed by forcing with T (because the two-step iteration of first adding T and then forcing with it is equivalent to adding a Cohen subset of κ), however not weakly compact (because there exists a κ-Suslin tree). □

A question

This means that *Choose* can also *first win* cut and choose games over *fairly small* inaccessible cardinals.

Question

Can *Choose* also first win cut and choose games over the *least* inaccessible cardinal?

Ideal Partitions

A natural generalization of cut and choose games is to allow *Cut* to cut into a larger number of pieces in each step, or even more generally, to fix an ideal I on κ and to let *Cut* play an *I-partition* in each move. It will turn out that we can use such generalized games to characterize central set theoretic notions. So let us fix a regular uncountable cardinal κ and a $<\kappa$-complete ideal I that contains all bounded subsets of κ.

Definition

An I-partition P of $X \in I^+$ is a maximal collection of I-positive subsets of X such that $a \cap b \in I$ whenever $a \neq b$ are both elements of P.

Generalized cut and choose games

For ν a cardinal, or $\nu = \infty$,

- let $\mathcal{G}_\nu(X, I, <\gamma)$ denote the cut and choose game of length γ where in each move, *Cut* presents an I-partition of size at most ν, or of arbitrary size if $\nu = \infty$, and *Choose* picks one of its elements. *Choose* wins in case at any stage $\delta < \gamma$, the intersection of their choices up to stage δ is in I^+;
- let $\mathcal{G}_\nu(X, I, \leq\gamma)$ denote the variant where for *Choose* to win, we also require that the intersection of all of their choices is nonempty;
- let $\mathcal{G}_\nu(X, I, \gamma)$ denote the variant where for *Choose* to win, we require that the intersection of all of their choices is in I^+.

For these generalized games, unlike for our basic cut and choose games, it is very interesting to consider the existence of winning strategies for *Cut*.

Weak compactness

bd_κ denotes the bounded ideal on κ.

Observation

A cardinal κ is weakly compact if and only if *Cut* does not win $\mathcal{G}_2(\kappa, \mathrm{bd}_\kappa, {<}\kappa)$.

The subscript 2 means that *Cut* plays *I*-partitions of size 2 in each of their moves, which is really just equivalent to cutting into 2 pieces, as we did in our earlier games.

Distributivity and Precipitousness

Observation

An ideal I on κ is (γ, ν)-distributive if and only if the Boolean algebra $P(\kappa)/I$ is (γ, ν)-distributive if and only if for any $X \in I^+$, *Cut* does not have a winning strategy in the game $\mathcal{G}_\nu(X, I, \gamma)$.

Theorem (essentially Jech)

An ideal I on κ is precipitous if and only if for any $X \in I^+$, *Cut* does not have a winning strategy in the game $\mathcal{G}_\infty(X, I, \leq\omega)$.

Precipitous games

Let $\mathcal{P}(I,\gamma)$ denote the game of length γ in which players *Empty* and *Nonempty* take turns to play I-positive sets that form a $\subseteq$-decreasing sequence. *Empty* starts, and *Nonempty* goes first at all limit stages. *Nonempty* wins if the intersection of all of their choices is nonempty, and *Empty* wins otherwise.

It is well-known that I is precipitous if and only if *Empty* does not win $\mathcal{P}(I,\omega)$. The following thus generalizes our earlier characterization of precipitousness via cut and choose games.

Theorem (Jech and Velickovic for $\gamma = \omega$)

The games $\mathcal{P}(I,\gamma)$ and $\mathcal{G}_\infty(X, I, \leq\gamma)$ are essentially equivalent, that is:

Empty wins $\mathcal{P}(I,\gamma)$ iff $\forall X \in I^+$ *Cut* wins $\mathcal{G}_\infty(X, I, \leq\gamma)$, and

Nonempty wins $\mathcal{P}(I,\gamma)$ iff $\forall X \in I^+$ *Choose* wins $\mathcal{G}_\infty(X, I, \leq\gamma)$.

