Large Cardinal Compactness

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Compactness

First order logic is *compact*:

Given any first order theory T, if every finite set of sentences from T is consistent (i.e., has a model), then T itself is consistent.

Non-compactness

Stronger logics usually fail to have this property.

For example, second order logic (variables for and quantifications over subsets of the domain of a given structure are allowed) is not compact.

To see this, consider the theory T that consists of the following statements (we let uppercase letters denote second order variables):

- $<$ is a linear ordering, (first order statement)
- $x_i > x_{i+1}$ for $i < \omega$,
- $<$ is a well-ordering: $\forall A \exists x \in A \forall y \in A \; x \leq y$.

Passing to higher cardinalities

Perhaps this is just a problem with respect to finiteness:

Definition 1

For a cardinal κ, we say that a theory T is $<\kappa$-consistent if every subset of T of size less than κ has a model.

Definition 2

κ is a *strong compactness cardinal* for second order logic $\mathcal{L}^2$ if whenever T is a $<\kappa$-consistent $\mathcal{L}^2$-theory, then T itself has a model.

Theorem (Magidor, 1971)

κ is a strong compactness cardinal for $\mathcal{L}^2$ if and only if there is an extendible cardinal $\nu \leq \kappa$. In particular, the least strong compactness cardinal for $\mathcal{L}^2$ is the least extendible cardinal.

A cardinal ν is *extendible* if $\forall \eta > \nu \, \exists \zeta \, \exists j \colon V_\eta \to V_\zeta \ \mathrm{crit}(j) = \nu$ and $j(\nu) > \eta$.

Strongly compact cardinals and more

By their very definition, a cardinal κ is *strongly compact* if κ is a strong compactness cardinal for the logic $\mathcal{L}_{\kappa,\kappa}$ that is first order logic together with infinitary conjunctions and disjunctions of size less than κ and simultaneous quantification over any number of less than κ many variables.

Weak compactness and *measurability* can also be characterized by compactness properties of $\mathcal{L}_{\kappa,\kappa}$, considering only theories of size κ.

Abstract Logic

An abstract logic $\mathcal{L}$ provides, for every given language τ, the class of τ-formulas of $\mathcal{L}$ and a corresponding satisfaction relation for these formulas, obeying a small number of fairly weak and natural axioms.

The most interesting (and least obvious) axiom is: There is a (least) cardinal o such that for every language τ, any τ-formula of $\mathcal{L}$ contains less than o symbols of the language.

- For first or second order logic, $o = \omega$.
- For $\mathcal{L}_{\kappa,\kappa}$ with κ regular, $o = \kappa$.
- $\mathcal{L}_{\infty,\omega}$ (arbitrary conjunctions and disjunctions) is not an abstract logic in the above sense.

Theorem (Makowsky, 1985)

Every abstract logic has a compactness cardinal if and only if Vopěnka's principle holds.

Vopěnka's principle is the statement that for any class of structures in a given signature, there's an elementary embedding between two of them.

Characterizing large cardinals via compactness

- Can we characterize other large cardinal properties of a given cardinal κ via certain compactness properties of generalized logics?

$\rightarrow$ that is, by statements of the form

Every theory that satisfies a certain property regarding the consistency of its $<\kappa$-sized fragments is itself consistent?

- Can we do so by using logics that are not parametrized by κ?

$\rightarrow$ Like $\mathcal{L}^2$ in Magidor's characterization of extendibility, or the class of all abstract logics, but not parametrized logics like $\mathcal{L}_{\kappa,\kappa}$?

Let's first see what we can do with second order logic!

A first step

Given a certain large cardinal property φ, let's try to find a sequence of $<\kappa$-consistent second order theories T_κ for cardinals κ so that T_κ is consistent if and only if some $\lambda \leq \kappa$ satisfies $\varphi(\lambda)$. Our language will be constant symbols c_x for $x \in V_{\kappa+1}$ and constant symbols d_γ for $\gamma \leq \kappa$.

