OUTWARD COMPACTNESS

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Abstract. We introduce and study a new type of compactness principle for strong logics that, roughly speaking, infers the consistency of a theory from the consistency of its small fragments in certain outer models of the set-theoretic universe. We refer to this type of compactness property as *outward compactness*, and we show that instances of this type of principle for second-order logic can be used to characterize various large cardinal notions between measurability and extendibility, directly generalizing a classical result of Magidor that characterizes extendible cardinals as the strong compactness cardinals of second-order logic. In addition, we generalize a result of Makowsky that shows that Vopěnka's Principle is equivalent to the existence of compactness cardinals for all abstract logics by characterizing the principle "Ord is Woodin" through outward compactness properties of abstract logics.

1. Introduction

The work presented in this paper contributes to the study of the deep connections between large cardinals and strong logics. Its starting point is a classical result of Magidor [13] that relates the existence of extendible cardinals to the compactness properties of second-order logic $\mathcal{L}^2$. Remember that an infinite cardinal κ is *extendible* if for every ordinal $\eta > \kappa$, there is an ordinal ζ and a non-trivial elementary embedding $j : V_\eta \longrightarrow V_\zeta$ satisfying $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \eta$. Moreover, given a cardinal κ, an $\mathcal{L}^2$-theory T is *$<\kappa$-consistent* if every subtheory of T of cardinality less than κ is consistent. Finally, an infinite cardinal κ is a *strong compactness cardinal for $\mathcal{L}^2$* if every $<\kappa$-consistent $\mathcal{L}^2$-theory is consistent. The next theorem summarizes Magidor's characterization of extendible cardinals through compactness properties of $\mathcal{L}^2$:

Theorem 1.1 (Magidor [13])**.** *A cardinal κ is a strong compactness cardinal for $\mathcal{L}^2$ if and only if there is an extendible cardinal less than or equal to κ.*

The results of this paper are motivated by the aim to obtain analogous characterizations for various well-studied large cardinal properties below extendibility, e.g., measurability, strongness and supercompactness. More precisely, for a given large cardinal property φ, we want to assign, uniformly in a parameter κ, natural and rich classes $\mathcal{T}_\kappa$ of $<\kappa$-consistent $\mathcal{L}^2$-theories to infinite cardinals κ in a way that ensures that for all such κ, all theories in $\mathcal{T}_\kappa$ are consistent if and only some $\lambda \leq \kappa$ satisfies $\varphi(\lambda)$. We start by presenting a characterization of measurability through

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compactness properties of $\mathcal{L}^2$ that illustrates the main concepts on which our more general results are based.

More precisely, we show that for every uncountable cardinal κ, there is a canonical $<\kappa$-consistent $\mathcal{L}^2$-theory T_κ whose consistency is equivalent to the existence of a measurable cardinal less than or equal to κ. To obtain this theory, first recall that there is an $\mathcal{L}^2$-sentence in the language of set theory that holds in a model using this language if and only if its $\in$-relation is well-founded. Next, work in the language that extends the language of set theory by constant symbols c_x for every $x \in V_{\kappa+1}$ and constant symbols d_γ for all $\gamma \leq \kappa$, and let T_κ be the $\mathcal{L}^2$-theory consisting of the union of the following theories:

- The elementary first-order diagram of $V_{\kappa+1}$, making use of the constant symbols c_x.
- All (first-order) sentences of the form $d_\beta \in d_\gamma \in c_\kappa$ for $\beta < \gamma \leq \kappa$.
- The (second-order) sentence stating that the $\in$-relation is well-founded.

The theory T_κ is obviously $<\kappa$-consistent because for every fragment F of T_κ of cardinality less than κ, we can easily find a model of F with underlying set $V_{\kappa+1}$. Moreover, if there is a measurable cardinal less than or equal to κ, then T_κ is consistent because this assumption allows us to use iterated ultrapowers (see [12, Section 19]) to find a transitive class M and a non-trivial elementary embedding $j : V \longrightarrow M$ with $j(\mathrm{crit}(j)) > \kappa$, and this allows us to construct a model N of T_κ with underlying set $j(V_{\kappa+1})$, $c_x^N = j(x)$ for all $x \in V_{\kappa+1}$ and $d_\gamma^N = \gamma$ for all $\gamma \leq \kappa$. Finally, if T_κ is consistent, then there exists a transitive set N and an elementary embedding $j : V_{\kappa+1} \longrightarrow N$ with $j(\kappa) > \kappa$, and the existence of such an embedding directly implies that some cardinal less than or equal to κ is measurable.

Following the approach outlined above, we now want to isolate a natural consistency property of a theory T with respect to infinite cardinals κ, that strengthens $<\kappa$-consistency, is possessed by T_κ at every infinite cardinal κ, and implies the consistency of T at cardinals κ that are greater than or equal to a measurable cardinal. Our definition of this property is motivated by the observation that for every cardinal $\lambda < \kappa$, the theory T_κ is not only $<\lambda$-consistent in our ground model V, but remains $<\lambda$-consistent when we pass to outer models[1] in which λ is still a cardinal, as witnessed by the structure $V^V_{\kappa+1}$,[2] by the absoluteness of well-foundedness. In addition, if we naïvely assumed that the property that an $\mathcal{L}^2$-theory T is $<\lambda$-consistent in every outer model in which λ is a cardinal could be uniformly expressed by a first-order formula with parameters T and λ, it would follow that for every cardinal κ greater than or equal to a measurable cardinal, every $\mathcal{L}^2$-theory T with this property is in fact consistent in V. We would argue as follows: By our assumptions, there is a cardinal λ with $T \in H_\lambda$ and an elementary embedding $j : V \longrightarrow M$ with $\mathrm{crit}(j) \leq \kappa$ and $j(\kappa) > \lambda$. Using our naïve assumption, we apply the elementarity of j to conclude that the $\mathcal{L}^2$-theory $j(T)$ is $<\lambda$-consistent in every outer model of M in which λ is still a cardinal. In particular, since V is such an outer model of M, it follows that $j(T)$ is a $<\lambda$-consistent $\mathcal{L}^2$-theory in V, and this implies that the pointwise image $j[T] \subseteq j(T)$ of T under j, which is of size less than λ, is a consistent $\mathcal{L}^2$-theory in V. But this conclusion also shows that T is consistent in V, because the finitary character of $\mathcal{L}^2$-formulae allows us to identify $j[T]$ with T via the renaming of the symbols of the given language induced by j.

Since the above assumption on the uniform first-order definability of statements about truth in outer models is easily seen to be too naïve, we will introduce several concepts that allow us to turn the above approach into a mathematically sound argument.

[1] I.e., models of ZFC in which V is a transitive class containing all ordinals.

[2] Whenever M is an inner or outer model of our set-theoretic universe V and α is an ordinal, we write $V_\alpha^M = \{x \in M \mid \mathrm{rank}(x) < \alpha\}$. In particular, when working in some outer model of V, we use the notation V_α^V for initial segments of (the ground model) V.

Definition 1.2. Given a fragment F of ZFC and a transitive set M, a transitive set $N \supseteq M$ is an *outer F-model of M* if F holds in N and the sets M and N have the same ordinals.

In the following, we let ZFC* denote the fragment of ZFC that contains all the single axioms of ZFC, together with the replacement and separation schemes restricted to Σ_2-formulae. Note that this theory proves the *Σ_2-Recursion Theorem* and therefore proves that all levels of the *von Neumann hierarchy* $\langle V_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ are sets. Moreover, it is strong enough to yield the existence of the second-order satisfaction relation $\models_{\mathcal{L}^2}$ for set-sized models,[3] in a way that for every second-order formula $\varphi(v_0,\dots,v_{m-1},W_0,\dots,W_{n-1})$ with first-order variables $v_0,\dots,v_{m-1}$ and second-order variables $W_0,\dots,W_{n-1}$,[4] ZFC* proves that for every non-empty set M, all $x_0,\dots,x_{m-1} \in M$ and all $Y_0,\dots,Y_{n-1} \in \mathcal{P}(M)$, the statement

$$\langle M, \in\rangle \models_{\mathcal{L}^2} \varphi(x_0\dots,x_{m-1},Y_0,\dots,Y_{n-1})$$

holds if and only if the first order formula

$$\varphi^M(x_0\dots,x_{m-1},Y_0,\dots,Y_{n-1})$$

holds, where φ^M denotes the *first-order relativization* of φ to M.[5] In particular, the theory ZFC* allows us to uniformly speak about the consistency of $\mathcal{L}^2$-theories.

Definition 1.3. (1) Given an infinite cardinal κ, an $\mathcal{L}^2$-theory T is *$<\kappa$-outward consistent* if for all cardinals $\lambda < \kappa$ and all sufficiently large cardinals $\vartheta > \kappa$ with $T \in V_\vartheta$, the partial order $\mathrm{Col}(\omega,\vartheta)$ forces that T is $<\lambda$-consistent in every outer ZFC*-model of V_ϑ^V in which λ is a cardinal.

(2) A cardinal κ is an *outward compactness cardinal for $\mathcal{L}^2$* if all $<\kappa$-outward consistent $\mathcal{L}^2$-theories are consistent.

In this definition, note that in any $\mathrm{Col}(\omega,\vartheta)$-generic extension $V[G]$, any outer ZFC*-model N of V_ϑ^V is countable in $V[G]$: N provides any of its levels V_α^N with some cardinality less than ϑ, and since ϑ is countable in $V[G]$, it follows that N is a countable union of countable sets, and thus itself countable. Before we state our characterization of measurable cardinals, we make some easy observations that relate outward compactness cardinals to compactness cardinals.

Proposition 1.4. *(1) If κ is an outward compactness cardinal for $\mathcal{L}^2$, then every cardinal greater than κ is an outward compactness cardinal for $\mathcal{L}^2$.*

(2) Given a limit cardinal κ, every $<\kappa$-outward consistent $\mathcal{L}^2$-theory is $<\kappa$-consistent.

(3) Every strong compactness cardinal for $\mathcal{L}^2$ is an outward compactness cardinal for $\mathcal{L}^2$.

Proof. (1) Let $\rho > \kappa$ be a cardinal and let T be a $<\rho$-outward consistent $\mathcal{L}^2$-theory. Let $\lambda < \kappa$ be a cardinal, let $\vartheta > \rho$ be a sufficiently large cardinal with $T \in V_\vartheta$, let G be $\mathrm{Col}(\omega,\vartheta)$-generic over V, and let N be an outer ZFC*-model of V_ϑ^V in $V[G]$ in which λ is a cardinal. Then, T is $<\lambda$-consistent in N. This shows that T is $<\kappa$-outward consistent, and hence T is consistent.

[3] This is because the second-order satisfaction relation can be defined by a recursion over subformulae that is based on a Σ_2-function: we proceed as for the first-order satisfaction relation, however we have to essentially assert that $\exists W\ W = \mathcal{P}(M)$ holds for each set-sized model with underlying set M, which is already a Σ_2-statement, and then we can bound all other relevant quantifiers by W. To prove that such a form of recursion can successfully be performed only requires replacement for Σ_2-formulae (a version of this argument for Σ_1-recursion can be found in [6, Theorem 6.4]).

[4] In the following, we will usually use uppercase letters for second-order variables and lowercase letters for first-order variables in $\mathcal{L}^2$-formulas.

[5] Inductively defined by $(v_0 \in v_1)^M \equiv v_0 \in v_1$, $(v \in W)^M \equiv v \in W$, $(\neg\varphi)^M \equiv \neg\varphi^M$, $(\varphi \wedge \psi)^M \equiv \varphi^M \wedge \psi^M$, $(\forall x\ \varphi(x))^M \equiv \forall x \in M\ \varphi^M(x)$ and $(\forall X\ \varphi(X))^M \equiv \forall x \subseteq M\ \varphi^M(x)$. Note that, violating our above convention, we consider W to be a first order variable after relativizing $v \in W$ to M (alternatively, we could replace W by a new and otherwise unused variable symbol w).

(2) Let T be an $\mathcal{L}^2$-theory that is $<\kappa$-outward consistent, and assume, for the sake of a contradiction, that it is not $<\kappa$-consistent. Fix an inconsistent subtheory T_0 of T of cardinality less than κ and a cardinal $\lambda < \kappa$ with $|T_0| < \lambda$. Pick a cardinal $\vartheta > \kappa$ such that V_ϑ is a model of ZFC* and T is an element of V_ϑ. Let G be $\mathrm{Col}(\omega, \vartheta)$-generic over V. Then, in $V[G]$, the set V_ϑ^V is an outer ZFC*-model of itself in which λ is a cardinal and hence T is $<\lambda$-consistent in V_ϑ^V. This shows that T_0 is consistent in V_ϑ^V and, by the nature of the satisfaction relation of $\mathcal{L}^2$, we can conclude that T_0 is consistent in V, contradicting our assumption.

(3) Let ρ be a strong compactness cardinal for $\mathcal{L}^2$. By Theorem 1.1, there exists an extendible cardinal $\kappa \leq \rho$. In this situation, Theorem 1.1 together with (2) implies that κ is an outward compactness cardinal for $\mathcal{L}^2$, and we can apply (1) to conclude that ρ also has this property. □

We now present the characterization of measurable cardinals that realizes the approach outlined above. The arguments appearing in its proof already contain many of the key ideas utilized in the later sections of this paper. Some parts of these arguments already appeared in a slightly different form in the naïve approach outlined above, but will be repeated within the proof below for the sake of clarity of presentation.

Theorem 1.5. *A cardinal κ is an outward compactness cardinal for $\mathcal{L}^2$ if and only if there exists a measurable cardinal less than or equal to κ.*

Proof. First, assume that κ is a measurable cardinal and T is a $<\kappa$-outward consistent $\mathcal{L}^2$-theory. Pick a cardinal $\lambda > \kappa$ with $T \in H_\lambda$. Using standard iteration arguments (see [12, Corollary 19.7(b)]), we find an inner model M and an elementary embedding $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. Pick a sufficiently large cardinal $\vartheta > j(\kappa)$ such that $j(\vartheta) = \vartheta$ and V_ϑ is a model of ZFC*. Elementarity ensures that in M, the $\mathcal{L}^2$-theory $j(T)$ is $<j(\kappa)$-outward consistent. Let G be $\mathrm{Col}(\omega, \vartheta)$-generic over V. Since V_ϑ^M is a model of ZFC*, the observation after Definition 1.3 shows that V_ϑ^M is countable in $M[G]$. In this situation, our setup ensures that, in $M[G]$, the $\mathcal{L}^2$-theory $j(T)$ is $<\lambda$-consistent in every countable outer ZFC*-model of V_ϑ^M in which λ is a cardinal. This statement can be formulated by a Π_1-formula using parameters contained in $H_{\aleph_1}^{M[G]}$ and is therefore provably equivalent to a $\mathbf{\Pi}^1_2$-statement whose parameters are real numbers coding the original parameters (see [11, Lemma 25.25]). Therefore, *Shoenfield absoluteness* implies that the given statement also holds in $V[G]$. But, in $V[G]$, the set V_ϑ^V is a countable outer ZFC*-model of V_ϑ^M in which λ is a cardinal, and hence $j(T)$ is $<\lambda$-consistent in V_ϑ^V. Now, note that $j[T] \subseteq j(T)$ is an element of V_ϑ^V and has cardinality less than λ in V_ϑ^V. Therefore, we can conclude that $j[T]$ is consistent in V_ϑ^V, and this implies that $j[T]$ is a consistent $\mathcal{L}^2$-theory in V. But this also shows that T is consistent in V, because the finitary character of $\mathcal{L}^2$-formulae ensures that we can identify $j[T]$ and T via a renaming of symbols. These computations show that κ is an outward compactness cardinal for $\mathcal{L}^2$ and we can now apply Proposition 1.4.(1) to see that every cardinal greater than κ is also an outward compactness cardinal for $\mathcal{L}^2$.

Next, assume that κ is an outward compactness cardinal for $\mathcal{L}^2$. Consider the $\mathcal{L}^2$-theory T_κ defined earlier, i.e.,, we extend the language of set theory by constant symbols c_x for every $x \in V_{\kappa+1}$ and constant symbols d_γ for all $\gamma \leq \kappa$, and define T_κ to consist of:

- The elementary first-order diagram of $V_{\kappa+1}$, making use of the constant symbols c_x.
- All (first-order) sentences of the form $d_\beta \in d_\gamma \in c_\kappa$ for $\beta < \gamma \leq \kappa$.
- The (second-order) sentence stating that the $\in$-relation is well-founded.