Strategic Closure

We can generalize precipitous games and cut and choose games to partial orders. Let Q be a partial order and $q \in Q$. In the precipitous game $\mathcal{P}(Q, \gamma)$ players *Empty* and *Nonempty* take turns playing increasingly stronger conditions in Q, and *Nonempty* wins in case they have a lower bound in Q. In the game $\mathcal{G}_\infty(q, Q, \gamma)$, *Cut* plays maximal antichains of Q, and *Choose* picks one of their elements. *Choose* wins if the set of their choices has a lower bound in Q. We have the same equivalence as before:

Theorem

The games $\mathcal{P}(Q, \gamma)$ and $\mathcal{G}_\infty(q, Q, \gamma)$ are essentially equivalent, that is:

Empty wins $\mathcal{P}(Q, \gamma)$ iff $\forall q \in Q$ *Cut* wins $\mathcal{G}_\infty(q, Q, \gamma)$, and

Nonempty wins $\mathcal{P}(Q, \gamma)$ iff $\forall q \in Q$ *Choose* wins $\mathcal{G}_\infty(q, Q, \gamma)$.

Note: *Nonempty* wins $\mathcal{P}(Q, \gamma)$ if and only if Q is $<\gamma^+$-strategically closed. By the above, we can thus characterize strategic closure in terms of cut and choose games on partial orders.

One out of four directions of proof

(Velickovic)

If *Choose* wins $\mathcal{G}_\infty(q, Q, \omega)$ for all $q \in Q$, then *Nonempty* wins $\mathcal{P}(Q, \omega)$.

Proof: Suppose that *Empty* starts a run of the game $\mathcal{P}(Q, \omega)$ by playing some $q_0 \in Q$. Let σ be a winning strategy for *Choose* in the game $\mathcal{G}_\infty(q_0, Q, \omega)$. We can identify σ with a function F which on input $\langle W_i \mid i \leq n \rangle$ for some $n < \omega$ considers the partial run in which the moves of *Cut* are given by the W_i, the moves of *Choose* at stages below n are given by the strategy σ, and $F(\langle W_i \mid i \leq n \rangle)$ produces a response $w_n \in W_n$ for *Choose* to this partial run. We describe a winning strategy for *Nonempty* in the game $\mathcal{P}(Q, \omega)$, making use of an auxiliary run of $\mathcal{G}_\infty(q_0, Q, \omega)$ according to σ. Let $Q(\leq q) = \{r \in Q \mid r \leq q\}$.

In order to define the first move of *Nonempty*, consider the set

$$\Sigma_\emptyset = \{F(\langle W\rangle) \mid W \text{ is a maximal antichain of } Q(\leq q_0)\}.$$

There is $r_0 \leq q_0$ such that $Q(\leq r_0) \subseteq \Sigma_\emptyset$, for otherwise the complement of $\Sigma_\emptyset$ is dense below q_0, and hence there is a maximal antichain W of $Q(\leq q_0)$ that is disjoint from $\Sigma_\emptyset$, however $F(\langle W\rangle) \in W \cap \Sigma_\emptyset$, which is a contradiction. Let *Nonempty* pick such r_0 as their first move.
In the next round, suppose that *Empty* plays $q_1 \leq r_0$. Let *Cut* play a maximal antichain W_0 of $Q(\leq q_0)$ such that $F(\langle W_0\rangle) = q_1$ as their first move in the game $\mathcal{G}_\infty(q_0, I, \gamma)$. Consider the set

$$\Sigma_{\langle W_0\rangle} = \{F(\langle W_0, W\rangle) \mid W \text{ is a maximal antichain of } Q \text{ below } q_0\}.$$

As before, there is $r_1 \leq q_1$ such that $Q(\leq r_1) \subseteq \Sigma_{\langle W_0\rangle}$, and we let *Nonempty* respond with such r_1.

Proceeding in this way, the choices of *Choose* are exactly the choices of *Empty*, and hence they have a lower bound in Q, for *Choose* was following their winning strategy σ. So *Nonempty* wins $\mathcal{P}(Q, \gamma)$, as desired. $\square$

Another open question

By classical results of Galvin, Jech and Magidor on precipitous games, and by the equivalences presented above, for all of the cut and choose games presented in this talk, the consistency strength of *Choose* winning them is that of a measurable cardinal.

Question

Can we find nontrivial examples of when we can separate the properties of *Choose* winning various of our cut and choose games?