If $\varphi(\lambda) \equiv$ "λ is measurable", T_κ contains:

- The elementary first order diagram of $V_{\kappa+1}$, making use of the c_x.
- All (first order) sentences of the form $d_\beta \in d_\gamma \in c_\kappa$ for $\beta < \gamma \leq \kappa$.
- The (second order) statement that the $\in$-relation is wellfounded.

If T_κ is consistent, this gives us an elementary embedding $j\colon V_{\kappa+1} \to N$, $x \mapsto (c_x)^N$ with a transitive structure N and with $\operatorname{crit}(j) \leq \kappa$.
On the other hand, the ultrapower embedding obtained from the measurability of some $\nu \leq \kappa$ easily yields the consistency of T_κ.

Strong cardinals

An analogous theory for strong cardinals: Fix some cardinal $\lambda > \kappa$. Our language will be constant symbols c_x for $x \in V_{\kappa+1}$ and constant symbols d_γ for $\gamma < \lambda$. T_κ^λ contains:

- The elementary first order diagram of $V_{\kappa+1}$, making use of the c_x.
- All (first order) sentences of the form $d_\beta \in d_\gamma \in c_\kappa$ for $\beta < \gamma < \lambda$.
- The (second order) statement that the $\in$-relation is wellfounded.
- The (second order) statement that for every $\gamma < \lambda$, the d_γ-th level of the cumulative hierarchy exists and is equal to V_{d_γ}.

If T_κ^λ is consistent, this gives us an elementary embedding $j \colon V_{\kappa+1} \to N$ with a transitive N, with $\mathrm{crit}(j) \leq \kappa$ and with $V_\lambda \subseteq N$. Using all $\lambda > \kappa$, this yields that some $\nu \leq \kappa$ is a strong cardinal. The reverse direction, starting from a strong cardinal $\nu \leq \kappa$, is again pretty much straightforward.

Supercompact cardinals

A similar approach also works for supercompact cardinals.

What's next?

Let's concentrate again on the case of measurable cardinals (strong and supercompact cardinals are handled similarly). We know that the theory T_κ defined there is $<\kappa$-consistent. We want to obtain a result of the following form:

Goal Theorem

For every cardinal κ, there is a certain (definable in κ) natural and rich class C of second order theories such that every theory in C is consistent if and only if there is a measurable cardinal that is $\leq\kappa$.

We can't take C to be the class of all $<\kappa$-consistent theories, for this would give us an extendible cardinal by Magidor's result. We could take $C = \{T_\kappa\}$, but that would not be a very natural class of theories, and it would certainly not be rich (in the sense of containing as many theories as possible).

Outward Compactness - Basic Idea

Reminder: If $\varphi(\lambda) \equiv$ "λ is measurable", T_κ contains:

- The elementary first order diagram of $V_{\kappa+1}$, making use of the c_x.
- All (first order) sentences of the form $d_\beta \in d_\gamma \in c_\kappa$ for $\beta < \gamma \leq \kappa$.
- The (second order) statement that the $\in$-relation is wellfounded.

T_κ is not only $<\kappa$-consistent, but is also $<\kappa$-consistent in all outer models of the universe V in which κ is still a cardinal.

(well-foundedness is absolute – we may just take the same witnessing structures as in the ground model)

Problem: This is not formalizable.

But: It almost is.

The main concept

Let ZFC^* denote the fragment of ZFC with the axioms of separation and replacement for Σ_2-formulae only.

Definition 1

An $\mathcal{L}^2$-theory T is *$<\kappa$-outward consistent* if for all cardinals $\lambda < \kappa$ and all $\theta > \kappa$ with $T \in V_\theta$, the partial order $\mathrm{Col}(\omega, V_\theta)$ forces that whenever $N \models \mathrm{ZFC}^*$ is an outer model of V_θ^V which preserves λ as a cardinal, T is $<\lambda$-consistent in N.

Definition 2

A cardinal κ is an *outward compactness cardinal* for $\mathcal{L}^2$ if all $<\kappa$-outward consistent theories are consistent.

Theorem 1

κ is an outward compactness cardinal for $\mathcal{L}^2$ if and only if there is a measurable cardinal $\nu \leq \kappa$. In particular, the least measurable cardinal is the least outward compactness cardinal for $\mathcal{L}^2$.