Claim. *The $\mathcal{L}^2$-theory T_κ is $<\kappa$-outward consistent.*

Proof of the Claim. Let $\lambda < \kappa$ be a cardinal, let $\vartheta > \kappa$ be a cardinal with $T_\kappa \in V_\vartheta$, let G be $\mathrm{Col}(\omega, \vartheta)$-generic over V and let $N \in V[G]$ be an outer ZFC*-model of V_ϑ^V in which λ is a cardinal. Then $V^V_{\kappa+1}$ is an element of N. Now, work in N and fix a subtheory T_0 of T_κ of cardinality less than λ. Since λ is a cardinal, the axioms of ZFC* allow us to construct a model of T_0 with domain $V^V_{\kappa+1}$ that interprets all constant symbols c_x appearing in T_0 by the corresponding element x, and interprets the (less than λ-many) constant symbols d_β appearing in T_0 as suitable elements of λ. □

Using the fact that κ is an outward compactness cardinal for $\mathcal{L}^2$, we thus obtain that T_κ is consistent. The definition of T_κ now ensures that T_κ has a transitive model M and, by sending $x \in V_{\kappa+1}$ to c^M_x, we obtain a non-trivial elementary embedding $j : V_{\kappa+1} \longrightarrow M$ with $j(\kappa) > d^M_\kappa \geq \kappa$. As above, the existence of such an embedding implies that there is a measurable cardinal less than or equal to κ. □

In Sections 2 and 3, we will refine the notion of outward compactness to isolate analogous characterizations for other types of large cardinal notions below extendibility. The main result of these sections provides a general correspondence between objects in this region of the large cardinal hierarchy and outward compactness principles for second-order logic (see Theorem 3.5) that we will afterwards apply to several well-studied notions (see Corollary 3.6). This correspondence generalizes both Magidor's characterization of extendibility in Theorem 1.1 and the characterization of measurability in Theorem 1.5. More specifically, using the more general notion of Ψ-*outward compactness* that will be introduced in Definition 3.3 below, we will obtain the following characterizations:

Theorem 1.6. *(1) A cardinal κ is a Ψ_{str}-outward compactness cardinal for $\mathcal{L}^2$ if and only if there is a strong cardinal less than or equal to κ.*

(2) A cardinal κ is a Ψ_{sc}-outward compactness cardinal for $\mathcal{L}^2$ if and only if there is a supercompact cardinal less than or equal to κ.

(3) A cardinal κ is a Ψ_{stc}-outward compactness cardinal for $\mathcal{L}^2$ if and only if κ is ω_1-strongly compact.

(4) A cardinal κ is a Ψ_{ext}-outward compactness cardinal for $\mathcal{L}^2$ if and only if there is an extendible cardinal less than or equal to κ.

We then discuss the naturalness of the obtained large cardinal characterizations in Section 4, and consider the question which other types of large cardinal properties below extendibility can be characterized through outward compactness principles. Finally, in Sections 5 and 6, we extend our concepts to arbitrary abstract logics and large cardinal properties beyond extendibility. The main results of these sections will provide a general correspondence between outward compactness for abstract logics and fragments of *Vopěnka's Principle* (see Theorems 6.9 and 6.10) that directly leads to outward compactness characterizations for Vopěnka's Principle and the principle "Ord *is Woodin*" (see Corollaries 6.11 and 6.12). The given correspondence directly generalizes a classical result of Makowsky in [15] that characterizes the validity of Vopěnka's Principle through the existence of strong compactness cardinals for all abstract logics (see Theorem 6.3). These results will be based on the (even more general) notion of F-Ψ-*outward compactness* introduced in Definition 6.4 below.

Theorem 1.7. *(1) The following schemes are equivalent over* ZFC*:*

(a) Vopěnka's Principle.

(b) For every natural number n and every abstract logic $\mathcal{L}$, there exists a ZFC_n-Ψ_{ext}*-outward compactness cardinal for $\mathcal{L}$.*

(c) For every natural number n and every abstract logic $\mathcal{L}$ with simple formulas, there exists a ZFC_n-Ψ_{sc}*-outward compactness cardinal for $\mathcal{L}$.*

(2) The following schemes are equivalent over ZFC*:*

(a) Ord *is Woodin.*

(b) *For every natural number n and every abstract logic $\mathcal{L}$ with simple formulas, there exists a* ZFC_n-Ψ_{str}*-outward compactness cardinal for* $\mathcal{L}$.

2. Ψ-large cardinals

We now introduce a broad framework for large cardinal properties between measurability and extendibility. Our results will later show that all notions that fall into this framework can be characterized through compactness properties of $\mathcal{L}^2$. In order to achieve this goal, we first define a property to express a certain amount of closeness between two transitive models $M \subseteq N$ of set theory. In the following, we say that a class is *closed under basic set operations* if it is closed under taking pairs, products and intersections. Note that for every class M closed under basic set operations and every limit ordinal λ, the set $M \cap V_\lambda$ is closed under basic set operations.

Definition 2.1. (1) A tuple $\langle N, M, \mu, \nu, \rho\rangle$ of sets is *suitable* if $M \subseteq N$ are transitive sets closed under basic set operations with $M \cap \mathrm{Ord} = N \cap \mathrm{Ord} \in \mathrm{Lim}$ and $\mu < \nu < \rho \in M$ are limit ordinals with the property that μ is a cardinal in N.

(2) A formula $\Psi(v_0, \ldots, v_4)$ in the language of set theory is a *measure of closeness* if ZFC proves the following statements:
   (A) If $\Psi(N, M, \mu, \nu, \rho)$ holds, then the tuple $\langle N, M, \mu, \nu, \rho\rangle$ is suitable.
   (B) If μ is a cardinal and $\nu < \rho < \theta$ are limit ordinals strictly greater than μ, then $\Psi(V_\theta, V_\theta, \mu, \nu, \rho)$ holds.
   (C) Given transitive sets $M \subseteq N$ closed under basic set operations with the property that $M \cap \mathrm{Ord} = N \cap \mathrm{Ord} \in \mathrm{Lim}$ and limit ordinals $\mu < \nu < \rho < \theta \in M$, we have

$$\Psi(N, M, \mu, \nu, \rho) \longleftrightarrow \Psi(N \cap V_\theta, M \cap V_\theta, \mu, \nu, \rho).$$

As a first example of the concept introduced above, let $\Psi_{ms}(v_0, \ldots, v_4)$ denote the canonical formula such that $\Psi_{ms}(N, M, \mu, \nu, \rho)$ holds if and only if the tuple $\langle N, M, \mu, \nu, \rho\rangle$ is suitable. It is easy to check that the formula Ψ_{ms} also satisfies the above properties (B) and (C):

Proposition 2.2. *The formula Ψ_{ms} is a measure of closeness.* □

We now continue by associating large cardinal properties to measures of closeness.

Definition 2.3. Given a formula $\Psi(v_0, \ldots, v_4)$ in the language of set theory, a cardinal κ is Ψ*-large* if for all limit ordinals $\kappa < \eta < \theta$, there are unboundedly many cardinals $\lambda \geq \eta$ with the property that there is a transitive set M and an elementary embedding $j : V_{\theta+1} \longrightarrow M$ such that $j(\kappa) > \lambda$ and $\Psi(V_{j(\theta)}, M \cap V_{j(\theta)}, \lambda, j(\kappa), j(\eta))$ holds.

In the following, we want to show that several well-known large cardinal properties can be characterized through the notion of Ψ-largeness, in the sense that we can associate a measure of closeness Ψ to the given large cardinal property so that we can prove that some cardinal κ is Ψ-large if and only if there is a cardinal with the given property that is less than or equal to κ. Before we start to derive these characterizations, we observe that no large cardinal property that is implication-wise stronger than extendibility can be characterized in this way.

Proposition 2.4. *If $\Psi(v_0, \ldots, v_4)$ is a measure of closeness, then every cardinal greater than or equal to an extendible cardinal is Ψ-large.*

Proof. Assume that μ is an extendible cardinal, $\kappa \geq \mu$ is a cardinal, $\kappa < \eta < \theta$ are limit ordinals and $\lambda > \theta$ is a cardinal. Then there exists an ordinal ρ and an elementary embedding $i : V_\lambda \longrightarrow V_\rho$ with $\mathrm{crit}(i) = \mu$ and $i(\mu) > \lambda$. If we now define

$$j = i \restriction V_{\theta+1} : V_{\theta+1} \longrightarrow V_{i(\theta)+1},$$

then $j(\kappa) > \lambda$ and (B) in Definition 2.1.2 ensures that $\Psi(V_{j(\theta)}, V_{j(\theta)}, \lambda, j(\kappa), j(\eta))$ holds. These computations allow us to conclude that κ is Ψ-large. □

Next, we use standard arguments to show that Ψ-largeness can also be characterized through the existence of elementary embeddings from V into inner models.

Lemma 2.5. *Given a measure of closeness $\Psi(v_0,\dots,v_4)$, a cardinal κ is Ψ-large if and only if for every limit ordinal $\eta>\kappa$, there are unboundedly many cardinals $\lambda\geq\eta$ with the property that there is an inner model M and an elementary embedding $j:V\longrightarrow M$ such that $j(\kappa)>\lambda$ and $\Psi(V_\vartheta,V^M_\vartheta,\lambda,j(\kappa),j(\eta))$ holds for all limit ordinals $\vartheta>j(\eta)$.*

Proof. First, assume that κ is Ψ-large and fix limit ordinals $\zeta>\eta>\kappa$. Pick a sufficiently large fixed point $\nu>\zeta$ of the $\beth$-function. By our assumption, we find a cardinal $\lambda\geq\zeta$, a transitive set M_0, and an elementary embedding $i:V_{\nu+1}\longrightarrow M_0$, such that $i(\kappa)>\lambda$ and $\Psi(V_{i(\nu)},M_0\cap V_{i(\nu)},\lambda,i(\kappa),i(\eta))$ holds. Since ν is a strong limit cardinal, we can use standard arguments (see [12, Lemma 26.1]) to find an inner model M with $V^M_{i(\nu)}=M_0\cap V_{i(\nu)}$, and an elementary embedding $j:V\longrightarrow M$ satisfying

$$j\restriction(V_\nu\cup\{\nu\}) \;=\; i\restriction(V_\nu\cup\{\nu\}).$$

That is, we define E to be the $(\kappa,i(\nu))$-extender derived from i, defining E_a for every $a\in[i(\nu)]^{<\omega}$ by

$$X\in E_a \iff X\in\mathcal{P}([\nu]^{|a|})\wedge a\in i(X),$$

and let $j:V\longrightarrow M$ be the ultrapower embedding of V induced by E. Let $j':V_{\nu+1}\longrightarrow M'$ be the ultrapower embedding of $V_{\nu+1}$ induced by E. Since V and $V_{\nu+1}$ have the same subsets of ν, it follows that M and M' agree up to $j(\nu)=j'(\nu)$, and that j and j' agree up to ν. Let $k:M'\longrightarrow M_0$ be the factor embedding satisfying that $i=k\circ j'$. Then, by [12, Lemma 26.1], we first obtain that $\mathrm{crit}(k)>j(\nu)$, yielding that $i(\nu)=j'(\nu)=j(\nu)$, and then consequently that $V^M_{j(\nu)}=V_{j(\nu)}\cap M_0$. If $\vartheta>j(\eta)$ is a limit ordinal, we apply (C) in Definition 2.1.2 to conclude that $\Psi(V_\vartheta,V^M_\vartheta,\lambda,j(\kappa),j(\eta))$ holds. Summing up, this shows that we have found our desired embedding $j:V\longrightarrow M$.

In the other direction, assume that for some limit ordinal $\eta>\kappa$, there are unboundedly many cardinals $\lambda\geq\eta$ such that there exists an inner model M and an elementary embedding $j:V\longrightarrow M$ for which $j(\kappa)>\lambda$ and $\Psi(V_\vartheta,V^M_\vartheta,\lambda,j(\kappa),j(\eta))$ holds for all limit ordinals $\vartheta>j(\eta)$. Fix a limit ordinal $\theta>\eta$ and an ordinal $\zeta>\eta$. Pick a cardinal $\lambda\geq\zeta$, an inner model M and an elementary embedding $j:V\longrightarrow M$ for which $j(\kappa)>\lambda$ and $\Psi(V_\vartheta,V^M_\vartheta,\lambda,j(\kappa),j(\eta))$ holds for all limit ordinals $\vartheta>j(\eta)$. Then, $M_0=V^M_{j(\theta)+1}$ is a transitive set, and $i=j\restriction V_{\theta+1}:V_{\theta+1}\longrightarrow M_0$ is an elementary embedding for which $i(\kappa)>\lambda$ and $\Psi(V_{i(\theta)},M_0\cap V_{i(\theta)},\lambda,i(\kappa),i(\eta))$ holds. These computations yield the converse implication. □

Using the above equivalence, we can now easily show that the formula Ψ_{ms} canonically corresponds to the large cardinal notion of measurability.

Corollary 2.6. *A cardinal κ is Ψ_{ms}-large if and only if there is a measurable cardinal less than or equal to κ.*

Proof. First, assume that κ is Ψ_{ms}-large. By Lemma 2.5, there is a transitive class M and an elementary embedding $j:V\longrightarrow M$ with $j(\kappa)>\kappa$. In particular, we know that the critical point $\mathrm{crit}(j)$ of j is a measurable cardinal less than or equal to κ. In the other direction, assume that there is a measurable cardinal less than or equal to κ and $\lambda>\kappa$ is a cardinal. Standard arguments about iterated ultrapowers (see [12, Corollary 19.7(b)]) now allow us to find an inner model M and an elementary embedding $j:V\longrightarrow M$ with $j(\kappa)>\lambda$. Then it is easy to see that $\Psi_{ms}(V_\vartheta,V^M_\vartheta,\lambda,j(\kappa),j(\eta))$ holds for every limit ordinal $\kappa<\eta\leq\lambda$ and every limit ordinal $\vartheta>j(\eta)$. By Lemma 2.5, this shows that the cardinal κ is Ψ_{ms}-large. □

In the remainder of this section, we will show that several other important large cardinal notions in the region between measurability and extendibility canonically correspond to measures of closeness.

2.1. **Strong cardinals.** Let $\Psi_{str}(v_0,\ldots,v_4)$ denote the canonical suitable formula in the language of set theory with the property that $\Psi_{str}(N,M,\mu,\nu,\rho)$ holds if and only if the tuple $\langle N,M,\mu,\nu,\rho\rangle$ is suitable and $N\cap V_\mu\subseteq M$ holds. The following statement is then immediate:

Proposition 2.7. *The formula Ψ_{str} is a measure of closeness.* □

In the following, we will show that the formula Ψ_{str} corresponds to the large cardinal concept of strongness. These and later arguments will make crucial use of the following well-known consequence of the *Kunen Inconsistency* (see, for example, [12, Corollary 23.14]):

Lemma 2.8. *Let M be a transitive class and let $j: V\longrightarrow M$ be a non-trivial elementary embedding. If λ is a limit ordinal of uncountable cofinality with $V_\lambda\subseteq M$, then there exists a natural number n with $j^n(\mathrm{crit}(j))\geq\lambda$.*

Proof. Assume, towards a contradiction, that the above conclusion fails and define

$$\zeta \;=\; \sup_{n<\omega} j^n(\mathrm{crit}(j)).$$

Then, ζ is a limit ordinal of countable cofinality with $j(\zeta)=\zeta$, and our assumptions ensure that $\zeta+2<\lambda$. But this implies that the map $j\restriction V_{\zeta+2}:V_{\zeta+2}\longrightarrow V_{\zeta+2}$ is a non-trivial elementary embedding, contradicting the *Kunen Inconsistency.* □

Lemma 2.9. *A cardinal κ is Ψ_{str}-large if and only if there is a strong cardinal less than or equal to κ.*

Proof. First, assume that some cardinal $\mu\leq\kappa$ is strong, and $\lambda>\kappa$ is a cardinal. In this situation, there exists an inner model M and an elementary embedding $j:V\longrightarrow M$ with $\mathrm{crit}(j)=\mu$, $j(\mu)>\lambda$ and $V_\lambda\subseteq M$. Then, $j(\kappa)>\lambda$, and it is easy to see that $\Psi_{str}(V_\vartheta,V^M_\vartheta,\lambda,j(\kappa),j(\eta))$ holds for all limit ordinals $\kappa<\eta\leq\lambda$ and all limit ordinals $\vartheta>j(\eta)$. By Lemma 2.5, this shows that κ is Ψ_{str}-large.

In the other direction, assume that κ is Ψ_{str}-large, and no cardinal less than or equal to κ is strong. Then, there exists a regular cardinal $\eta>\kappa$ with the property that no cardinal less than or equal to κ is η-strong. Using Lemma 2.5, we find a cardinal $\lambda>\eta$, an inner model M with $V_\lambda\subseteq M$ and an elementary embedding $j:V\longrightarrow M$ with $j(\kappa)>\lambda$. Set $\mu=\mathrm{crit}(j)<\kappa$ and $\mu_i=j^i(\mu)$ for all $i<\omega$. Lemma 2.8 yields a least natural number n with $\mu_n>\eta$. Since $\mu<\eta$, we know that $n>0$, and the fact that μ is not η-strong then implies that $n>1$.

