# Failures of the Forcing Theorem

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# Set Forcing

Forcing is a way to obtain new models of (perhaps fragments of) *ZFC*, starting from old ones. Given a ctm $M$ of *ZFC*, we consider some partial order $P \in M$ and a *generic filter* $G \subseteq P$ that meets all dense subsets of $P$ that are elements of $M$. We form the generic extension $M[G]$, which is the collection of all $\sigma^G$ for $\sigma \in M$, with $\sigma^G$ defined inductively as

$$\sigma^G = \{\tau^G \mid \exists\, p \in G\ (\tau, p) \in \sigma\}.$$

If $M$ is a model of *ZFC*, so is $M[G]$.
Moreover the *Forcing Theorem* holds.

# The Forcing Theorem

We say that $p \Vdash \phi$ iff whenever $G$ is $P$-generic over $M$ with $p \in G$ then $M[G] \models \phi$. The Forcing Theorem consists of two parts:

**Lemma (Definability of the Forcing Relation)**

*For every formula $\varphi(x_1, \ldots, x_n)$ there is a formula $\psi$ s.t. for any $\vec{a} \in M$ and $p \in P$,*

$$p \Vdash \varphi(a_1, \ldots, a_n) \iff M \models \psi(a_1, \ldots, a_n, p, P, \leq_P).$$

**Lemma (Truth Lemma)**

*For every formula $\varphi(x_1, \ldots, x_n)$ and $\vec{a} \in M$, if $M[G] \models \varphi(a_1^G, \ldots, a_n^G)$ with $G$ generic over $M$, then $\exists p \in G \; p \Vdash \varphi(a_1, \ldots, a_n)$.*

# Class Forcing

A different way to obtain new models of $ZFC$ (or, at least fragments of that theory) is by class forcing. That is, we do not require that $P \in M$, but only that $P$ is a definable subset of $M$, or a class of $M$, and we require a $P$-generic filter over $M$ to intersect with all dense $M$-subclasses of $P$. We still form $M[G]$ as the collection of all $\sigma^G$ for $\sigma \in M$ as before.

There's some degree of freedom in the choice of second order objects for the generic extension: one may want to add a class predicate for the ground model $M$, one for the generic filter $G$, and maybe even more... but this is mostly irrelevant for this talk (except for a few remarks that follow right now).

Does $M[G]$ satisfy the axioms of $ZFC$? Maybe at least if we do not allow for the generic filter $G$ as a predicate (in the Replacement and Separation axioms)?

# Destroying Replacement

Consider the forcing $Coll(\omega, Ord)$, with conditions $p \colon n \to Ord$, $n \in \omega$, ordered by reverse inclusion. A generic filter $G$ for this forcing gives rise to a sequence of length $\omega$ that is cofinal in the ordinals. Thus $M[G]$ does not satisfy Replacement if we allow reference to the predicate $G$. However $Coll(\omega, Ord)$ does not add any new sets (we will verify this later), so $M[G] = M$ satisfies the axioms of $ZFC$ if we do not allow reference to $G$.

One can modify $Coll(\omega, Ord)$ so that the generic predicate is coded into the values of the continuum function (by adding Cohen subsets of suitable infinite successor cardinals), thus obtaining a model $M[G]$ – with $G$ generic over $M$ for this modification – so that $M[G]$ does not satisfy Replacement, even if we do not allow reference to the predicate $G$.

Conclusion: Class Forcing can sometimes destroy Replacement.

Let us look again at $Coll(\omega, Ord)$. It is quite easy to verify that $Coll(\omega, Ord)$ satisfies the Forcing Theorem (we will later hint at a possible proof). If $g \colon \omega \to Ord$ is the generic function, then an easy density argument shows that $\{n \in \omega \mid g(n) \text{ even}\}$ is not in $V$. However, $Coll(\omega, Ord)$ does not add any new sets, so this means that Separation fails in $Coll(\omega, Ord)$-generic extensions (if we allow reference to the generic predicate; as before, this can also be eliminated). In fact, the following holds:

**Theorem**

*Assume P is a class forcing that satisfies the Forcing Theorem and preserves Separation. Then P preserves Replacement.*

Preservation of Replacement (*pretameness*) implies the Forcing Theorem.

**Question**

Does preservation of Separation imply the Forcing Theorem to hold?

# Definability and Truth

**Observation**

Let $P$ be a class forcing. If the Definability Lemma holds for $P$, then the Truth Lemma holds for $P$.

