Canonical Function Coding over a Stationary Set

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May 21, 2013

Abstract

Canonical function coding at κ as defined in [2] was claimed in [2], Remark (a) after the proof of Theorem 39, to be $<\kappa$-directed closed. There is a minor gap in that argument and we want to take the opportunity here to provide a corrected and additionally simpler approach to that claim, i.e. present a short and self-contained improved version of Theorem 39 of [2]. Thus we provide a $<\kappa$-directed closed (in fact even $<\kappa$-linked closed), κ^+-cc forcing to introduce a boldface definable wellorder of H_{κ^+}, assuming κ is regular, $\kappa^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$. The main idea for the simplification (and in fact the whole proof) is essentially contained in [1]. This result slightly improves parts of a result in [3], where a $\mathbf{\Delta}^1_2$-definable wellorder of H_{κ^+} is introduced by $<\kappa$-closed (and in fact $<\kappa$-directed closed, but not $<\kappa$-linked closed), κ^+-cc forcing (however without assuming $2^\kappa = \kappa^+$).

Definition 1 *Assume P is a partially ordered set and κ is a cardinal.*

- *$D \subseteq P$ is* directed *if any two conditions in D have a common extension in D.*
- *$D \subseteq P$ is* linked *if any two conditions in D have a common extension in P.*
- *P is $<\kappa$-directed closed if any directed subset of P of size less than κ has a lower bound in P.*
- *P is $<\kappa$-linked closed if any linked subset of P of size less than κ has a lower bound in P.*

Assume κ is regular, $\kappa^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$. For every $\gamma \in [\kappa, \kappa^+)$, let $f_\gamma \colon \kappa \to \gamma$ be a bijection. We will define a forcing P to introduce a boldface definable wellorder of H_{κ^+} which will be $<\kappa$-linked closed and κ^+-cc. P is an iteration of length κ^+ with $<\kappa$-support. P_0 and P_1 both denote the forcing to add a Cohen subset of κ. Let B denote the generic subset of κ added by P_0, let S denote the generic subset of κ added by P_1. P is trivial in the interval $[2, \kappa)$, so $P_{<\kappa}$, the iteration up to κ, is equivalent to just $P_0 * P_1$.

We will inductively define P_α for $\alpha \in [\kappa, \kappa^+)$ and a predicate $A \colon [\kappa, \kappa^+) \to 2$. As this notation already suggests, we will identify predicates or sets of ordinals and their characteristic functions. For every $\alpha \in (\kappa, \kappa^+]$, $A \restriction \alpha$ will be a $P_{<\alpha}$-name. The definitions will be such that for any $\alpha \in [\kappa, \kappa^+)$, P_α can be defined given $A \restriction \alpha$. We fix a wellorder $\mathcal{W}$ of H_{κ^+} of order-type κ^+. Given $\alpha \in [\kappa, \kappa^+)$, $A(\alpha)$ is a $P_{<\alpha+1}$-name for either 0 or 1 such that in any $P_{<\alpha+1}$-generic extension,

$A(\alpha)$ is evaluated to 1 iff $\alpha = \prec \beta, \gamma, \delta \succ$, $\dot{x}$ is the β^{th} (in the sense of $\mathcal{W}$) $P_{<\gamma}$-name for a subset of κ, $\delta < \kappa$ and the induced $P_{<\gamma}$-generic decides that $\dot{x}(\delta) = 1$.

If $i \in [\kappa, \kappa^+)$, P_i is the forcing defined in the $P_{<i}$-generic extension as follows. A condition t in P_i is a κ-Cohen condition s.t. $\{\eta < |t| \mid t(\eta) = 1\}$ is a closed, bounded subset of κ and[1]

$$\forall \eta \in (t \cap S)\ B(\text{ot}\, f_i[\eta]) = A(i).^2$$

Conditions in P_i are ordered by end-extension.