Claim. *In M, the cardinal μ_1 is superstrong with target μ_n.*[6]

Proof of the Claim. By induction, we show that, in M, for all $0<i<n$, the cardinal μ_1 is superstrong with target μ_{i+1}. Since $n>1$, the embedding j witnesses that μ is superstrong with target μ_1 in V and hence elementarity of j ensures that, in M, the cardinal μ_1 is superstrong with target μ_2. Next, assume that $i<n-1$ has the property that the cardinal μ_1 is superstrong with target μ_{i+1} in M. Then there exists a transitive class $N\subseteq M$ with $V^M_{\mu_{i+1}}\subseteq N$ and an elementary embedding $i:M\longrightarrow N$ with $\mathrm{crit}(i)=\mu_1$ and $i(\mu_1)=\mu_{i+1}$. Since $i+1<n$, we have $\mu_{i+1}\leq\eta$ and hence $V_{\mu_{i+1}}\subseteq N$. Moreover, the map $i\circ j:V\longrightarrow N$ is an elementary embedding with $\mathrm{crit}(i\circ j)=\mu$ and $(i\circ j)(\mu)=\mu_{i+1}$. In particular, this map witnesses that μ is superstrong with target μ_{i+1} in V. This allows us to conclude that, in M, the cardinal μ_1 is superstrong with target μ_{i+2}. This argument completes the proof of the claim. □

[6]Remember that a cardinal κ is *superstrong with target* λ if there is a transitive class M with $V_\lambda\subseteq M$ and an elementary embedding $j:V\longrightarrow M$ with $\mathrm{crit}(j)=\kappa$ and $j(\kappa)=\lambda$.

By the above claim, there exists a transitive class $N \subseteq M$ with $V^M_{\mu_n} \subseteq N$ and an elementary embedding $i : M \longrightarrow N$ with $\mathrm{crit}(i) = \mu_1$ and $i(\mu_1) = \mu_n$. But then, $V_\lambda \subseteq N$ and $i \circ j : V \longrightarrow N$ is an elementary embedding with $\mathrm{crit}(i \circ j) = \mu$ and $(i \circ j)(\mu) > \eta$, yielding that μ is η-strong, a contradiction. □

2.2. **Supercompact cardinals.** Let $\Psi_{sc}(v_0, \dots, v_4)$ denote the canonical formula in the language of set theory with the property that $\Psi_{sc}(N, M, \mu, \nu, \rho)$ holds if and only if the tuple $\langle N, M, \mu, \nu, \rho\rangle$ is suitable and ${}^\mu\rho \cap N \subseteq M$ holds.

Proposition 2.10. *The formula Ψ_{sc} is a measure of closeness.* □

We now show that this formula corresponds to the large cardinal concept of supercompactness.

Lemma 2.11. *A cardinal κ is Ψ_{sc}-large if and only if there is a supercompact cardinal less than or equal to κ.*

Proof. First, assume that $\mu \leq \kappa$ is a supercompact cardinal, and $\lambda > \kappa$ is a cardinal. Then, there exists a transitive class M with ${}^\lambda M \subseteq M$ and an elementary embedding $j : V \longrightarrow M$ with $\mathrm{crit}(j) = \mu$ and $j(\mu) > \lambda$. If $\kappa < \eta \leq \lambda$ is a limit ordinal, and $\vartheta > j(\eta)$ is a limit ordinal, then ${}^{<\lambda}j(\eta) \subseteq V^M_\vartheta$ and this shows that $\Psi_{sc}(V_\vartheta, V^M_\vartheta, \lambda, j(\kappa), j(\eta))$ holds. By Lemma 2.5, this shows that κ is Ψ_{sc}-large.

Now, assume that κ is Ψ_{sc}-large, and no cardinal less than or equal to κ is supercompact. Pick a fixed point $\eta > \kappa$ of the $\beth$-function of uncountable cofinality with the property that no cardinal less than or equal to κ is η-supercompact, and apply Lemma 2.5 to find a cardinal $\lambda > \eta$, a transitive class M with ${}^\lambda j(\eta) \subseteq M$ and an elementary embedding $j : V \longrightarrow M$ with $j(\kappa) > \lambda$. Define $\mu = \mathrm{crit}(j) < \kappa$, and $\mu_i = j^i(\mu)$ for all $i < \omega$. Then, our choice of η ensures that $V_\eta \subseteq M$, and we can use Lemma 2.8 to find a minimal natural number $n > 1$ with $\mu_n > \eta$.

Claim. *The cardinal μ is (n-1)-huge with targets $\mu_1, \dots, \mu_{n-1}$.*[7]

Proof. Since we have $\mu_{n-1} \leq \eta < \lambda$, the fact that ${}^\lambda j(\eta) \subseteq M$ implies that $j[\mu_{n-1}] \in M$, and hence, we know that

$$\{A \subseteq \mu_{n-1} \mid j[\mu_{n-1}] \in j(A)\}$$

is a $<\mu$-complete normal ultrafilter over $P(\mu_{n-1})$ that contains the set

$$\{A \subseteq \mu_{n-1} \mid \mathrm{ot}(A \cap \mu_i) = \mu_{i-1} \text{ for all } 0 < i < n\}.$$

By [12, Theorem 24.8], this implies the statement of the claim. □

By the above claim, we know that μ is huge with target μ_{n-1}, and hence, elementarity implies that, in M, the cardinal μ_1 is huge with target μ_n. Therefore, we find a transitive class $N \subseteq M$ definable in M with $M \cap {}^{\mu_n}N \subseteq N$, and an elementary embedding $i : M \longrightarrow N$ definable in M with $\mathrm{crit}(i) = \mu_1$ and $i(\mu_1) = \mu_n$. Since $j[\eta] \in {}^\eta j(\eta) \subseteq M$ and $\eta < \mu_n$, we then know that

$$(i \circ j)[\eta] \ = \ i[j[\eta]] \ \in \ M \cap {}^\eta N \ \subseteq \ N.$$

This shows that $i \circ j : V \longrightarrow N$ is an elementary embedding with $\mathrm{crit}(i \circ j) = \mu$, $(i \circ j)(\mu) > \eta$ and $(i \circ j)[\eta] \in N$. But this contradicts the fact that μ is not η-supercompact. □

[7] Remember that, given a natural number $n > 0$, a cardinal κ is *n-huge with targets* $\lambda_1, \dots, \lambda_n$ if there is a transitive class M with ${}^{\lambda_n}M \subseteq M$ and an elementary embedding $j : V \longrightarrow M$ with $\mathrm{crit}(j) = \kappa$ and $j(\lambda_m) = \lambda_{m+1}$ for all $m < n$.

2.3. **Extendible cardinals.** Let $\Psi_{ext}(v_0, \ldots, v_4)$ denote the canonical formula in the language of set theory with the property that $\Psi_{ext}(N, M, \mu, \nu, \rho)$ holds if and only if the tuple $\langle N, M, \mu, \nu, \rho\rangle$ is suitable and $N \cap V_\rho \subseteq M$ holds.

Proposition 2.12. *The formula Ψ_{ext} is a measure of closeness.* □

The following result shows that the formula Ψ_{ext} corresponds to extendibility. In order to motivate the concepts that will be introduced in the next chapter, we use Theorem 1.1 to prove the backward implication of the stated equivalence.[8]

Lemma 2.13. *A cardinal κ is Ψ_{ext}-large if and only if there is an extendible cardinal less than or equal to κ.*

Proof. First, note that by Proposition 2.4, every cardinal greater than or equal to an extendible cardinal is Ψ_{ext}-large. In the other direction, assume that κ is Ψ_{ext}-large and let T be a $<\kappa$-consistent $\mathcal{L}^2$-theory. Pick a cardinal $\eta > \kappa$ with $T \in V_\eta \prec_{\Sigma_2} V$. By our assumption, we can apply Lemma 2.5 to find a cardinal $\lambda > \eta$, a transitive class M and an elementary embedding $j : V \longrightarrow M$ such that $j(\kappa) > \lambda$ and $V_{j(\eta)} \subseteq M$. We then know that $V_{j(\eta)} = V^M_{j(\eta)} \prec_{\Sigma_2} M$, and $j(T) \in V_{j(\eta)}$ is a $<j(\kappa)$-consistent $\mathcal{L}^2$-theory in M. Moreover, our assumptions on M ensure that $j[T]$ is an element of M and has cardinality less than $j(\kappa)$ in M. But this implies that $j[T]$ is consistent in M, and the above observations show that $V_{j(\eta)}$ contains a structure that is a model of $j[T]$ in M. We conclude that $j[T]$ is also consistent in V and, since we can identify the theories T and $j[T]$ through a renaming of symbols, this shows that T is consistent in V. These computations show that κ is a strong compactness cardinal for $\mathcal{L}^2$, and hence, Theorem 1.1 ensures that there exists an extendible cardinal less than or equal to κ. □

2.4. **ω_1-strongly compact cardinals.** In [3] and [4], Bagaria and Magidor defined a cardinal $\kappa > \omega_1$ to be ω_1-*strongly compact* if for every set I, every $<\kappa$-complete filter on I can be extended to a countably complete ultrafilter on I. Note that every cardinal above an ω_1-strongly compact cardinal is also ω_1-strongly compact. In the following, we show that the naïve attempt to characterize strongly compact cardinals through the concept of Ψ -largeness (motivated by [12, Theorem 22.17]) actually leads to a characterization of ω_1-strongly compact cardinals. This characterization differs strongly from the ones presented above, because e.g., the first ω_1-strongly compact cardinal can be singular (see [3, Theorem 6.1]). Let $\Psi_{stc}(v_0, \ldots, v_4)$ denote the canonical formula in the language of set theory with the property that $\Psi_{stc}(N, M, \mu, \nu, \rho)$ holds if and only if the tuple $\langle N, M, \mu, \nu, \rho\rangle$ is suitable and for every $d \in N \cap P(\rho)$ with the property that N contains no injection of μ into d, there exists $c \in M \cap P(\rho)$ such that $d \subseteq c$ and M does not contain an injection of ν into c.[9]

Proposition 2.14. *The formula Ψ_{stc} is a measure of closeness.*

Proof. The formula Ψ_{stc} trivially satisfies (A) and (B) in Definition 2.1.2. Assume that $M \subseteq N$ are transitive sets closed under basic set operations with $M \cap \mathrm{Ord} = N \cap \mathrm{Ord} \in \mathrm{Lim}$ and $\mu < \nu < \rho < \theta \in M$ are limit ordinals. Since M and $M \cap V_\theta$ (respectively, N and $N \cap V_\theta$) contain the same subsets of ρ, the same injections from μ into ρ and the same injections from ν into ρ, it follows directly that $\Psi_{stc}(N, M, \mu, \nu, \rho)$ holds if and only if $\Psi_{stc}(N \cap V_\theta, M \cap V_\theta, \mu, \nu, \rho)$ holds. This shows that Ψ_{stc} also satisfies (C) in Definition 2.1.2. □

[8] It would also be possible to establish this equivalence with the help of [16, Theorem 2.28], which shows that a cardinal κ is extendible if and only if it is *jointly λ-supercompact and λ-superstrong* for all $\lambda > \kappa$ (see [16, Definition 2.24]). However, the proof of the equivalence that we present, using the connections between second-order logic and extendibility, provides an elegant alternative to an adaptation of the proof of Lemma 2.11 to these large cardinal properties.

[9] Note that these assumptions are phrased in this way because Definition 2.1.2 only makes minimal assumptions on the closure properties of the sets M and N.

Lemma 2.15. *A cardinal is ω_1-strongly compact if and only if it is Ψ_{stc}-large.*

Proof. First, assume that κ is an ω_1-strongly compact cardinal, and let $\lambda > \kappa$ be a cardinal. The proof of [3, Theorem 4.7] then shows that there is a σ-complete fine ultrafilter $\mathcal{U}$ on $P_\kappa(\lambda)$. Let $j : V \longrightarrow M$ denote the ultrapower embedding induced by $\mathcal{U}$, and fix $d \subseteq j(\lambda)$ of cardinality at most λ. Then there exists a sequence

$$\langle f_\gamma : P_\kappa(\lambda) \longrightarrow \lambda \mid \gamma < \lambda\rangle$$

of functions with $d = \{[f_\gamma]_\mathcal{U} \mid \gamma < \lambda\}$. Define

$$F : P_\kappa(\lambda) \longrightarrow P_\kappa(\lambda);\ a \longmapsto \{f_\gamma(a) \mid \gamma \in a\}.$$

Given $\gamma < \lambda$, we then have

$$\{a \in P_\kappa(\lambda) \mid f_\gamma(a) \in F(a)\} \ \supseteq\ \{a \in P_\kappa(\lambda) \mid \gamma \in a\} \ \in\ \mathcal{U},$$

and this implies that $[f_\gamma]_\mathcal{U} \in [F]_\mathcal{U}$. Moreover, since $F(a)$ has cardinality less than κ for each $a \in P_\kappa(\lambda)$, we know that $[F]_\mathcal{U}$ has cardinality less than $j(\kappa)$ in M. In particular, if d has cardinality λ, then $\lambda = |d| \leq |d|^M \leq |[F]_\mathcal{U}|^M < j(\kappa)$. These computations show that $\Psi_{stc}(V_\vartheta, V_\vartheta^M, \lambda, j(\kappa), j(\eta))$ holds for every limit ordinal $\kappa < \eta \leq \lambda$ and every limit ordinal $\vartheta > j(\eta)$. Using Lemma 2.5, this allows us to conclude that κ is Ψ_{stc}-large.

Now, assume that κ is Ψ_{stc}-large, and let $\eta > \kappa$ be a limit ordinal. Lemma 2.5 allows us to find a cardinal $\lambda > \eta$, a transitive class M and an elementary embedding $j : V \longrightarrow M$ such that $j(\kappa) > \lambda$ and $\Psi_{stc}(V_\vartheta, V_\vartheta^M, \lambda, j(\kappa), j(\eta))$ holds for all limit ordinals $\vartheta > j(\eta)$. Since $j[\eta]$ is a subset of $j(\eta)$ of cardinality less than λ, we can now find $b \in P(j(\eta))^M$ such that $j[\eta] \subseteq b$ and b has cardinality less than $j(\kappa)$ in M. By [3, Theorem 4.7], these computations show that κ is ω_1-strongly compact. □

3. $\mathcal{L}^2$-CHARACTERIZATIONS

We now connect the concepts introduced in the previous section to compactness properties of second-order logic. The next definition provides the first step to establish these connections. In this definition, we consider simply definable measures of closeness with the additional property that closeness of a transitive model M to $V_{M\cap\mathrm{Ord}}$ is expressible over M by an $\mathcal{L}^2$-statement. Remember that, given an $\in$-theory F and a natural number $n > 0$, a set-theoretic formula $\varphi(v_0, \ldots, v_{m-1})$ is a *Δ_n^F-formula* if there is a Σ_n-formula $\psi_0(v_0, \ldots, v_{m-1})$ and a Π_n-formula $\psi_1(v_0, \ldots, v_{m-1})$ with the property that

$$F \ \vdash\ \forall x_0, \ldots, x_{m-1}\ [\varphi(x_0, \ldots, x_{m-1}) \longleftrightarrow \psi_i(x_0, \ldots, x_{m-1})]$$

holds for all $i < 2$.

Definition 3.1. A formula $\Psi(v_0, \ldots, v_4)$ in the language of set theory is an *$\mathcal{L}^2$-measure of closeness* if the following statements hold:

(1) Ψ is a measure of closeness.
(2) Ψ is both a $\Delta_1^{\mathrm{ZFC}^*}$- and a $\Delta_1^{\mathrm{ZFC}^-}$-formula.[10]
(3) There exists an $\mathcal{L}^2$-formula $\Psi^*(v_0, v_1, v_2)$ in the language of set theory such that ZFC^* proves that for all limit ordinals $\mu < \nu < \rho < \theta$ and every transitive set M with $\theta \subseteq M \subseteq V_\theta$, the statement $\Psi(V_\theta, M, \mu, \nu, \rho)$ holds if and only if $\langle M, \in\rangle \models_{\mathcal{L}^2} \Psi^*(\mu, \nu, \rho)$ holds.

Proposition 3.2. *The formulas Ψ_{ms}, Ψ_{str}, Ψ_{sc}, Ψ_{stc} and Ψ_{ext} are all $\mathcal{L}^2$-measures of closeness.*

[10] As usual, we let ZFC^- denote the axioms of ZFC without the Powerset axiom and with the Collection scheme instead of the Replacement scheme.