Strong cardinals

Definition 3

An $\mathcal{L}^2$-theory T is *weakly $<\kappa$-outward consistent* if for all cardinals $\lambda < \kappa$ and all $\theta > \kappa$ with $T \in V_\theta$ and all infinite cardinals $\lambda < \kappa$, the partial order $\mathrm{Col}(\omega, V_\theta)$ forces that whenever $N \models \mathrm{ZFC}^*$ is an outer model of V_θ^V with $V_\lambda^N = V_\lambda^V$ which preserves κ as a cardinal, T is $<\kappa$-consistent in N.

Definition 4

A cardinal κ is a *strong outward compactness cardinal* for $\mathcal{L}^2$ if all weakly $<\kappa$-outward consistent theories are consistent.

Theorem 2

κ is a strong outward compactness cardinal for $\mathcal{L}^2$ if and only if there is a strong cardinal $\nu \leq \kappa$. In particular, the least strong cardinal is the least outward compactness cardinal for $\mathcal{L}^2$.

Further results

- There's a highly analogous result for supercompact cardinals.
- There's an analogous result for extendible cardinals (but there's already Magidor's compactness characterization of extendible cardinals, so perhaps this isn't overly interesting).
- There's a somewhat similar result for ω_1-strongly compact cardinals (exact characterization, not just for the least ω_1-strongly compact).
- We also characterize when Ord is Woodin by a compactness property of abstract logics.
- Similarly for Vopěnka's principle, but as there's Makowsky's result, this is perhaps not as interesting.

Proof for measurable cardinals, Part 1

Assume that κ is a measurable cardinal, let T be an $\mathcal{L}^2$-theory, and assume that T is $<\kappa$-outward consistent. We need to show that T is consistent. Let $j\colon V \to M$ be a suitable iterate of a measurable ultrapower embedding for κ such that $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > |T|$ is a cardinal (of V). Pick a sufficiently large strong limit cardinal θ of cofinality greater than κ, so that V_θ satisfies ZFC^*, and $j(\theta) = \theta$. By elementarity, $j(T)$ is $<j(\kappa)$-outward consistent in M. It follows that in every $\mathrm{Col}(\omega, V_\theta^M)$-generic extension of M, whenever N is an outer model of V_θ^M that satisfies ZFC^*, then N satisfies the following first order statement $\psi(j(\kappa), j(T))$:

$$\text{whenever } \lambda < j(\kappa) \text{ is a cardinal, } j(T) \text{ is } {<}\lambda\text{-consistent.}$$

Let $\mathfrak{r} \subseteq \omega$ be a real that codes $\langle V_\theta^M, \in\rangle$ in a $\mathrm{Col}(\omega, V_\theta^M)$-generic extension of M.

Proof for measurable cardinals, Part 2

The above property of V_θ^M in this extension is now a $\mathbf{\Pi}^1_2$-property of $\mathfrak{r}$ (saying that whenever $\mathfrak{n}$ codes an extensional wellfounded binary relation on ω that is isomorphic to an outer model of the model coded by $\mathfrak{r}$, and this model satisfies ZFC_2, then it satisfies a certain first order statement), and is thus absolute to any $\mathrm{Col}(\omega, V_\theta^M)$-generic extension of V containing $\mathfrak{r}$ as an element. But V_θ is an outer model of V_θ^M that satisfies ZFC^* in such an extension, and $j(\kappa)$ is a cardinal in V_θ, as it is a cardinal in V by our choice of embedding j. We may thus conclude that $j(T)$ is $<j(\kappa)$-consistent in V_θ.

Now note that $j[T] \subseteq j(T)$ is of size less than $j(\kappa)$ by our choice of embedding j, and thus that $j[T]$ is consistent in V_θ. By the nature of second order logic, it follows that $j[T]$ is also consistent in V. Finally, note that we may identify $j[T]$ and T via a renaming of symbols, using the finitary character of $\mathcal{L}^2$-formulae. This yields that in fact, T is consistent, as desired.