*Proof.* A brief inspection of the usual proof of the Forcing Theorem in set forcing, for example in Kunen's book. □

# Why should the forcing relation ever be non-definable?

For set forcing, we can build a formula that defines the forcing relation for a given formula $\varphi$ by induction on the complexity of $\varphi$. But the crucial point, for generalizing this to class forcing, is in fact the definability of the forcing relation for atomic formulas, that is formulas of the form $\sigma = \tau$ and $\sigma \in \tau$. If the forcing relation is definable for those, then as for set forcing one can show that it is definable for all first order formulas. Back to atomic formulas, the following equivalence holds (and is used to define the forcing relation for set forcing):

$$p \Vdash \sigma \in \tau \iff \forall q \leq p \, \exists r \leq q \, \exists \langle \rho, s \rangle \in \tau \; (r \leq s \, \wedge \, r \Vdash \sigma = \rho).$$

In a similar way, the validity of $p \Vdash \sigma = \tau$ can be reduced to checking the validity of the $\in$-relation to hold for names of ranks smaller than those of $\sigma$ and $\tau$. However note that we introduce an unbounded quantifier at each induction step. So this does not result in a $\Sigma_n$-formula defining $\{(p, \sigma, \tau) \mid p \Vdash \sigma \in \tau\}$, for any $n \in \omega$.

# Sy Friedman's forcing...

We work in a ctm $M$ of $ZFC$.

## Definition

Let $\mathbb{F}$ denote the forcing consisting of triples $\langle d, e, f\rangle$ with the following properties:

1. $d \subseteq \omega$ is a finite set;
2. $e$ is an acyclic relation on $d$;
3. $f\colon dom\, f \to M$, where $dom\, f \in \{\emptyset, d\}$;
4. $f$ is injective and if $dom\, f = d$ then for $i, j \in d$ we have

$$\langle i, j\rangle \in e \iff f(i) \in f(j),$$

i.e. $f$ is a partial isomorphism between $\langle \omega, e\rangle$ and $\langle M, \in\rangle$.

The ordering is given by $\langle d', e', f'\rangle \leq \langle d, e, f\rangle$ iff $d' \supseteq d$, $e' \supseteq e$, $e' \upharpoonright d = e$ and $f' \supseteq f$.

$\mathbb{F}$ adds a bijection $F$ between $\omega$ and $M$ and a relation $E$ on $\omega$ so that

$$\langle M, \in \rangle \models \varphi(x) \iff \exists n \in \omega \ \left[F(n) = x \wedge \varphi^{\langle \omega, E \rangle}(n)\right]$$

$$\iff \exists n \in \omega \, \exists p = \langle d, e, f \rangle \in \mathbb{F} \ f(n) = x \wedge p \Vdash \varphi^{\langle \check{\omega}, \dot{E} \rangle}(\check{n}).$$

**Lemma (Friedman, unpublished)**

*The forcing relation for first order formulas with quantification only over $\omega$ is not uniformly definable for $\mathbb{F}$.*

*Proof.* Otherwise $M$ has a definable truth definition (for formulas with parameters) by the above equivalence, contradicting Tarski. □

However, we do not know if $\mathbb{F}$-generic extensions possess a formula that is universal for $\Delta_0$ formulas, thus we cannot immediately infer that there is a single formula for which $\mathbb{F}$ fails the Definability Lemma.

# ...does not satisfy the Forcing Theorem,

We can however replace $\Delta_0$ formulas with quantification over $\omega$ by equivalent $\mathcal{L}_{\infty,0}$-formulas with junctors over $\omega$. Thus the failure of the forcing theorem for $\mathbb{F}$ follows from:

**Theorem**

*Assume $P$ is a class forcing for which the forcing relation for the formula $v_0 = v_1$ is definable. Then there is a uniform definition of the forcing relation for $\mathcal{L}_{\infty,0}$ formulas with parameters.*

*Idea of Proof:* We construct, by induction on formula complexity, a definable class $\{(\varphi, \mu_\varphi, \nu_\varphi) \mid \varphi \in \mathcal{L}_{\infty,0}\}$ with the property that

$$\mathbf{1} \Vdash \varphi \leftrightarrow [\mu_\varphi = \nu_\varphi] .$$

$\square$

# even in a very strong sense.

We can even do better and find fixed names $\nu$ and $\mu$ such that the forcing relation for the formula $\nu = \mu$, i.e. $\{p \in \mathbb{F} \mid p \Vdash \nu = \mu\}$ is not definable.