Proof. First, note that there is a first-order sentence in the language of set theory that holds in a transitive set if and only if the given set is closed under basic set operations. Moreover, it is easy to find an $\mathcal{L}^2$-formula $\varphi(v)$ with the property that ZFC* proves that for every transitive set M closed under basic set operations and every $a \in M$, we have $\langle M, \in\rangle \models_{\mathcal{L}^2} \varphi(a)$ if and only if a is an element of $M \cap \mathrm{Ord}$ that is a infinite cardinal in V. In combination with the fact that the formalized satisfaction relation can be defined by both $\Delta_1^{\mathrm{ZFC}^*}$- and $\Delta_1^{\mathrm{ZFC}^-}$-formulas, this shows that Ψ_{ms} is an $\mathcal{L}^2$-measure of closeness. There is a canonical $\mathcal{L}^2$-formula $\varphi(v_0, v_1)$ such that ZFC* proves that for every transitive set M closed under basic set operations and all $\alpha, \beta \in M \cap \mathrm{Ord}$, we have $\langle M, \in\rangle \models_{\mathcal{L}^2} \varphi(\alpha, \beta)$ if and only if the set ${}^{\alpha}\beta$ is a subset of M, and this shows that Ψ_{sc} is an $\mathcal{L}^2$-measure of closeness. There is an $\mathcal{L}^2$-formula $\varphi(v_0, v_1, v_2)$ such that ZFC* proves that for every transitive set M closed under basic set operations and all $\alpha, \beta, \gamma \in M \cap \mathrm{Ord}$, we have $\langle M, \in\rangle \models_{\mathcal{L}^2} \varphi(\alpha, \beta, \gamma)$ if and only if for every subset d of γ of cardinality less than α, there exists $c \in M$ such that $d \subseteq c \subseteq \gamma$ and M contains no injection from β into c, and this directly shows that Ψ_{stc} is an $\mathcal{L}^2$-measure of closeness.

For the remaining formulas, recall that an argument of Magidor in [13, Proof of Theorem 2] shows that there is an $\mathcal{L}^2$-sentence ψ in the language of set theory with the property that for every transitive set M, we have $\langle M, \in\rangle \models_{\mathcal{L}^2} \psi$ if and only if $M = V_{M \cap \mathrm{Ord}}$. In addition, this $\mathcal{L}^2$-sentence ψ can be chosen in a way that this equivalence is provable in ZFC* (see [7, Proof of Fact 2.1]). This directly allows us to find an $\mathcal{L}^2$-formula $\varphi(v_0, v_1)$ in the language of set theory with the property that ZFC* proves that for every transitive set M and all $a, b \in M$, we have $\langle M, \in\rangle \models_{\mathcal{L}^2} \varphi(a, b)$ if and only if $a \in M \cap \mathrm{Ord}$ and $b = V_a$. In combination with the fact that the rank function is definable by both $\Delta_1^{\mathrm{ZFC}^*}$- and $\Delta_1^{\mathrm{ZFC}^-}$-formulas, we can conclude that the formulas Ψ_{str} and Ψ_{ext} are both $\mathcal{L}^2$-measures of closeness. □

The second step to connect Ψ-large cardinals to second-order logic is given by the family of compactness properties introduced in the next definition.

Definition 3.3. Let $\Psi(v_0, \ldots, v_4)$ be a formula in the language of set theory and let κ be an infinite cardinal.

(1) An $\mathcal{L}^2$-theory T is Ψ-*outward consistent at* κ if there is a limit ordinal $\eta > \kappa$ such that for all infinite cardinals $\lambda < \kappa$ and all cardinals $\vartheta > \eta$ with $T \in H_\vartheta$, the partial order $\mathrm{Col}(\omega, \vartheta)$ forces that T is $<\lambda$-consistent in every outer ZFC*-model N of V_ϑ^V with the property that $\Psi(N, V_\vartheta^V, \lambda, \kappa, \eta)$ holds.
(2) A cardinal κ is a Ψ-*outward compactness cardinal for* $\mathcal{L}^2$ if all $\mathcal{L}^2$-theories that are Ψ-outward consistent at κ are consistent.

As noted after Definition 1.3 above, if ϑ is a limit ordinal, G is $\mathrm{Col}(\omega, \vartheta)$-generic over V and N is an outer ZFC*-model of V_ϑ^V in $V[G]$, then both N and V_ϑ^V are countable in $V[G]$. Moreover, it is easy to see that, given an $\mathcal{L}^2$-theory T and an infinite cardinal κ, the theory T is $<\kappa$-outward consistent (as defined in Definition 1.3) if and only if it is Ψ_{ms}-outward consistent at κ. In the following, we make a further basic observation about the above notions that relate them to strong compactness cardinals for $\mathcal{L}^2$.

Proposition 3.4. *Let $\Psi(v_0, \ldots, v_4)$ be a Δ_1^{ZFC}-formula that is a measure of closeness. If κ is a limit cardinal, then every $\mathcal{L}^2$-theory that is Ψ-outward consistent at κ is $<\kappa$-consistent. In particular, every cardinal greater than or equal to an extendible cardinal is a Ψ-outward compactness cardinal for $\mathcal{L}^2$.*

Proof. Let T be an $\mathcal{L}^2$-theory that contains an inconsistent subtheory T_0 of cardinality less than κ, and let $\eta > \kappa$ be a limit ordinal. Pick a cardinal $\vartheta > \eta$ such that $T \in H_\vartheta = V_\vartheta$ and V_ϑ is sufficiently elementary in V. In addition, pick a cardinal $\lambda < \kappa$ with $|T_0| < \lambda$. Then (B) in Definition 2.1.2 ensures that $\Psi(V_\vartheta, V_\vartheta, \lambda, \kappa, \eta)$ holds. Moreover, since T is an element of V_ϑ and V_ϑ was chosen to be sufficiently elementary in V, we also know that T is not $<\lambda$-consistent in V_ϑ. Now, let G be $\mathrm{Col}(\omega, \vartheta)$-generic over V. Then V_ϑ^V is an outer ZFC*-model of V_ϑ^V in $V[G]$. Moreover, since Ψ is a Δ_1^{ZFC}-formula, we can use Σ_1-upwards absoluteness to infer that $\Psi(V_\vartheta^V, V_\vartheta^V, \lambda, \kappa, \eta)$ holds in $V[G]$. Therefore, T is not Ψ-outward consistent at κ in V. □

We now combine the above two concepts to prove a general duality theorem that connects Ψ-largeness with Ψ-outwards compactness. Together with our earlier results, this can be seen as a generalization of Theorem 1.5 (see also Corollary 3.6 below). For the proof of this result, we inductively define *the second-order relativisation* $\Psi|v$ of an $\mathcal{L}^2$-formula in the language of set theory to some first-order variable v by setting $(v_0 \in v_1)|v \equiv v_0 \in v_1 \in v$, $(u \in W)|v \equiv u \in W \subseteq v$, $(\neg\varphi)|v \equiv \neg(\varphi|v)$, $(\varphi \wedge \psi)|v \equiv (\varphi|v) \wedge (\psi|v)$, $(\forall x\ \varphi(x))|v \equiv \forall x \in v\ (\varphi|v)(x)$ and $(\forall X\ \varphi(X))|v \equiv \forall X \subseteq v\ (\varphi|v)(X)$. Note that we consider the resulting formulas in this inductive definition as $\mathcal{L}^2$-formulas in each case. Moreover, if $\varphi(v_0, \ldots, v_{n-1})$ is an $\mathcal{L}^2$-formula, then ZFC* proves that $\langle M, \in\rangle \models_{\mathcal{L}^2} (\varphi|v_n)(c_0, \ldots, c_{n-1}, d)$ is equivalent to $\langle d, \in\rangle \models_{\mathcal{L}^2} \varphi(c_0, \ldots, c_{n-1})$ whenever M is a transitive set, $d \in M$ is transitive and $c_0, \ldots, c_{n-1} \in d$.

Theorem 3.5. *Given an $\mathcal{L}^2$-measure of closeness Ψ, an infinite cardinal κ is Ψ-large if and only if it is a Ψ-outward compactness cardinal for $\mathcal{L}^2$.*

Proof. First, assume that κ is Ψ-large. Let T be an $\mathcal{L}^2$-theory that is Ψ-outward consistent at κ, and let $\eta > \kappa$ be a limit ordinal witnessing this. Using Lemma 2.5, we can find a cardinal $\lambda > \eta$ with $T \in H_\lambda$, an inner model M and an elementary embedding $j : V \longrightarrow M$ such that $j(\kappa) > \lambda$ and $\Psi(V_\vartheta, V_\vartheta^M, \lambda, j(\kappa), j(\eta))$ holds for all limit ordinals $\vartheta > j(\eta)$. Pick a sufficiently large cardinal $\vartheta > j(\lambda)$ such that $j(\vartheta) = \vartheta$, $H_\vartheta = V_\vartheta$ and the set V_ϑ is a model of ZFC*. Elementarity now implies that, in M, the cardinal $j(\eta)$ witnesses that the $\mathcal{L}^2$-theory $j(T)$ is Ψ-outward consistent at $j(\kappa)$. Let G be $\mathrm{Col}(\omega, \vartheta)$-generic over V. Since V_ϑ^M is a model of ZFC*, V_ϑ^M is countable in $M[G]$. Moreover, our setup ensures that, in $M[G]$, the $\mathcal{L}^2$-theory $j(T)$ is $<\lambda$-consistent in every countable outer ZFC*-model N of V_ϑ^M with the property that $\Psi(N, V_\vartheta^M, \lambda, j(\kappa), j(\eta))$ holds. Since our assumptions on Ψ imply that this statement can be formulated by a Π_1-formula with parameters in $H_{\aleph_1}^{M[G]}$ and is therefore provably equivalent to a $\mathbf{\Pi}^1_2$-statement whose parameters are real numbers canonically coding the original parameters (see [11, Lemma 25.25]), *Shoenfield absoluteness* implies that the given statement also holds in $V[G]$. But Σ_1-upwards absoluteness implies that $\Psi(V_\vartheta^V, V_\vartheta^M, \lambda, j(\kappa), j(\eta))$ also holds in $V[G]$ and, since V_ϑ^V is a countable outer ZFC*-model of V_ϑ^M in $V[G]$, we conclude that $j(T)$ is $<\lambda$-consistent in V_ϑ^V. Now, note that $j[T] \subseteq j(T)$ is an element of V_ϑ^V, and this set has cardinality less than λ in V_ϑ^V. Thus, the theory $j[T]$ is consistent in V_ϑ^V and, by the nature of $\mathcal{L}^2$, we now know that $j[T]$ is a consistent $\mathcal{L}^2$-theory in V. But this also shows that T is consistent in V, because the finitary character of $\mathcal{L}^2$-formulae ensures that we can identify $j[T]$ and T via a renaming of symbols induced by j.

Now, assume that κ is a Ψ-outward compactness cardinal for $\mathcal{L}^2$. Let $\theta > \eta > \kappa$ be limit ordinals and let $\zeta > \eta$ be a cardinal. In addition, let $\Psi^*(v_0, v_1, v_2)$ be the $\mathcal{L}^2$-formula corresponding to Ψ as in Definition 3.1.3 and let $\Psi'(v_0, \ldots, v_3)$ denote the second-order relativisation $\Psi^*|v_3$ of $\Psi^*(v_0, v_1, v_2)$ to v_3. Consider the language that extends the language of set theory by a constant symbol b, constant symbols c_x for all elements x of $V_{\theta+1}$ and constant symbols d_γ for all $\gamma \leq \zeta$. Let T denote the second-order theory consisting of the following:

(1) The first-order elementary diagram of $V_{\theta+1}$, using the constant symbols c_x for x in $V_{\theta+1}$.
(2) All first-order sentences of the form "$d_\beta < d_\gamma < c_\kappa$" for $\beta < \gamma \leq \zeta$.
(3) The second-order sentence stating that the $\in$-relation is well-founded.
(4) The first-order sentence "$d_\zeta < b < c_\kappa$".
(5) The second-order sentence $\Psi'(b, c_\kappa, c_\eta, c_{V_\theta})$.

Claim. *The ordinal η witnesses that T is Ψ-outwards consistent at κ.*

Proof of the Claim. Let $\lambda < \kappa$ be a cardinal, let $\vartheta > \eta$ be a cardinal with $T \in H_\vartheta$, let G be $\mathrm{Col}(\omega, \vartheta)$-generic over V and let $N \in V[G]$ be an outer ZFC*-model of V_ϑ^V with the property that $\Psi(N, V_\vartheta^V, \lambda, \kappa, \eta)$ holds in $V[G]$. Since N is a model of ZFC* and $T \in H_\vartheta^V$ implies that $\theta < \vartheta$, we can apply (C) in Definition 2.1.2 to infer that $\Psi(V_\theta^N, V_\theta^V, \lambda, \kappa, \eta)$ holds in $V[G]$. Moreover, since Ψ is a $\Delta_1^{\mathrm{ZFC}^*}$-formula and all occurring parameters are elements of N, we know that $\Psi(V_\theta^N, V_\theta^V, \lambda, \kappa, \eta)$ also holds in N. In addition, the fact that N is a model of ZFC* ensures that

$$\langle V_\theta^V, \in\rangle \models_{\mathcal{L}^2} \Psi^*(\lambda, \kappa, \eta)$$

holds in N and hence we know that

$$\langle V_{\theta+1}^V, \in\rangle \models_{\mathcal{L}^2} \Psi'(\lambda, \kappa, \eta, V_\theta^V)$$

holds in N.

Now, let T_0 be a subtheory of T in N that has cardinality less than λ in N. Since λ is a cardinal in N, we can construct a model of T_0 with domain $V_{\theta+1}^V$ in N that interprets b as λ and all constant symbols of the form d_γ that appear in sentences in T_0 as ordinals less than λ. □

Our setup now ensures that T is consistent. Hence, we find a transitive set M and an elementary embedding $j : V_{\theta+1} \longrightarrow M$ such that $j(\kappa) > \zeta$ and

$$\langle M, \in\rangle \models_{\mathcal{L}^2} \Psi'(\zeta, j(\kappa), j(\eta), j(V_\theta)).$$

Elementarity then implies that $M \cap \mathrm{Ord} = j(\theta) + 1$ and $j(V_\theta) = M \cap V_{j(\theta)}$ is transitive. In particular, we know that

$$\langle M \cap V_{j(\theta)}, \in\rangle \models_{\mathcal{L}^2} \Psi^*(\zeta, j(\kappa), j(\eta)),$$

and therefore we can conclude that $\Psi(V_{j(\theta)}, M \cap V_{j(\theta)}, \zeta, j(\kappa), j(\eta))$ holds. These computations allow us to conclude that κ is Ψ-large. □

In combination with the results of the previous section, the above theorem now shows that all of the large cardinal properties that we considered so far can be characterized through compactness properties of second-order logic.

Corollary 3.6. *(1) A cardinal κ is a Ψ_{ms}-outward compactness cardinal for $\mathcal{L}^2$ if and only if there is a measurable cardinal less than or equal to κ.*

(2) A cardinal κ is a Ψ_{str}-outward compactness cardinal for $\mathcal{L}^2$ if and only if there is a strong cardinal less than or equal to κ.

(3) A cardinal κ is a Ψ_{sc}-outward compactness cardinal for $\mathcal{L}^2$ if and only if there is a supercompact cardinal less than or equal to κ.

(4) A cardinal κ is a Ψ_{stc}-outward compactness cardinal for $\mathcal{L}^2$ if and only if κ is ω_1-strongly compact.

(5) A cardinal κ is a Ψ_{ext}-outward compactness cardinal for $\mathcal{L}^2$ if and only if there is an extendible cardinal less than or equal to κ. □

By Proposition 3.4, extendible cardinals are Ψ-outward compactness cardinals for $\mathcal{L}^2$ whenever Ψ is an $\mathcal{L}^2$-measure of closeness. In particular, it is not possible to use Theorem 3.5 to characterize large cardinal notions stronger than extendibility. Therefore, we may view Ψ_{ext} as the strongest $\mathcal{L}^2$-measure of closeness. This view is further supported by the following observation:

Proposition 3.7. *If κ is an infinite cardinal, then every $<\kappa$-consistent $\mathcal{L}^2$-theory is Ψ_{ext}-outward consistent.*

Proof. Let T be a $<\kappa$-consistent $\mathcal{L}^2$-theory. Pick a cardinal $\eta > \kappa$ with the property that $T \in H_\eta = V_\eta$ and V_η is sufficiently elementary in V. Fix a cardinal $\lambda < \kappa$ and a cardinal $\vartheta > \eta$, and let G be $\mathrm{Col}(\omega, \vartheta)$-generic over V. Pick an outer ZFC*-model $N \in V[G]$ of V_ϑ^V with the property that $\Psi_{ext}(N, V_\vartheta^V, \lambda, \kappa, \eta)$ holds in $V[G]$. We then have $V_\eta^N = V_\eta^V$. Let T_0 be a subtheory of T of cardinality less than λ in N. Then, T_0 is contained in V and has cardinality less than λ in V. Hence, we know that T_0 is consistent in V and our setup ensures that T_0 is also consistent in V_η^V. Since $V_\eta^N = V_\eta^V$, this shows that T_0 is also consistent in N. These computations prove that T is $<\lambda$-consistent in N. □

4. On the naturalness of $\mathcal{L}^2$-characterizations

The results of the previous two sections naturally raise the question which other large cardinal notions between measurability and extendibility can be characterized through outward compactness properties of $\mathcal{L}^2$. In particular, Corollary 3.6.4 directly motivates the question whether strong compactness can be characterized in this way. Below, we will show that this is indeed possible. But, we will also argue that the presented characterization lacks several desirable features that the characterizations listed in Corollary 3.6 possess.