If $\alpha \leq \kappa^+$, p is a condition in $P_{<\alpha}$ and $i \in [\kappa, \alpha)$, we denote $p(i)$ by p^{**}_i. We write $b(p)$ to denote $p(0)$ and we write $s(p)$ to denote $p(1)$. We define the *club support of* p as $\text{C-supp}(p) = \{i \mid p^{**}_i \neq \mathbf{1}\}$. Let G be P-generic.

Claim 2 *Asume $\lambda < \kappa$ and D is a linked set of conditions in $P_{<\alpha}$. Let r be the componentwise union of D, i.e. r is a sequence of length α such that $b(r) = \bigcup_{p \in D} b(p)$, $s(r) = \bigcup_{p \in D} s(p)$ and $r^{**}_\gamma = \bigcup_{p \in D} (p)^{**}_\gamma$ for $\gamma \in \text{C-supp}(r) := \bigcup_{p \in D} \text{C-supp}(p)$. If $p \restriction \gamma \Vdash |b(r)| = |s(r)| = |r^{**}_\gamma|$ for every $\gamma \in \text{C-supp}(r)$, then D has a lower bound in $P_{<\alpha}$.*

Proof: Let $\xi = |b(r)|$. We build q out of r by setting $b(q) = b(r) \cup \{(\xi, 0)\}$, $s(q) = s(r) \cup \{(\xi, 0)\}$ and $q^{**}_\gamma = r^{**}_\gamma \cup \{\xi\}$ for every $\gamma \in \text{C-supp}(r)$. Using that $q \restriction \gamma^{\oplus}$ forces that either $\sup(r^{**}_\gamma) = \xi$ or $\exists p \in D\ \sup r^{**}_\gamma \in p^{**}_\gamma$ and using that $\mathbf{1} \Vdash s(q)(\xi) = 0$ it is trivial to check that q is a condition in $P_{<\alpha}$ extending each $p \in D$. $\square$

Claim 3 *Suppose $\kappa \leq \alpha \leq \kappa^+$. Then the following hold:*

1. *$P_{<\alpha}$ has a dense subset $D_{<\alpha}$ of conditions p such that $p^{**}_\gamma \in \mathbf{V}$ for every $\gamma \in \text{C-supp}(p)$.*
2. *The following set is dense in $D_{<\alpha}$:*

$$E_{<\alpha} = \{p \in D_{<\alpha} \mid \forall \gamma \in \text{C-supp}(p)\ p \restriction \gamma \Vdash |b(p)| = |s(p)| = |p^{**}_\gamma|\}.$$

3. *$E_{<\alpha}$ is $<\kappa$-linked closed.*

Proof of 1: If $\alpha = \beta + 1$ is a successor ordinal and given any condition $p \in P_{<\alpha}$, we use 2 and 3 inductively to decide p^{**}_β. If α is a limit ordinal of cofinality κ or $\alpha = \kappa^+$, the result follows inductively by 1 as any condition $p \in P_{<\alpha}$ has support bounded in α. Assume that α is a limit ordinal of cofinality $\lambda < \kappa$ and p is a condition in $P_{<\alpha}$. Let $\langle \alpha_i \mid i < \lambda \rangle$ be increasing, continuous and cofinal in α. Build a decreasing sequence of conditions $\langle p^i \mid i < \lambda \rangle$ such that $p^0 = p$ and for every $i < \lambda$, $p^{i+1} \restriction \alpha_i \in E_{<\alpha_i}$ and $p^{i+1}[\alpha_i, \kappa^+) = p^i[\alpha_i, \kappa^+)$. It follows by Claim 2 that $\langle p^i \mid i < \xi \rangle$ has a lower bound q for every $\xi \leq \lambda$ and in fact the construction of q in the proof of that claim shows that $q \restriction \alpha_\xi \in E_{<\alpha_\xi}$. Hence we

[1] We write $\eta \in t$ to abbreviate $t(\eta) = 1$. We write $\sup(t)$ for $\sup(\{\eta \mid t(\eta) = 1\})$. Using predicates giving rise to closed, bounded subsets of κ instead of closed, bounded subsets of κ themselves is the necessary correction to make the proof work, as mentioned in the abstract.