This is based on a parametrized version of the following:

**Lemma**

*There exist $\mu$, $\nu$ and $\{(\varphi, q_\varphi) \mid \varphi \in \mathcal{L}_\in\} \in M$ such that*

$$M \models \varphi \iff q_\varphi \Vdash \mu = \nu.$$

We consistently also obtain non-amenability of the forcing relation:

**Lemma**

*If $M$ is a model in which every element is lightface definable, then*

$$X = \{q_\varphi \mid \varphi \in L_\in, q_\varphi \Vdash \mu = \nu\}$$

*is not an element of $M$.*

# Forcing Definable Ordinals

We will make use of the following.

**Theorem (Hamkins-Linetsky-Reitz)**

*If $M$ is a ctm of ZFC then there is a tame (ZFC-preserving) class forcing $\mathbb{P}$ and a $\mathbb{P}$-generic extension of $M$ in which every ordinal is lightface definable.*

If $M \models V = L$, $\mathbb{P}$ can simply be taken to be the two-step iteration of the forcing that adds a Cohen subset $U$ of $Ord$, followed by the reverse Easton iteration that codes $U$ into the continuum function. By a careful choice of $U$, one can make sure that every ordinal of $M$ is lightface definable in $\langle M, \in, U \rangle$, and thus obtain a generic extension as desired.

Lemma (Hamkins-Linetsky-Reitz)

*If $M$ is a ctm of Kelley-Morse class theory, then $M$ does not have lightface definable ordinals.*

*Proof:* Using that a truth predicate for $M$ is a second order object of $M$, every ordinal being lightface definable in $M$ implies that we obtain a bijection between $\omega$ and $Ord$ as a second order object, contradicting the axiom of Replacement. □

Lemma (Antos-Kuby)

*If $P$ is a tame class forcing for a Kelley-Morse model $M$ and $G$ is Kelley-Morse generic for $P$ over $M$, in the sense that $G$ meets all dense subsets of $P$ that are second-order objects of $M$, then $M[G]$ is the underlying set of a model of Kelley-Morse class theory.*

In particular, if $M$ is the underlying set of a model of Kelley-Morse class theory, then for every $p \in \mathbb{P}$ (the forcing from the previous slide), there is a $\mathbb{P}$-generic $G \ni p$ so that in $M[G]$, not every ordinal is lightface definable.

# A Failure of the Truth Lemma

**Theorem**

*If $M$ is the underlying set of a ctm of Kelley-Morse, then the two-step iteration $\mathbb{P} * \dot{\mathbb{F}}$ does not satisfy the truth lemma over $M$.*

This will follow from the following:

## Theorem

*Assume $M$ is a ctm of ZFC and $P$ is a class forcing for $M$ so that*

1. *$P$ is tame.*
2. *There is a $P$-generic filter $G$ over $M$ so that $M[G]$ has lightface definable ordinals.*
3. *For every $p \in G$ there is a $P$-generic filter $\bar{G}$ such that $M[\bar{G}]$ does not have lightface definable ordinals.*

*Then the truth lemma fails for $P * \dot{\mathbb{F}}$.*

*Proof-Idea:* Forcing with $\mathbb{F}$ makes the property of an intermediate $P$-generic extension $N$ to have lightface definable ordinals expressible by an infinitary, quantifier-free formula in the final generic extension, namely by the formula

$$\Phi \equiv \bigwedge_{n \in \omega} \bigvee_{\varphi} \bigwedge_{m \in \omega} \Big( Ord(F(n)) \rightarrow [\varphi^{\langle \omega, E \rangle}(m) \iff m = n] \Big).$$

But if we choose a $P * \dot{\mathbb{F}}$-generic $G * H$ with $G$ as in (2), then by (3), no condition in $G * H$ forces that $\Phi$ holds.

# A Positive Result...

### Definition

We say that a class forcing $P$ has the *set decision property* if for every subset $A$ of $P$ and every $p \in P$ there is $q \leq p$ such that for every $a \in A$, either $q \leq a$ or $q \perp a$ ($\leftrightarrow q$ decides $\dot{G} \cap A$).