Given a formula $\psi(v_0, v_1, v_2)$ in the language of set theory, we let $\psi^\uparrow(v_0, \dots, v_4)$ denote the canonical formula in the language of set theory with the property that $\psi^\uparrow(N, M, \mu, \nu, \rho)$ holds if and only if the tuple $\langle N, M, \mu, \nu, \rho\rangle$ is suitable and either $N \cap V_\rho \subseteq M$ or $\psi(\mu, \nu, \rho)$ holds in $M \cap V_{\rho+\omega}$.

Proposition 4.1. *Given a formula $\psi(v_0, v_1, v_2)$ in the language of set theory, the formula $\psi^\uparrow$ is an $\mathcal{L}^2$-measure of closeness.*

Proof. The above definition directly ensures that $\psi^\uparrow$ satisfies (A) and (B) in Definition 2.1.2. Fix transitive sets $M \subseteq N$ closed under basic set operations with $M \cap \mathrm{Ord} = N \cap \mathrm{Ord} \in \mathrm{Lim}$ and limit ordinals $\mu < \nu < \rho < \theta \in M$. First, assume that $\psi^\uparrow(N, M, \mu, \nu, \rho)$ holds. In the first case, if $N \cap V_\rho \subseteq M$ holds, then $N \cap V_\theta \cap V_\rho \subseteq M \cap V_\theta$ holds and it follows that $\psi^\uparrow(N \cap V_\theta, M \cap V_\theta, \mu, \nu, \rho)$ is also true. Next, assume that $\psi(\mu, \nu, \rho)$ holds in $V_{\rho+\omega} \cap M$. Since $\rho+\omega \leq \theta$ and $V_{\rho+\omega} \cap (M \cap V_\theta) = V_{\rho+\omega} \cap M$, we then know that $\psi^\uparrow(N \cap V_\theta, M \cap V_\theta, \mu, \nu, \rho)$ also holds in this case. Since an analogous case distinction also shows that $\psi^\uparrow(N \cap V_\theta, M \cap V_\theta, \mu, \nu, \rho)$ implies that $\psi^\uparrow(N, M, \mu, \nu, \rho)$ holds, we can now conclude that $\psi^\uparrow$ satisfies (C) in Definition 2.1.2 and hence we know that (1) of Definition 3.1 holds. By arguing as in the proof of Proposition 3.2, we see that (2) of Definition 3.1 holds. Finally, it is easy to find an $\mathcal{L}^2$-formula $\varphi(v, W)$ in the language of set theory such that ZFC* proves that for every transitive set D and every ordinal α in D, the set $D \cap V_\alpha$ is the unique element X of $\mathcal{P}(D)$ with the property that $\langle D, \in\rangle \models_{\mathcal{L}^2} \varphi(\alpha, X)$ holds. Therefore, it follows that $\psi^\uparrow$ also satisfies (3) of Definition 3.1. □

We now show how the above concept can be used to obtain characterizations of strongly compact cardinals through compactness properties of $\mathcal{L}^2$. Let $\psi_{stc}(v_0, v_1, v_2)$ denote the canonical formula in the language of set theory with the property that $\psi_{stc}(\mu, \nu, \rho)$ holds if and only if for some cardinal $\mu < \kappa \leq \nu$, there exists a fine ultrafilter on $\mathcal{P}_\kappa(\rho)$.

Proposition 4.2. *A cardinal κ is $\psi^\uparrow_{stc}$-large if and only if there is a strongly compact cardinal less than or equal to κ. In particular, a cardinal κ is a $\psi^\uparrow_{stc}$-outward compactness cardinal for $\mathcal{L}^2$ if and only if there is a strongly compact cardinal less than or equal to κ.*

Proof. First, assume that μ is a strongly compact cardinal and $\kappa \geq \mu$ is a cardinal. Fix a limit ordinal $\eta > \kappa$ and a cardinal $\lambda \geq \eta$. By [12, Corollary 22.18], there exists a fine ultrafilter on $\mathcal{P}_\mu(\eta)$ and therefore $\psi_{stc}(\delta, \kappa, \eta)$ holds in $V_{\eta+\omega}$ for every cardinal $\delta < \mu$. Since μ is measurable, we can find a transitive class M and an elementary embedding $j : V \longrightarrow M$ with $j(\mu) > \lambda$.

By elementarity, we know that $\psi_{stc}(\lambda, j(\kappa), j(\eta))$ holds in $V^M_{j(\eta)+\omega}$. This directly implies that $\psi^{\uparrow}_{str}(V_\vartheta, V^M_\vartheta, \lambda, j(\kappa), j(\eta))$ holds for all limit ordinals $\vartheta > j(\eta)$. Using Lemma 2.5, we can now conclude that κ is $\psi^{\uparrow}_{stc}$-large.

Now, assume, towards a contradiction, that κ is a $\psi^{\uparrow}_{stc}$-large cardinal and no cardinal less than or equal to κ is strongly compact. Then there exists an ordinal $\alpha > \kappa$ with the property that for every cardinal $\mu \leq \kappa$, there is no fine ultrafilter on $\mathcal{P}_\mu(\alpha)$. Fix a limit ordinal $\eta > \alpha$. By Lemma 2.5, we can now find a cardinal $\lambda > \eta$, an inner model M and an elementary embedding $j : V \longrightarrow M$ with $j(\kappa) > \lambda$ and the property that $\psi^{\uparrow}_{stc}(V_{j(\eta)+\omega}, V^M_{j(\eta)+\omega}, \lambda, j(\kappa), j(\eta))$ holds. Since $\eta > \alpha$, we know that for all cardinals $\mu \leq \kappa$, there is no fine ultrafilter on $\mathcal{P}_\mu(\eta)$. This shows that for all $\delta < \kappa$, the statement $\psi_{stc}(\delta, \kappa, \eta)$ does not hold in $V_{\eta+\omega}$. By elementarity, $\psi_{stc}(\lambda, j(\kappa), j(\eta))$ fails in $V^M_{j(\eta)+\omega}$ and it follows that $\Psi_{ext}(V_{j(\eta)+\omega}, V^M_{j(\eta)+\omega}, \lambda, j(\kappa), j(\eta))$ holds. In particular, we have $V_{j(\eta)} \subseteq M$. These computations show that for every limit ordinal $\eta > \kappa$, there is an inner model M and an elementary embedding $j : V \longrightarrow M$ with $j(\kappa) > \eta$ and $V_{j(\eta)} \subseteq M$. This directly implies that κ is Ψ_{ext}-large and hence Lemma 2.13 ensures that there is an extendible cardinal less than or equal to κ, contradicting our assumption. □

Note that the above arguments can be modified to obtain analogous characterizations for various types of large cardinal properties defined through the existence of certain filters. Nevertheless, we consider the outward compactness characterizations obtained in this way as unsatisfying, because, except for ensuring the suitability of certain tuples, the measures of closeness $\psi^{\uparrow}$ utilized in these characterization only rely on properties of the ground model V and are therefore almost completely unaffected by the relationship between V and the given outer model. In the remainder of this section, we aim to isolate criteria for the naturalness of outward compactness characterizations that allow us to separate the characterizations obtained in Section 3 from the characterization of strong compactness derived above.

Our first criterion is motivated by Propositions 3.4 and 3.7 which, in conjunction, show that $<\kappa$-consistency provably coincides with Ψ_{ext}-outward consistency at all limit cardinals κ. Moreover, if Ψ is an $\mathcal{L}^2$-measure of closeness and κ is a Ψ-large cardinal that is not extendible, then Theorems 1.1 and 3.5 show that there is a $<\kappa$-consistent $\mathcal{L}^2$-theory that is not Ψ-outward consistent at κ. Therefore, it seems reasonable to expect that $\mathcal{L}^2$-measures of closeness Ψ that canonically correspond to large cardinal notions strictly weaker than extendibility have the property that Ψ-outwards compactness differs from $<\kappa$-consistency at some cardinal κ. This intuition is captured in the following definition:

Definition 4.3. We say that a formula $\Psi(v_0, \ldots, v_4)$ in the language of set theory *naturally induces a large cardinal property below extendibility* if ZFC proves that for some infinite cardinal κ, there is a $<\kappa$-consistent $\mathcal{L}^2$-theory that is not Ψ-outward consistent at κ.

A short argument now shows that the formulae introduced in Section 2 that correspond to large cardinal notions below extendibility fulfill the above criterion:

Proposition 4.4. *If Ψ is one of the formulae Ψ_{ms}, Ψ_{str}, Ψ_{stc} or Ψ_{sc}, then there is an $\mathcal{L}^2$-sentence φ in the language of set theory such that $\{\varphi\}$ is a consistent $\mathcal{L}^2$-theory that is not Ψ-outward consistent at any uncountable cardinal κ.*

Proof. There are $\mathcal{L}^2$-sentences φ_0 and φ_1 in the language of set theory such that ZFC* proves the following statements:

- If $i < 2$ and $\langle M, E \rangle \models_{\mathcal{L}^2} \varphi_i$, then E is a well-founded and extensional relation on E and the corresponding transitive collapse maps M to V_{ω_2+1}.
- We have

$$2^{\aleph_1} = \aleph_2 \iff \langle V_{\omega_2+1}, \in \rangle \models_{\mathcal{L}^2} \varphi_0 \iff \langle V_{\omega_2+1}, \in \rangle \models_{\mathcal{L}^2} \neg\varphi_1.$$

Now, fix $i < 2$ with $\langle V_{\omega_2+1}, \in\rangle \models_{\mathcal{L}^2} \varphi_i$. Note that there is a σ-closed partial order $\mathbb{P}$ with the property that whenever G is $\mathbb{P}$-generic over V, then $\langle V^{V[G]}_{\omega_2+1}, \in\rangle \models \neg\varphi_i$ holds in $V[G]$.

Assume, towards a contradiction, that η is a limit ordinal that witnesses that the theory $\{\varphi_i\}$ is Ψ-outward consistent at some uncountable cardinal κ. Pick a cardinal $\vartheta > \eta$ such that $\mathbb{P}$ is an element of H_ϑ and $\mathbb{P}$ forces that V_ϑ is a model of ZFC*. Let H be $\mathrm{Col}(\omega, \vartheta)$-generic over V. Then we can find $G \in V[H]$ that is $\mathbb{P}$-generic over V and our setup ensures that $V^{V[G]}_\vartheta$ is an outer ZFC*-model of V^V_ϑ in $V[H]$. Moreover, since $\mathbb{P}$ is σ-closed in V, it is easy to check that $\Psi(V^{V[G]}_\vartheta, V^V_\vartheta, \omega, \kappa, \eta)$ holds in $V[H]$. It now follows that $\{\varphi_i\}$ is consistent in $V^{V[G]}_\vartheta$ and this implies that $\langle V^{V[G]}_{\omega_2+1}, \in\rangle \models_{\mathcal{L}^2} \varphi_i$ holds in $V[G]$, a contradiction. □

Corollary 4.5. *The formulae Ψ_{ms}, Ψ_{str}, Ψ_{stc} and Ψ_{sc} all naturally induce a large cardinal property below extendibility.* □

As a second criterion for the naturalness of outward compactness characterizations, we want to demand that the given characterization actually depends on the consistency of theories in outer models and does not only check the validity of statements in initial segments of V. This criterion will be given by the negation of the property introduced in the next definition. It will turn out to be a weakening of our first criterion.

Definition 4.6. Given a formula $\Psi(v_0, \ldots, v_4)$ in the language of set theory, we say that *Ψ-outward compactness trivializes at a cardinal κ* if there is a proper class of limit ordinals $\eta > \kappa$ with the property that for all cardinals $\lambda < \kappa$ and all limit ordinals $\vartheta > \eta$, the partial order $\mathrm{Col}(\omega, \vartheta)$ forces that $N \cap V^{V[G]}_\eta \subseteq V$ holds for every outer ZFC*-model N of V^V_ϑ with the property that $\Psi(N, V^V_\vartheta, \lambda, \kappa, \eta)$ holds.

As a first observation, we show that the above property causes Ψ-largeness to be equivalent to Ψ_{ext}-largeness:

Proposition 4.7. *Let $\Psi(v_0, \ldots, v_4)$ be an $\mathcal{L}^2$-measure of closeness. If Ψ-outward compactness trivializes at a Ψ-large cardinal κ, then there exists an extendible cardinal less than or equal to κ.*

Proof. Assume, towards a contradiction, that no cardinal less than or equal to κ is extendible. By [12, Proposition 23.15], we can find an ordinal $\alpha > \kappa$ with the property that for every ordinal β, there is no non-trivial elementary embedding $j : V_\alpha \longrightarrow V_\beta$ with $\mathrm{crit}(j) \leq \kappa$. Then there exists a limit ordinal $\eta > \alpha$ such that for all cardinals $\lambda < \kappa$ and all limit ordinals $\vartheta > \eta$, the partial order $\mathrm{Col}(\omega, \vartheta)$ forces that $N \cap V^{V[G]}_\eta \subseteq V$ holds for every outer ZFC*-model N of V^V_ϑ with the property that $\Psi(N, V^V_\vartheta, \lambda, \kappa, \eta)$ holds. Using Lemma 2.5, we now find a cardinal $\lambda > \eta$, an inner model M and an elementary embedding $j : V \longrightarrow M$ such that $j(\kappa) > \lambda$ and $\Psi(V_\vartheta, V^M_\vartheta, \lambda, j(\kappa), j(\eta))$ holds for every limit ordinal $\vartheta > j(\eta)$. Pick a limit ordinal $\vartheta > j(\eta)$ with the property that $j(\vartheta) = \vartheta$ and V_ϑ is a model of ZFC*. Let G be $\mathrm{Col}(\omega, \vartheta)$-generic over V. By using the elementarity of j and Shoenfield absoluteness as in the proof of Theorem 3.5, it now follows that, in $V[G]$, we have $N \cap V^{V[G]}_{j(\eta)} \subseteq V^M_\vartheta$ whenever N is an outer ZFC*-model of V^M_ϑ with the property that $\Psi(N, V^M_\vartheta, \lambda, j(\kappa), j(\eta))$ holds. Since $\alpha < j(\eta)$ and our assumptions on Ψ ensure that $\Psi(V^V_\vartheta, V^M_\vartheta, \lambda, j(\kappa), j(\eta))$ holds in both V and $V[G]$, we can now conclude that $V^V_{j(\eta)} \subseteq M$ and $j \restriction V_\alpha : V_\alpha \longrightarrow V_{j(\alpha)}$ is a non-trivial elementary embedding with $\mathrm{crit}(j \restriction V_\alpha) \leq \kappa$, contradicting our initial assumption. □

We now show how our two criteria are related:

Lemma 4.8. *Let $\Psi(v_0, \ldots, v_4)$ be a formula in the language of set theory. If Ψ-outward compactness trivializes at an infinite cardinal κ, then every $<\kappa$-consistent $\mathcal{L}^2$-theory is Ψ-outward consistent at κ.*

Proof. Let T be a $<\kappa$-consistent $\mathcal{L}^2$-theory. Then there exists a limit ordinal $\eta > \kappa$ such that the following statements hold:

- $T \in H_\rho$ for some cardinal $\rho < \eta$.
- Every subtheory of T of cardinality less than κ has a model in V_η.
- For all cardinals $\lambda < \kappa$ and all limit ordinals $\vartheta > \eta$, the partial order $\mathrm{Col}(\omega, \vartheta)$ forces that $N \cap V_{\eta+1}^{V[G]} \subseteq V$ holds for every outer ZFC*-model N of V_ϑ^V with the property that $\Psi(N, V_\vartheta^V, \lambda, \kappa, \eta)$ holds.

Now, let $\lambda < \kappa$ be a cardinal, let $\vartheta > \eta$ be a cardinal, let G be $\mathrm{Col}(\omega, \vartheta)$-generic over V and let N be an outer ZFC*-model of V_ϑ^V in $V[G]$ with the property that $\Psi(N, V_\vartheta^V, \lambda, \kappa, \eta)$ holds in $V[G]$. We then know that $N \cap V_\eta^{V[G]} \subseteq V$ and therefore every subtheory of T of cardinality less than κ in N is contained in V and has cardinality less than κ in V. In particular, the theory T is $<\lambda$-consistent in N. □

Corollary 4.9. *If $\Psi(v_0, \ldots, v_4)$ is a formula in the language of set theory that naturally induces a large cardinal property below extendibility, then* ZFC *proves that Ψ-outward compactness does not trivialize at all infinite cardinals.* □

We are now ready to show that, in contrast to the characterizations derived in Section 3, the characterization of strong compactness given by Proposition 4.2 does not meet the above criteria.