[2] This constitutes the simplification mentioned in the abstract - we don't demand this kind of coding property for all $\eta \in t$, but only for $\eta \in (t \cap S)$, where S is the stationary subset of κ previously added by Cohen forcing.

can perform the above construction and if q is the lower bound of $\langle p^i \mid i < \lambda\rangle$ as obtained in the proof of Claim 2, then $q \in E_{<\alpha} \subseteq D_{<\alpha}$.

Proof of 2: Immediate by 1 and just lengthening components by zeroes.

Proof of 3: Immediate by Claim 2. □

Claim 4 *P is κ^+-cc.*

Proof: If $\alpha < \kappa^+$, $\{p \in D_{<\alpha} \mid p(0)$ decides $p(1)\}$ is dense in $P_{<\alpha}$ and has size κ. $P_{<\kappa^+}$ is the direct limit of $\langle P_{<\alpha} \mid \alpha < \kappa^+\rangle$ and thus is κ^+-cc. □

Claim 5 *Any condition $p \in P$ has an extension q such that for any given $\xi < \kappa$ and any $\zeta \in [\kappa, \kappa^+)$, $q\restriction\zeta \Vdash \sup q^{**}_\zeta > \xi$.*

Proof: Let X be an antichain of $P_{<\zeta}$ below $p\restriction\zeta$ deciding $\sup(p^{**}_\zeta)$. Choose $\xi' > \xi$ such that $p(0) \Vdash \xi' \notin |s(p)|$. Choose q to extend p such that $s(q) \supseteq s(p) \cup \{(\xi', 0)\}$ and choose q^{**}_ζ such that whenever $x \in X$ forces that $\sup(p^{**}_\zeta) \leq \xi$, then x forces that $q^{**}_\zeta = p^{**}_\zeta \cup \{\xi'\}$ and such that x forces $q^{**}_\zeta = p^{**}_\zeta$ otherwise. □

Claim 6 *A is definable from S and B over $H^{\mathbf{V}[G]}_{\kappa^+}$.*

Proof: An easy density argument using Claim 5 shows that in $H^{\mathbf{V}[G]}_{\kappa^+}$,

$$\gamma \in A \iff \exists C \text{ club } \forall \delta \in C \cap S \text{ ot } f_\gamma[\delta] \in B.$$

Moreover the same is true with f_γ replaced by any bijection between κ and γ. □

Claim 7 *In $\mathbf{V}[G]$, $H_{\kappa^+} = L_{\kappa^+}[A]$.*

Proof: An obvious density argument. □

Theorem 8 *Forcing with P introduces a $\boldsymbol{\Delta}^1_1$-definable wellorder of H_{κ^+}.*

Proof: By Claim 7, using the standard $\Delta^1_1(A)$-wellorder of $L_{\kappa^+}[A]$ and using that A is definable from S and B over $H^{\mathbf{V}[G]}_{\kappa^+}$ by Claim 6. □

Note: As in Theorem 39 of [2], it is easily possible to additionally make any given ground model subset of H_{κ^+} $\boldsymbol{\Delta}^1_1$-definable over $H^{\mathbf{V}[G]}_{\kappa^+}$.

References

[1] David Asperó and Sy-David Friedman. *Large cardinals and locally defined well-orders of the universe.* Annals of Pure and Applied Logic 157, no. 1, pp 1–15, 2009.

[2] Sy-David Friedman and Peter Holy. *Condensation and Large Cardinals.* Fundamenta Mathematicae 215, no. 2, pp 133-166, 2011.

[3] Philipp Lücke. Σ^1_1*-definability at uncountable regular cardinals.* Journal of Symbolic Logic 77, no. 3, pp 1011–1046, 2012.