### Lemma

*If $P$ has the set decision property, then $P$ does not add new sets.*

*Proof:* Let $\sigma \in M$ be a $P$-name. Let $A$ be the set of all conditions appearing within the name $\sigma$. Let $p \in P$. By the set decision property, we can find $q \leq p$ such that for every $a \in A$, either $q \leq a$ or $q \perp a$. But such $q$ decides the evaluation $\sigma^G$ of $\sigma$ by $G$. □

# ...about class forcing that doesn't add new sets

Being able to densely decide all names leads to the following:

**Lemma**

*If $P$ has the set decision property, then $P$ satisfies the forcing theorem for atomic, and hence for all formulas.*

Perhaps slightly more surprising is the following:

**Lemma**

*If $P$ is a separative class forcing which does not add new sets (i.e. no $P$-generic extension does so), then $P$ has the set decision property.*

*Idea of proof:* Let $A \subseteq P$ be a set of conditions and let $p \in P$. Let $\sigma = \{(\check{a}, a) \mid a \in A\}$. Assuming that no $q \leq p$ decides $\dot{G} \cap A$, we construct (from the outside) a generic filter $G$ such that $\sigma^G \notin M$. $\square$

**Corollary**

*Class forcing that doesn't add new sets satisfies the Forcing Theorem.*

# $Coll(\omega, Ord)$ satisfies the set decision property

**Definition**

We say that a class forcing $P$ has the *set decision property* if for every subset $A$ of $P$ and every $p \in P$ there is $q \leq p$ such that for every $a \in A$, either $q \leq a$ or $q \perp a$ ($\leftrightarrow q$ decides $\dot{G} \cap A$).

**Claim**

$Coll(\omega, Ord)$ satisfies the set decision property.

*Proof:* Let $A$ be a subset of $P := Coll(\omega, Ord)$ and let $p \in P$. Let $\alpha^*$ be the supremum of all values of $a(n)$ for $a \in A$ and $n \in dom(a)$. Let $q = p \cup \{(dom(p), \alpha^* + 1)\}$. Then for every $a \in A$, $q \leq a$ or $q \perp a$. $\square$

**Corollary**

*$Coll(\omega, Ord)$ satisfies the forcing theorem and doesn't add new sets.*

# Unions of complete subforcings

Another sufficient condition for the forcing theorem to hold is the following:

**Theorem (Zarach,1972)**

*If $P$ is the increasing union of a sequence $\langle P_\alpha \mid \alpha \in Ord \rangle$ of complete subforcings, then the Forcing Theorem holds for $P$.*

This in particular implies that the Forcing Theorem holds for any iteration or product of set sized forcings.

**Note**

The above condition on $P$ is not sufficient for $ZF^-$ preservation.

We want to find a weaker condition that still implies the Forcing Theorem to hold, but is more widely applicable.

# Approachability by Projections...

We say that a (class) notion of forcing $P$ is *approachable by projections* if $P$ can be written as a union $P = \bigcup_{\alpha \in Ord} P_\alpha$ for a sequence $\langle P_\alpha \mid \alpha \in Ord \rangle$ of sets for which there exists a sequence of maps $\langle \pi_{\alpha+1} \mid \alpha \in Ord \rangle$ so that $\pi_{\alpha+1} \colon P \to P_{\alpha+1}$ and for every $\alpha$, the following hold:

1. $\pi_{\alpha+1}(\mathbf{1}) = \mathbf{1}$,
2. $\forall p, q \in P \quad p \leq q \to \pi_{\alpha+1}(p) \leq \pi_{\alpha+1}(q)$,
3. $\forall p \in P \, \forall q \leq_{P_{\alpha+1}} \pi_{\alpha+1}(p) \, \exists r \leq p \quad \pi_{\alpha+1}(r) \leq q$,
4. $\forall p \in P_\alpha \, \forall q \in P \quad \pi_{\alpha+1}(q) \leq p \to q \leq p$ and
5. $\pi_{\alpha+1}$ is the identity on $P_\alpha$.

Note that (in order to justify our terminology), each $\pi_\alpha$, $\alpha$ a successor ordinal, is (in particular) a projection from $P$ to $P_\alpha$ – conditions (1)-(3) are the definition of a projection. Also, using (1) and (3), it follows that each $\pi_{\alpha+1}$ is a dense embedding and thus $\pi''_{\alpha+1} G$ is $P_{\alpha+1}$-generic whenever $G$ is $P$-generic.

# ...implies the Forcing Theorem to hold

**Theorem**

*If $P$ is approachable by projections, then the Forcing Theorem holds and every new set lies in a set-generic extension of the ground model.*

This is a weakening of Zarach's condition, in the following sense.

**Lemma**

*If $P$ is the increasing union of a sequence of set-sized complete subforcings, then it is equivalent to a forcing that is approachable by projections.*

Examples: $Coll(\omega, Ord)$ and $Coll_*(\omega, Ord)$ are approachable by projections, where the latter is like the former, however domains are arbitrary finite subsets of $\omega$ - all uncountable cardinals are collapsed.

**Note**

Not every forcing that satisfies the forcing theorem is approachable by projections - Example: Jensen Coding.