Proposition 4.10. *If there are no strongly compact cardinals, then $\psi_{stc}^{\uparrow}$-outward compactness trivializes at all infinite cardinals.*

Proof. Fix an infinite cardinal κ. Then there is a limit ordinal $\eta > \kappa$ with the property that for every infinite cardinal $\mu \leq \kappa$, there is no fine ultrafilter on $\mathcal{P}_\mu(\eta)$. Let $\lambda < \kappa$ be a cardinal, let $\vartheta > \kappa$ be a cardinal, let G be $\mathrm{Col}(\omega, \vartheta)$-generic over V and let N be an outer ZFC*-model of V_ϑ^V in $V[G]$ with the property that $\psi_{stc}^{\uparrow}(N, V_\vartheta^V, \lambda, \kappa, \eta)$ holds in $V[G]$. Since $\psi_{stc}(\lambda, \kappa, \eta)$ does not hold in $V_{\eta+\omega}^V$, we can now conclude that $N \cap V_\eta^{V[G]} \subseteq V$ holds. □

Corollary 4.11. *If the formula $\psi_{str}^{\uparrow}$ naturally induces a large cardinal property below extendibility, then* ZFC *is inconsistent.* □

5. Fragments of Vopěnka's Principle

We now aim to extend the equivalence established in Theorem 3.5 to obtain characterizations of stronger large cardinal assumptions. For this purpose, we start by introducing a natural strengthening of the the notion of Ψ-largeness that is motivated by classical characterizations of Woodin cardinals (see [12, Theorem 26.14]) and related variations of supercompactness and extendibility (see [12, p. 339]).

Definition 5.1. Given a formula $\Psi(v_0, \ldots, v_4)$ in the language of set theory and a class A,[11] a cardinal κ is Ψ*-large for A* if for all limit ordinals $\kappa < \eta < \theta$, there are unboundedly many cardinals $\lambda \geq \eta$ with the property that there is a transitive set M and an elementary embedding $j : V_{\theta+1} \longrightarrow M$ such that $j(\kappa) > \lambda$,

$$j(A \cap V_\kappa) \cap V_\lambda \;=\; A \cap M \cap V_\lambda$$

and $\Psi(V_{j(\theta)}, M \cap V_{j(\theta)}, \lambda, j(\kappa), j(\eta))$ holds.

In the following, we present the first example of a large cardinal principle that we aim to represent through the above concept. Remember that *Vopěnka's Principle* is a scheme of axioms stating that for every proper class $\mathcal{C}$ of structures of the same type, there exist $A, B \in \mathcal{C}$ with

[11] Note that, since we are working in a ZFC context, all classes are definable.

$A \neq B$ and an elementary embedding from A to B. Using results from [1] and [2], it is easy to show that Vopěnka's Principle provides various examples of cardinals with the given property.

Proposition 5.2. *If Vopěnka's Principle holds, then for every measure of closeness $\Psi(v_0, \ldots, v_4)$ and every class A, there is a proper class of cardinals that are Ψ-large for A.*

Proof. Pick a natural number $n > 0$, a Σ_n-formula $\varphi(v_0, v_1)$, and a set y with the property that $A = \{x \mid \varphi(x, y)\}$, and pick an ordinal ξ. Using [2, Corollary 6.9], we find a cardinal $\kappa > \xi$, with $y \in V_\kappa$, that is $C^{(n)}$-*extendible*, i.e., the cardinal κ has the property that for every $\lambda > \kappa$ with $V_\lambda \prec_{\Sigma_n} V$, there exists an ordinal ζ and an elementary embedding $j : V_\lambda \longrightarrow V_\zeta$ with $\operatorname{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $V_{j(\kappa)} \prec_{\Sigma_n} V$ (see [1, Definition 3.2]). We can now use [2, Proposition 3.4] to see that $V_\kappa \prec_{\Sigma_{n+2}} V$ and hence $A \cap V_\kappa = \{x \in V_\kappa \mid V_\kappa \models \varphi(x, y)\}$. In the following, fix limit ordinals $\theta > \eta > \kappa$ and a cardinal $\lambda > \theta$ with $V_\lambda \prec_{\Sigma_n} V$. In this situation, we can find an ordinal ζ and an elementary embedding $j : V_\lambda \longrightarrow V_\zeta$ with $\operatorname{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $V_{j(\kappa)} \prec_{\Sigma_n} V$. Elementarity then implies that

$$j(A \cap V_\kappa) \cap V_\lambda \ = \ \{x \in V_\lambda \mid V_{j(\kappa)} \models \varphi(x, y)\} \ = \ A \cap V_\lambda.$$

Since (B) in Definition 2.1.2 ensures that $\Psi(V_{j(\theta)}, V_{j(\theta)}, \lambda, j(\kappa), j(\eta))$ holds, we can conclude that the embedding $j \restriction V_{\theta+1} : V_{\theta+1} \longrightarrow V_{j(\theta)+1}$ possesses the desired properties with respect to η and λ. □

Next, we prove the direct analog of Lemma 2.5 for Definition 5.1.

Lemma 5.3. *Given a measure of closeness $\Psi(v_0, \ldots, v_4)$ and a class A, a cardinal κ is Ψ-large for A if and only if for every limit ordinal $\eta > \kappa$, there exist unboundedly many cardinals $\lambda \geq \eta$ with the property that there is an inner model M and an elementary embedding $j : V \longrightarrow M$ such that $j(\kappa) > \lambda$, $j(A \cap V_\kappa) \cap V_\lambda = A \cap M \cap V_\lambda$ and $\Psi(V_\vartheta, V_\vartheta^M, \lambda, j(\kappa), j(\eta))$ holds for all limit ordinals $\vartheta > j(\eta)$.*

Proof. First, assume that κ is Ψ-large for A and $\zeta > \eta > \kappa$ are limit ordinals. We find a sufficiently large fixed point $\nu > \zeta$ of the $\beth$-function, a cardinal $\lambda \geq \zeta$, a transitive set M_0 and an elementary embedding $i : V_{\nu+1} \longrightarrow M_0$ such that $i(\kappa) > \lambda$, $i(A \cap V_\kappa) \cap V_\lambda = A \cap M \cap V_\kappa$ and $\Psi(V_{i(\nu)}, M_0 \cap V_{i(\nu)}, \lambda, i(\kappa), i(\eta))$ holds. By repeating the proof of Lemma 2.5, we can construct an inner model M with $V^M_{i(\nu)} = M_0 \cap V_{i(\nu)}$ and an elementary embedding $j : V \longrightarrow M$ with $j \restriction (V_\nu \cup \{\nu\}) = i \restriction (V_\nu \cup \{\nu\})$. In particular, we know that $j(\kappa) > \lambda$ and $j(A \cap V_\kappa) \cap V_\lambda = A \cap M \cap V_\lambda$. Moreover, for every limit ordinal $\vartheta \geq j(\eta)$, we can use (C) in Definition 2.1.2 to show that $\Psi(V_\vartheta, V_\vartheta^M, \lambda, j(\kappa), j(\eta))$ holds.

Now, assume that for every limit ordinal $\eta > \kappa$, there exist unboundedly many cardinal $\lambda \geq \eta$ with the property that there is a transitive class M and an elementary embedding $j : V \longrightarrow M$ with $j(\kappa) > \lambda$, $j(A \cap V_\kappa) \cap V_\lambda = A \cap V_\lambda$ and the property that $\Psi(V_\vartheta, V_\vartheta^M, \lambda, j(\kappa), j(\eta))$ holds for all limit ordinals $\vartheta > j(\eta)$. Pick limit ordinals $\theta > \eta > \kappa$ and an ordinal $\zeta > \eta$. Then there is a cardinal $\lambda > \zeta$, an inner model M and an elementary embedding $j : V \longrightarrow M$ such that the above statements hold with respect to κ, η and λ. We then know that $M \cap V_{j(\theta)+1}$ is a transitive set and the map $j \restriction V_{\theta+1} : V_{\theta+1} \longrightarrow M \cap V_{j(\theta)+1}$ possesses all of the desired properties. □

5.1. **Vopěnka's Principle.** We now show how the validity of Vopěnka's Principle can be characterized through the existence of cardinals that are Ψ-large for certain classes. Note that the results of [1, Section 4] show that, over ZFC, Vopěnka's Principle is equivalent to the scheme that only requires the instances of Vopěnka's Principle for classes of finite signatures that are defined by formulas without parameters to hold. The following lemma is a direct adaptation of the results discussed in [12, p. 339] to our setting.

Lemma 5.4. *The following schemes are equivalent over* ZFC*:*

(1) Vopěnka's Principle.
(2) For every class A, there is a proper class of cardinals that are Ψ_{ext}-large for A.
(3) For every parameter-free formula $\varphi(v)$ in the language of set theory, there is a cardinal that is Ψ_{sc}-large for the class $\{x \mid \varphi(x)\}$.

Proof. First, note that the implication from (1) to (2) already follows from Proposition 5.2, and the implication from (2) to (3) is trivial. Now, assume that (3) holds. Fix a proper class $\mathcal{C}$ of structures of the same finite signature that is definable by a formula $\varphi(v)$ without parameters. Let κ be a cardinal that is Ψ_{sc}-large for $\mathcal{C}$. By our assumptions, there is a structure B in $\mathcal{C}$ that is not an element of H_κ. Fix a cardinal $\eta > \kappa$ with $B \in H_\eta = V_\eta$ and a cardinal $\theta > \eta$ such that the formula $\varphi(v)$ is absolute between V_θ and V. In this situation, we can find a cardinal $\lambda \geq \eta$, a transitive set M and an elementary embedding $j : V_{\theta+1} \longrightarrow M$ such that $j(\kappa) > \lambda$, $j(\mathcal{C}\cap V_\kappa)\cap V_\lambda = \mathcal{C}\cap V_\lambda$, and $\Psi_{sc}(V_{j(\theta)}, M\cap V_{j(\theta)}, \lambda, j(\kappa), j(\eta))$ holds. We then know that $B \in H^M_{j(\kappa)}$ and hence elementarity implies that $j(B) \neq B$. This shows that $j(B)$ is a structure of the given signature and the embedding j induces an elementary embedding of B into $j(B)$. Moreover, our setup ensures that the set M contains both B and the given elementary embedding of B into $j(B)$. In addition, we know that

$$B \in \mathcal{C} \cap V_\lambda \subseteq j(\mathcal{C} \cap V_\kappa) = j(\{x \in V_\kappa \mid V_\theta \models \varphi(x)\}).$$

By elementarity, this implies that, in $V_{\theta+1}$, there is an elementary embedding of a structure A of the given signature into B such that $A \neq B$ and $V_\theta \models \varphi(A)$. This shows that, in V, there is a structure $A \in \mathcal{C}$ with $A \neq B$ and an elementary embedding of A into B. As discussed earlier, the results of [1, Section 4] show that these computations ensure that Vopěnka's Principle holds. □

5.2. **Ord is Woodin.** Remember that "Ord *is Woodin*" is the scheme of axioms stating that for every class A, there exists a cardinal κ that is *strong for A*, i.e., for every ordinal λ, κ is *λ-strong for A*: there is an inner model M with $V_\lambda \subseteq M$ and an elementary embedding $j : V \longrightarrow M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $A \cap V_\lambda = j(A \cap V_\kappa) \cap V_\lambda$ (see [5] and [17]). The following lemma shows how this principle is related to the concepts introduced above. Its proof is based on the *Σ_n-Product Reflection Principle* introduced by Bagaria and Wilson in [5].

Remember that for any set S of relational structures of the same type, the *set-theoretic product* $\prod S$ is defined to be the structure whose domain is the set of all functions f with domain S and the property that $f(\mathcal{A}) \in A$ holds for every structure $\mathcal{A} = \langle A, \ldots\rangle$ in S, and whose interpretation of relation symbols is defined pointwise, i.e., for every $(n+1)$-ary relation symbol R in the given language, we have $\dot{R}^{\prod S}(f_0, \ldots, f_n)$ if and only if $R^{\mathcal{A}}(f_0(\mathcal{A}), \ldots, f_n(\mathcal{A}))$ holds for all $\mathcal{A} \in S$. For a class $\mathcal{C}$ of relational structures of the same type, we then let $\mathrm{PRP}_{\mathcal{C}}$ denote the statement that there exists a non-empty subset S of $\mathcal{C}$ with the property that for every $\mathcal{A}$ in $\mathcal{C}$, there exists a homomorphism[12] of $\prod S$ into $\mathcal{A}$. Given a natural number $n > 0$, Bagaria and Wilson then defined the *Σ_n-Product Reflection Principle* to be the statement that $\mathrm{PRP}_{\mathcal{C}}$ holds for every non-empty class $\mathcal{C}$ of graphs that is definable by a Σ_n-formula without parameters (see [5, Definition 3.1]). The results of [5, Section 5] show that Ord is Woodin if and only if the Σ_n-Product Reflection Principle holds for all $n > 0$.

Lemma 5.5. *The following schemes are equivalent over* ZFC*:*

(1) Ord *is Woodin.*
(2) For every class A, there is a proper class of cardinals that are Ψ_{str}-large for A.
(3) For every parameter-free formula $\varphi(v)$ in the language of set theory, there is a cardinal that is Ψ_{str}-large for the class $\{x \mid \varphi(x)\}$.

[12]As defined in [10, Section 1.2].

Proof. First, assume that (1) holds. Fix an ordinal ξ and pick a formula $\varphi(v_0,\ldots,v_m)$ and parameters $y_0,\ldots,y_{m-1}$ with $A = \{x \mid \varphi(x,y_0,\ldots,y_{m-1})\}$. Define

$$A_* \;=\; \{\langle x_0,\ldots,x_m\rangle \mid \varphi(x_0,\ldots,x_m)\}.$$

Using [5, Theorem 5.13], we find a cardinal $\kappa > \xi$ that satisfies $y_0,\ldots,y_{m-1} \in V_\kappa$ and is strong for A_*. Fix a cardinal $\lambda > \kappa$. We can then find an inner model M with $V_\lambda \subseteq M$ and an elementary embedding $j : V \longrightarrow M$ with $\operatorname{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $A_* \cap V_\lambda = j(A_* \cap V_\kappa) \cap V_\lambda$. Given $x \in V_\lambda \subseteq M$, we then know that

$$\begin{aligned} x \in A \iff \langle x, y_0,\ldots,y_{m-1}\rangle \in A_* \iff \langle x, y_0,\ldots,y_{m-1}\rangle \in j(A_* \cap V_\kappa) \\ \iff M \models \varphi(x,y_0,\ldots,y_{m-1}) \iff x \in j(A \cap V_\kappa). \end{aligned}$$

These equivalences show that $j(A \cap V_\kappa) \cap V_\lambda = A \cap V_\lambda$ holds. Moreover, it follows that $\Psi_{str}(V_\vartheta, V_\vartheta^M, \lambda, j(\kappa), j(\eta))$ holds for all limit ordinals $\kappa < \eta \leq \lambda$ and all limit ordinals $\vartheta > j(\eta)$. By Lemma 5.3, this shows that the cardinal $\kappa > \xi$ is Ψ_{str}-large for A. Hence, we have shown that (2) holds in this case.

The implication from (2) to (3) is immediate.

Now, assume that (3) holds. Let $\mathcal{C}$ denote a non-empty class of graphs that is defined by a formula $\varphi(v)$ without parameters. By our assumptions, there is a cardinal κ that is Ψ_{str}-large for $\mathcal{C}$. We then know that $\mathcal{C} \cap V_\kappa \neq \emptyset$. Fix a graph $\mathcal{G}$ in $\mathcal{C}$ and pick a cardinal $\eta > \kappa$ with $\mathcal{G} \in H_\eta = V_\eta$. Using Lemma 5.3, we now find a cardinal $\lambda > \eta$, a transitive class M with $V_\lambda \subseteq M$ and an elementary embedding $j : V \longrightarrow M$ satisfying $j(\kappa) > \lambda$ and $j(\mathcal{C} \cap V_\kappa) \cap V_\lambda = \mathcal{C} \cap V_\lambda$. In this situation, we know that $\mathcal{G} \in j(\mathcal{C} \cap V_\kappa)$. Since the graph $j(\prod(\mathcal{C} \cap V_\kappa))$ is a subgraph of the product graph $\prod(j(\mathcal{C} \cap V_\kappa))$ and therefore the embedding j induces a homomorphism of $\prod(\mathcal{C} \cap V_\kappa)$ into $\prod(j(\mathcal{C} \cap V_\kappa))$, we can combine this homomorphism with the projection from $\prod(j(\mathcal{C} \cap V_\kappa))$ to $\mathcal{G}$ to find a homomorphism from $\prod(\mathcal{C} \cap V_\kappa)$ to $\mathcal{G}$. These computations show that the Σ_n-Product Reflection Principle holds for every natural number $n > 0$ and therefore the results of [5, Section 5] allow us to conclude that (1) holds in this case. □

6. Abstract logics

In order to connect the existence of cardinals that are Ψ-strong for certain classes to compactness properties of strong logics, we now fix a notion of *abstract logic*. This notion, as introduced below, is taken from [8]. As argued in [8, Section 4], results of Makowsky in [15] show that Vopěnka's Principle is equivalent to the assumption that every abstract logic has a strong compactness cardinal (see Theorem 6.3 below), i.e., a direct analogue of Theorem 1.1 that replaces second-order logic with arbitrary abstract logics and the existence of an extendible cardinal with the validity of Vopěnka's Principle. The goal of this section is to formulate and verify similar analogues of Theorem 3.5 and Corollary 3.6 for abstract logics.

Definition 6.1. (1) A *language* is a tuple

$$\tau \;=\; \langle \mathfrak{F}_\tau, \mathfrak{R}_\tau, \mathfrak{C}_\tau, a_\tau\rangle$$

consisting of disjoint sets $\mathfrak{F}_\tau$ (the *function symbols* in τ), $\mathfrak{R}_\tau$ (the *relation symbols* in τ) and $\mathfrak{C}_\tau$ (the *constant symbols* in τ) together with a function $a_\tau : \mathfrak{F}_\tau \cup \mathfrak{R}_\tau \longrightarrow \omega$ (the *arity function* for τ).

(2) Given a language τ, a τ-*structure* is a tuple

$$M \;=\; \langle |M|, \langle f^M \mid f \in \mathfrak{F}_\tau\rangle, \langle r^M \mid r \in \mathfrak{R}_\tau\rangle, \langle c^M \mid c \in \mathfrak{C}_\tau\rangle\rangle,$$

where $|M|$ is a non-empty set (the *universe of* M), each f^M is an $a_\tau(f)$-ary function on $|M|$, each r^M is an $a_\tau(r)$-ary relation on $|M|$, and each c^M is an element of $|M|$. We let Str_τ denote the collection of all τ-structures.

(3) A *renaming* of a language σ into a language τ is a bijection

$$r : \mathfrak{F}_\sigma \cup \mathfrak{R}_\sigma \cup \mathfrak{C}_\sigma \longrightarrow \mathfrak{F}_\tau \cup \mathfrak{R}_\tau \cup \mathfrak{C}_\tau$$

satisfying $r[\mathfrak{F}_\sigma] = \mathfrak{F}_\tau$, $r[\mathfrak{R}_\sigma] = \mathfrak{R}_\tau$, $r[\mathfrak{C}_\sigma] = \mathfrak{C}_\tau$ and $a_\sigma(s) = a_\tau(r(s))$ for all $s \in \mathfrak{F}_\sigma \cup \mathfrak{R}_\sigma$. In this setting, we let $r^* : Str_\sigma \longrightarrow Str_\tau$ denote the canonical bijective class function induced by r that fixes the universes of the given σ-structures.

Given the above definitions, there is a canonical notion of a *sublanguage* of a given language. Moreover, given a sublanguage σ of a language τ, there is a canonical notion of a σ-reduct $M \restriction \sigma \in Str_\sigma$ of a τ-structure M. Finally, given a language τ, there is a canonical notion of an isomorphism of τ-structures.

Definition 6.2. An *abstract logic* $\langle \mathcal{L}, \models_\mathcal{L} \rangle$ consists of a class function $\mathcal{L}$ and a binary class relation $\models_\mathcal{L}$ that satisfy the following conditions:

(1) The domain of $\mathcal{L}$ is the class of all languages.
(2) If $M \models_\mathcal{L} \phi$, then there is a language τ with $M \in Str_\tau$ and $\phi \in \mathcal{L}(\tau)$.
(3) (Monotonicity) If σ is a sublanguage of τ, then $\mathcal{L}(\sigma) \subseteq \mathcal{L}(\tau)$.
(4) (Expansion) If σ is a sublanguage of τ, $M \in Str_\tau$ and $\phi \in \mathcal{L}(\sigma)$, then

$$M \models_\mathcal{L} \phi \iff M \restriction \sigma \models_\mathcal{L} \phi.$$

(5) (Isomorphism) Given a language τ and isomorphic τ-structures M and N, we have

$$M \models_\mathcal{L} \phi \iff N \models_\mathcal{L} \phi$$

for all $\phi \in \mathcal{L}(\tau)$.
(6) (Renaming) Every renaming r of a language σ into a language τ induces a unique bijection $r_* : \mathcal{L}(\sigma) \longrightarrow \mathcal{L}(\tau)$ with the property that

$$M \models_\mathcal{L} \phi \iff r^*(M) \models_\mathcal{L} r_*(\phi)$$

holds for every $M \in Str_\sigma$ and every $\phi \in \mathcal{L}(\sigma)$.
(7) There exists a cardinal $\mathscr{o}$ with the property that for every language τ and every $\phi \in \mathcal{L}(\tau)$, there exists a sublanguage σ of τ with $|\mathfrak{F}_\sigma \cup \mathfrak{R}_\sigma \cup \mathfrak{C}_\sigma| < \mathscr{o}$ and $\phi \in \mathcal{L}(\sigma)$.

In the remainder of this section, we will usually write $\mathcal{L}$ instead of $\langle \mathcal{L}, \models_\mathcal{L} \rangle$ to denote an abstract logic. Given an abstract logic $\mathcal{L}$, we call $\models_\mathcal{L}$ the *satisfaction relation* of $\mathcal{L}$ and, given a language τ, we say that $\mathcal{L}(\tau)$ is the set of τ*-sentences*. In addition, we call the least cardinal $\mathscr{o}$ witnessing (7) in Definition 6.2 the *occurrence number* of $\mathcal{L}$. We say that a set T is an $\mathcal{L}$*-theory* if there is a language τ with $T \subseteq \mathcal{L}(\tau)$. We say that an $\mathcal{L}$-theory T is *consistent* if there is a language τ with $T \subseteq \mathcal{L}(\tau)$ and a τ-structure M with $M \models_\mathcal{L} \phi$ for all $\phi \in T$. Given an infinite cardinal κ, we say that an $\mathcal{L}$-theory T is $<\kappa$*-consistent* if every subset of T of cardinality less than κ is consistent. Finally, we say that an infinite cardinal κ is a *strong compactness cardinal for* $\mathcal{L}$ if every $<\kappa$-consistent $\mathcal{L}$-theory is consistent. Makowsky's result from [15] can now be formulated in the following way:

Theorem 6.3 (Makowsky [15])**.** *The following schemes are equivalent over* ZFC*:*

(1) Vopěnka's Principle.
(2) Every abstract logic has a strong compactness cardinal.

We now generalize the notions introduced in Section 3 to obtain characterizations of the validity of fragments of Vopěnka's Principle through compactness properties of abstract logics. In the following, we say that an abstract logic $\mathcal{L}$ is defined by formulas $\varphi_0(v_0, v_1, v_2)$ and $\varphi_1(v_0, v_1, v_2)$ together with a parameter z if $\mathcal{L} = \{\langle x, y \rangle \mid \varphi_0(x, y, z)\}$ and $\models_\mathcal{L} = \{\langle x, y \rangle \mid \varphi_1(x, y, z)\}$.

Definition 6.4. Let F be an $\in$-theory, let $\Psi(v_0, \ldots, v_4)$ be a formula in the language of set theory, let κ be an infinite cardinal and let $\mathcal{L}$ be an abstract logic that is defined by formulas φ_0 and φ_1 together with a parameter z.

(1) An $\mathcal{L}$-theory T is F-Ψ-*outward consistent at* κ if there is a limit ordinal $\eta > \kappa$ such that for all cardinals $\lambda < \kappa$ and all cardinals $\vartheta > \eta$ with $T, z \in H_\vartheta$, the partial order $\mathrm{Col}(\omega, \vartheta)$ forces that T is a $<\lambda$-consistent $\mathcal{L}'$-theory in N whenever N is an outer F-model of V_ϑ^V, $\mathcal{L}'$ is an abstract logic in N and the following statements hold:
   (a) $\Psi(N, V_\vartheta^V, \lambda, \kappa, \eta)$ holds.
   (b) The formulas φ_0 and φ_1 are absolute from V to N with respect to parameters contained in V_λ^V.[13]
   (c) The formulas φ_0 and φ_1 together with the parameter z define $\mathcal{L}'$ in N.
   (d) T is an $\mathcal{L}'$-theory in N.

(2) A cardinal κ is an F-Ψ-*outward compactness cardinal for* $\mathcal{L}$ if all $\mathcal{L}$-theories that are F-Ψ-outward consistent at κ are consistent.

In order to motivate the above concept, we now discuss its relation to the compactness properties of $\mathcal{L}^2$ introduced in Section 3. For this purpose, we make the following definition, which will also be relevant later in this section.

Definition 6.5. A measure of closeness $\Psi(v_0, \ldots, v_4)$ is *strong* if it is a $\Delta_1^{\mathrm{ZFC}^-}$-formula with

$$\mathrm{ZFC} \vdash \forall M, N, \mu, \nu, \rho\ [\Psi(N, M, \mu, \nu, \rho) \longrightarrow \Psi_{str}(N, M, \mu, \nu, \rho)].$$

Proposition 6.6. *Given a strong measure of closeness Ψ, an $\mathcal{L}^2$-theory T is Ψ-outward consistent at a cardinal κ if and only if it is* ZFC^*-Ψ*-outward consistent at κ. In particular, a cardinal κ is a Ψ-outward compactness cardinal for $\mathcal{L}^2$ if and only if it is a* ZFC^*-Ψ*-outward compactness cardinal for $\mathcal{L}^2$.*

Proof. Our definitions directly ensure that every $\mathcal{L}^2$-theory that is Ψ-outward consistent at a cardinal κ is also ZFC^*-Ψ-outward consistent at the given cardinal κ. In the other direction, assume that a limit ordinal η witnesses that an $\mathcal{L}^2$-theory T is ZFC^*-Ψ-outward consistent at a cardinal κ. Fix a cardinal $\lambda < \kappa$ and a cardinal $\vartheta > \eta$ with $T \in H_\vartheta$. In addition, let G be $\mathrm{Col}(\omega, \vartheta)$-generic over V and pick an outer ZFC^*-model N of V_ϑ^V in $V[G]$ with the property that $\Psi(N, V_\vartheta^V, \lambda, \kappa, \eta)$ holds in $V[G]$. We then know that the canonical formulas defining $\mathcal{L}^2$ define an abstract logic $\mathcal{L}'$ in N and T is an $\mathcal{L}'$-theory in N. Moreover, these formulas are absolute between V and V_λ as well as between N and V_λ^N. Since our setup ensures that $V_\lambda^N = V_\lambda^V$, we can now conclude that T is $<\lambda$-consistent in N. □

As a further motivation for the results proven below, we now derive an analogue of Proposition 3.4 for the above compactness property.

Proposition 6.7. *Let Ψ be a Δ_1^{ZFC}-formula that is a measure of closeness and let $n > 1$ be a natural number. If $\mathcal{L}$ is an abstract logic and κ is a limit cardinal, then every $\mathcal{L}$-theory that is* ZFC_n-Ψ*-outward consistent at κ is $<\kappa$-consistent. In particular, if Vopěnka's Principle holds, then every abstract logic has a* ZFC_n-Ψ*-outward compactness cardinal.*

Proof. Fix formulas $\varphi_0(v_0, v_1, v_2)$ and $\varphi_1(v_0, v_1, v_2)$ and a parameter z defining an abstract logic $\mathcal{L}$ and let η be a limit ordinal witnessing that an $\mathcal{L}$-theory T is ZFC_n-Ψ-outward consistent at a cardinal κ. Assume, towards a contradiction, that there is a subset T_0 of T of cardinality less than κ that is an inconsistent $\mathcal{L}$-theory. Pick a cardinal $\lambda < \kappa$ with $|T_0| < \lambda$ and a cardinal $\vartheta > \eta$ with $T, z \in H_\vartheta = V_\vartheta$ and V_ϑ sufficiently elementary in V. Then (B) in Definition 2.1.2

[13] I.e., the statement $\varphi_i(x_0, x_1, x_2)$ holds in N for all $i < 2$ and $x_0, x_1, x_2 \in V_\lambda^V$ with the property that $\varphi_i(x_0, x_1, x_2)$ holds in V.

implies that $\Psi(V_\vartheta, V_\vartheta, \lambda, \kappa, \eta)$ holds. In addition, our choice of ϑ ensured that the formulas φ_0 and φ_1 are absolute between V and V_ϑ. Finally, we also know that, in V_ϑ, the formulas φ_0 and φ_1 together with the parameter z define an abstract logic $\mathcal{L}'$, and that T is an $\mathcal{L}'$-theory that is not $<\lambda$-consistent. If G is $\mathrm{Col}(\omega, \vartheta)$-generic over V, then V_ϑ^V is an outer ZFC_n-model of V_ϑ^V in $V[G]$, and the fact that Ψ is a Δ_1^{ZFC}-formula ensures that $\Psi(V_\vartheta^V, V_\vartheta^V, \lambda, \kappa, \eta)$ holds in $V[G]$. However, this contradicts that T is ZFC_n-Ψ-outward consistent at κ in V. □

In the following, we will restrict ourselves to abstract logics with the property that both its formulas and the translations of formulas induced by renamings are produced from the given signatures and renamings by some simple recursive procedure. It is easy to see that all *finitely generated* logics (see [15, §1]) possess the property introduced below. In particular, we can apply the results below to second-order logic.

Definition 6.8. An abstract logic $\mathcal{L}$ with occurrence number ϱ has *simple formulas* if there exists a Σ_1-formula $\varphi(v_0, \ldots, v_3)$ and a set z with the property that whenever r is a renaming of a language σ with less than ϱ-many symbols into a language τ and $r_* : \mathcal{L}(\sigma) \longrightarrow \mathcal{L}(\tau)$ is the bijection induced by r, then

$$\{\langle \phi, r_*(\phi)\rangle \mid \phi \in \mathcal{L}(\sigma)\} \ = \ \{\langle x, y\rangle \mid \varphi(r, x, y, z)\}.$$

Note that if a Σ_1-formula $\varphi(v_0, \ldots, v_3)$ and a set z witness that an abstract logic $\mathcal{L}$ with occurrence number ϱ has simple formulas, then

$$\mathcal{L}(\sigma) \ = \ \{x \mid \varphi(\mathrm{id}_\sigma, x, x, z)\}$$

holds for every language σ with less than ϱ-many symbols, where id_σ denotes the trivial renaming of σ into itself. This shows that the restriction of the class function $\mathcal{L}$ to languages with less than ϱ-many symbols is also definable by a Σ_1-formula with parameter z in this case. The next theorem generalizes the forward direction of Theorem 3.5 to the setting of this section.

Theorem 6.9. *Let Ψ be a strong measure of closeness. Assume that for every class A, there is a proper class of cardinals that are Ψ-strong for A. Then for every natural number n, every abstract logic with simple formulas has a* ZFC_n*-Ψ-outward compactness cardinal.*

Proof. Let $\mathcal{L}$ be an abstract logic with simple formulas that is defined by formulas φ_0 and φ_1 together with a parameter z_0. In the following, let ϱ denote the occurrence number of $\mathcal{L}$. Fix a Σ_1-formula $\varphi(v_0, \ldots, v_3)$ and a parameter z_1 witnessing that $\mathcal{L}$ has simple formulas. Next, note that for every language τ, the identity $\mathrm{id}_{\mathcal{L}(\tau)}$ on $\mathcal{L}(\tau)$ is the unique (as in (6) of Definition 6.2) permutation π of $\mathcal{L}(\tau)$ corresponding to the trivial renaming of τ into itself. In particular, we know that for every language τ and every non-trivial[14] permutation π of $\mathcal{L}(\tau)$, there exists $\chi \in \mathcal{L}(\tau)$ and a τ-structure O with

$$O \models_{\mathcal{L}} \chi \iff O \not\models_{\mathcal{L}} \pi(\chi). \tag{1}$$

We can now pick a cardinal ρ above ϱ that satisfies $z_0, z_1 \in H_\rho = V_\rho$ and has the property that for every language τ in H_ϱ and every non-trivial permutation π of $\mathcal{L}(\tau)$, we have $\mathcal{L}(\tau) \in H_\rho$ and there exists $\chi \in \mathcal{L}(\tau)$ and a τ-structure O in H_ρ with (1). Let A be a class that codes the classes $\{\rho\}$, $\{z_0\}$, $\{z_1\}$, $\{\langle y_0, y_1, y_2\rangle \mid \varphi_0(y_0, y_1, y_2)\}$ and $\{\langle y_0, y_1, y_2\rangle \mid \varphi_1(y_0, y_1, y_2)\}$ in some canonical way. Pick a cardinal $\kappa > \rho$ that is Ψ-large for A. Fix a natural number $n > 1$. Pick an $\mathcal{L}$-theory T that is ZFC_n-Ψ-outward consistent at κ, and let $\eta > \kappa$ be a limit ordinal witnessing this. Let σ be a language with $T \subseteq \mathcal{L}(\sigma)$. In this situation, we can apply Lemma 5.3 to find a cardinal $\lambda > \eta$ with $\sigma, \mathcal{L}(\sigma) \in H_\lambda$, an inner model M and an elementary embedding $j : V \longrightarrow M$ such that $j(\kappa) > \lambda$, $j(A \cap V_\kappa) \cap V_\lambda = A \cap M \cap V_\lambda$ and $\Psi(V_\vartheta, V_\vartheta^M, \lambda, j(\kappa), j(\eta))$ holds for all limit

[14] I.e., not equal to the identity on $\mathcal{L}(\tau)$.

ordinals $\vartheta > j(\eta)$. Our assumptions on Ψ then ensure that $V_\lambda \subseteq M$. Moreover, by our choice of A, we know that $j(\rho) = \rho$, $j(z_0) = z_0$ and $j(z_1) = z_1$. Given $i < 2$, we define

$$B_i \ = \ \{\langle y_0, y_1, y_2\rangle \in V_\lambda^3 \mid \varphi_i(y_0, y_1, y_2)\}.$$

Our setup then ensures that

$$B_i \ = \ j(B_i \cap V_\kappa) \cap V_\lambda \ = \ \{\langle y_0, y_1, y_2\rangle \in V_\lambda^3 \mid M \models \varphi_i(y_0, y_1, y_2)\}$$

holds for all $i < 2$. Finally, we set $\mu = \mathrm{crit}(j)$.

Claim. *$\mu > \rho$ and $z_0, z_1 \in V_\mu$.*

Proof of the Claim. Assume that one of the statements of the claim fails. Then there is a limit ordinal $\mu < \zeta < \lambda$ with $j(\zeta) = \zeta$. But this implies that the map $j \restriction V_{\zeta+2} : V_{\zeta+2} \longrightarrow V_{\zeta+2}$ is a non-trivial elementary embedding, contradicting the *Kunen Inconsistency.* □

Next, note that $j(\sigma)$ is a language in V.

Claim. *$j(T) \subseteq \mathcal{L}(j(\sigma))$.*

Proof of the Claim. Fix $\phi \in T$. We can find a sublanguage $\bar{\sigma}$ of σ with less than $\mathscr{o}$-many symbols that satisfies $\phi \in \mathcal{L}(\bar{\sigma})$. By our assumptions, we know that $\varphi(\mathrm{id}_{\bar{\sigma}}, \phi, \phi, z_1)$ holds in V and hence elementarity implies that $\varphi(\mathrm{id}_{j(\bar{\sigma})}, j(\phi), j(\phi), z_1)$ holds in M. But then Σ_1-upwards absoluteness ensures that this statement also holds in V. Since our first claim shows that $j(\bar{\sigma})$ is a sublanguage of $j(\sigma)$ with less than $\mathscr{o}$-many symbols, $j(\phi) \in \mathcal{L}(j(\bar{\sigma})) \subseteq \mathcal{L}(j(\sigma))$. □

Notice that the elementarity of j ensures that, in M, the formulas φ_0 and φ_1 together with the parameter z_0 define an abstract logic $\mathcal{L}_M$ and the limit ordinal $j(\eta)$ witnesses that the $\mathcal{L}_M$-theory $j(T)$ is ZFC_n-Ψ-outward consistent at $j(\kappa)$.

Claim. *The $\mathcal{L}$-theory $j[T]$ is consistent in V.*

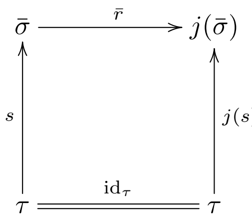
Proof of the Claim. Let $\vartheta > j(\lambda)$ be a cardinal such that $j(\vartheta) = \vartheta$, the set V_ϑ is sufficiently elementary in V and the set V_ϑ^M is sufficiently elementary in M. Let G be $\mathrm{Col}(\omega, \vartheta)$-generic over V. Then V_ϑ^M is countable in $M[G]$ and we know that, in $M[G]$, $j(T)$ is a $<\lambda$-consistent $\mathcal{L}'$-theory in N whenever N is a countable outer ZFC_n-model of V_ϑ^M, $\mathcal{L}'$ is an abstract logic in N and the following statements hold:

(a) $\Psi(N, V_\vartheta^M, \lambda, j(\kappa), j(\eta))$ holds.
(b) If $i < 2$ and $\langle y_0, y_1, y_2\rangle \in B_i$, then $\varphi_i(y_0, y_1, y_2)$ holds in N.
(c) The formulas φ_0 and φ_1 together with the parameter z_0 define $\mathcal{L}'$ in N.
(d) $j(T)$ is an $\mathcal{L}'$-theory in N.

Since the given statement can be expressed by a Π_1-formula with parameters in $H_{\aleph_1}^{M[G]}$, *Shoenfield Absoluteness* ensures that it also holds in $V[G]$. Thus, Σ_1-upwards absoluteness implies that $\Psi(V_\vartheta^V, V_\vartheta^M, \lambda, j(\kappa), j(\eta))$ holds in $V[G]$. Moreover, since V_ϑ^V was chosen sufficiently elementary in V, we know that for all $i < 2$ and all $\langle y_0, y_1, y_2\rangle \in B_i$, the statement $\varphi_i(y_0, y_1, y_2)$ holds in V_ϑ^V. In addition, since the previous claim shows that $j(T)$ is an $\mathcal{L}$-theory in V, our choice of ϑ also ensures that the formulas φ_0 and φ_1 together with the parameter z_0 define an abstract logic $\mathcal{L}'$ in V_ϑ^V and $j(T)$ is an $\mathcal{L}'$-theory in V_ϑ^V. This allows us to conclude that $j(T)$ is a $<\lambda$-consistent $\mathcal{L}'$-theory in V_ϑ^V. Since $j[T]$ is an element of V_ϑ^V and this set has cardinality less than λ in V_ϑ^V, we now know that $j[T]$ is a consistent $\mathcal{L}'$-theory in V_ϑ^V. In this situation, the fact that V_ϑ^V was chosen to be sufficiently elementary in V allows us to conclude that $j[T]$ is a consistent $\mathcal{L}$-theory in V. □

Let σ_j denote the sublanguage of $j(\sigma)$ that is given by the pointwise image of the symbols in σ under j, and let r denote the canonical renaming from σ into σ_j induced by j. Since the occurrence number ϱ of $\mathcal{L}$ is smaller than the critical point μ of j, we know that $j[T] \subseteq \mathcal{L}(\sigma_j)$. Moreover, the previous claim shows that there is a σ-structure N with the property that the σ_j-structure $r^*(N)$ obtained from N using the renaming r is a model of $j[T]$. Assume, towards a contradiction, that N is not a model of T and fix ϕ in T such that $N \not\models_{\mathcal{L}} \phi$. Pick a sublanguage $\bar{\sigma}$ of σ that contains less than ϱ-many symbols and has the property that ϕ is an element of $\mathcal{L}(\bar{\sigma})$. Then j induces a renaming $\bar{r}$ of $\bar{\sigma}$ into the sublanguage $j(\bar{\sigma})$ of σ_j, and this function is equal to the restriction of r to $\bar{\sigma}$. Let $\bar{N}$ denote the $\bar{\sigma}$-reduct of N. Then $\bar{N} \not\models_{\mathcal{L}} \phi$. Moreover, the $j(\bar{\sigma})$-structure $\bar{r}^*(\bar{N})$ obtained from $\bar{N}$ using $\bar{r}$ is equal to the $j(\bar{\sigma})$-reduct of $r^*(N)$ and therefore we know that $\bar{r}^*(\bar{N}) \models_{\mathcal{L}} j(\phi)$ and $\bar{r}^*(\bar{N}) \not\models_{\mathcal{L}} \bar{r}_*(\phi)$. In particular, it follows that $\bar{r}_*(\phi) \neq j(\phi)$.

Now, pick a renaming s of a language $\tau \in H_{\varrho}$ into $\bar{\sigma}$. Then $j(s)$ is a renaming of τ into $j(\bar{\sigma})$ and our setup ensures that $j(s) = \bar{r} \circ s$, i.e., we know that the following diagram commutes:



Since $V_\lambda \subseteq M$, we know that the sets $\bar{\sigma}$, $j(\bar{\sigma})$ and τ as well as the maps s and $j(s)$ are all contained in M, and this implies that the renaming $\bar{r}$ is also contained in M. In addition, our setup ensures that $\mathcal{L}(\bar{\sigma}) = \mathcal{L}_M(\bar{\sigma})$, $\mathcal{L}(j(\bar{\sigma})) = \mathcal{L}_M(j(\bar{\sigma}))$ and $\mathcal{L}(\tau) = \mathcal{L}_M(\tau)$. Let $s_* : \mathcal{L}(\tau) \longrightarrow \mathcal{L}(\bar{\sigma})$ denote the bijection induced by s in V. Then s_* is contained in M and Σ_1-upwards absoluteness implies that s_* is also the bijection between $\mathcal{L}_M(\bar{\sigma})$ and $\mathcal{L}_M(\tau)$ that is induced by s in M. Since $j(s_*)$ is the bijection between $\mathcal{L}_M(\tau)$ and $\mathcal{L}_M(j(\bar{\sigma}))$ induced by $j(s)$ in M, we know that $j(s_*) \circ s_*^{-1}$ is the bijection between $\mathcal{L}_M(\bar{\sigma})$ and $\mathcal{L}_M(j(\bar{\sigma}))$ induced by $\bar{r}$ in M.

Claim. $j(s_*) \circ s_*^{-1} = j \restriction \mathcal{L}(\bar{\sigma})$.

Proof of the Claim. Assume, towards a contradiction, that the statement of the claim fails. Since $j \restriction \mathcal{L}(\bar{\sigma})$ is a bijection between $\mathcal{L}(\bar{\sigma})$ and $\mathcal{L}(j(\bar{\sigma}))$, we then know that the map $j(s_*)^{-1} \circ j \circ s_*$ is a non-trivial permutation of $\mathcal{L}(\tau)$. In this situation, our setup yields $\chi \in \mathcal{L}(\tau)$ and a τ-structure $O \in H_\rho$ with

$$O \models_{\mathcal{L}} \chi \iff O \not\models_{\mathcal{L}} (j(s_*)^{-1} \circ j \circ s_*)(\chi).$$

Note that $\mathcal{L}(\tau) = \mathcal{L}_M(\tau)$ and, since $\mathcal{L}(\tau) \subseteq V_\lambda$, our setup ensures that

$$O \models_{\mathcal{L}} \upsilon \iff O \models_{\mathcal{L}_M} \upsilon$$

holds for every $\upsilon \in \mathcal{L}(\tau)$. Moreover, if $s^*(O)$ denotes the $\bar{\sigma}$-structure obtained from O using s, then, in M, the $j(\bar{\sigma})$-structure $j(s^*(O))$ is obtained from O using the renaming $j(s)$. Since $j(s_*)$ is the bijection between $\mathcal{L}_M(\tau)$ and $\mathcal{L}_M(j(\bar{\sigma}))$ induced by $j(s)$ in M, we then know that

$$O \models_{\mathcal{L}_M} j(s_*)(\upsilon) \iff j(s^*(O)) \models_{\mathcal{L}_M} \upsilon$$

holds for all $\upsilon \in \mathcal{L}_M(j(\bar{\sigma}))$. By combining these equivalences with the elementarity of j, we can now conclude that

$$\begin{aligned} O \models_{\mathcal{L}} \chi &\iff s^*(O) \models_{\mathcal{L}} s_*(\chi) \iff j(s^*(O)) \models_{\mathcal{L}_M} j(s_*(\chi)) \\ &\iff O \models_{\mathcal{L}_M} (j(s_*)^{-1} \circ j \circ s_*)(\chi) \iff O \models_{\mathcal{L}} (j(s_*)^{-1} \circ j \circ s_*)(\chi), \end{aligned}$$

a contradiction. □

The above claim shows that, in M, the map $j \restriction \mathcal{L}(\bar{\sigma})$ is the bijection between $\mathcal{L}_M(\bar{\sigma})$ and $\mathcal{L}_M(j(\bar{\sigma}))$ induced by $\bar{r}$. This directly implies that $\varphi(\bar{r}, \phi, j(\phi), z_1)$ holds in M. By Σ_1-upwards absoluteness, we know that this statement holds in V and we can conclude that $\bar{r}_*(\phi) = j(\phi)$, a contradiction. □

The following result provides a generalization of the backward direction of Theorem 3.5 that will turn out to be suitable for our characterizations of fragments of Vopěnka's Principle.

Theorem 6.10. *Let Ψ be an $\mathcal{L}^2$-measure of closeness. Assume that for every natural number n, every abstract logic that extends second-order logic and has simple formulas has a ZFC_n-Ψ-outward compactness cardinal. Then, for every formula $\varphi(v)$ in the language of set theory, there exists a cardinal that is Ψ-strong for the class $\{x \mid \varphi(x)\}$.*

Proof. Let A denote the class $\{x \mid \varphi(x)\}$. We define an abstract logic $\mathcal{L}_\varphi$ that extends second-order logic by adding atomic formulas $\phi_E(v_0, v_1)$ for every binary relation symbol E and all variables v_0 and v_1, and defining $M \models_{\mathcal{L}_\varphi} \phi_E(c_0, c_1)$ to hold for constant symbols c_0 and c_1, whenever E^M is a well-founded and extensional relation on $|M|$ and the corresponding transitive collapse $\pi : |M| \longrightarrow N$ sends c_0^M to an ordinal λ and c_1^M to a set X with the property that

$$V_\lambda \cap X \ = \ \{x \in N \cap V_\lambda \mid \varphi(x)\}.$$

Let $\varphi_0(v_0, v_1)$ and $\varphi_1(v_0, v_1)$ denote the canonical formulas defining $\mathcal{L}_\varphi$ (without using additional parameters), and pick a natural number $n > 1$ such that ZFC_n proves that these formulas define the abstract logic $\mathcal{L}_\varphi$. Finally, since it is immediate that $\mathcal{L}_\varphi$ has simple formulas, our assumptions ensure that there is a ZFC_n-Ψ-outward compactness cardinal κ for $\mathcal{L}_\varphi$.

Now, let $\theta > \eta > \kappa$ be limit ordinals and fix a cardinal $\zeta > \eta$. Let $\Psi^*(v_0, v_1, v_2)$ be an $\mathcal{L}^2$-formula in the language of set theory witnessing that Ψ is an $\mathcal{L}^2$-measure of closeness, and let $\Psi'(v_0, \ldots, v_3)$ denote the relativisation of $\Psi^*(v_0, v_1, v_2)$ to v_3, i.e., Ψ' has the property that ZFC^* proves that $\langle M, \in\rangle \models_{\mathcal{L}^2} \Psi'(a, b, c, d)$ is equivalent to $\langle d, \in\rangle \models_{\mathcal{L}^2} \Psi^*(a, b, c)$ whenever M is a transitive set, $d \in M$ is transitive and $a, b, c \in d$. Consider the language that extends the language of set theory by a constant symbol b, constant symbols c_x for all elements x of $V_{\theta+1}$ and constant symbols d_γ for all $\gamma \leq \zeta$. Let T denote the $\mathcal{L}_\varphi$-theory consisting of the following:

(1) The first-order elementary diagram of $V_{\theta+1}$, using the constant symbols c_x for x in $V_{\theta+1}$.
(2) The sentence "$d_\zeta < b < c_\kappa$" and all sentences of the form "$d_\beta < d_\gamma < c_\kappa$" for $\beta < \gamma \leq \zeta$.
(3) The sentence $\Psi'(b, c_\kappa, c_\eta, c_{V_\theta})$.
(4) The sentence $\phi_\in(b, c_{A \cap V_\kappa})$.

Claim. *The ordinal η witnesses that T is ZFC_n-Ψ-outward consistent at κ.*

Proof of the Claim. Fix a cardinal $\lambda < \kappa$ and a cardinal $\vartheta > \eta$ with $T \in H_\vartheta$. We then know that $\vartheta > \theta$. Let G be $\mathrm{Col}(\omega, \vartheta)$-generic over V, and let N be an outer ZFC_n-model of V_ϑ^V in $V[G]$ such that $\Psi(N, V_\vartheta^V, \lambda, \kappa, \eta)$ holds in $V[G]$, and the formulas φ_0 and φ_1 are absolute from V to N with respect to parameters in V_λ^V. We then know that T is an $\mathcal{L}_\varphi$-theory in N. Next, we apply (C) in Definition 2.1.2 to show that $\Psi(V_\theta^N, V_\theta^V, \lambda, \kappa, \eta)$ holds in $V[G]$ and, using the fact that Ψ is a $\Delta_1^{\mathrm{ZFC}^*}$-formula, we can conclude that this statement also holds in N. We then know that

$$\langle V_{\theta+1}^V, \in\rangle \models_{\mathcal{L}_\varphi} \Psi'(\lambda, \kappa, \eta, V_\theta^V)$$

holds in N.

For all $\alpha < \lambda$, our setup ensures that

$$\langle V_{\alpha+1}, \in\rangle \models_{\mathcal{L}_\varphi} \phi_\in(\alpha, A \cap V_\alpha)$$

holds in V and, since all parameters appearing in this statement are elements of V_λ, our assumptions on N ensure that it also holds in N. Hence, we know that $A \cap V_\lambda^N$ is equal to the set of all

$x \in V_\lambda^N \cap V_\lambda^V$ with the property that $\varphi(x)$ holds in N, and this allows us to conclude that

$$\langle V_{\theta+1}^V, \in\rangle \models_{\mathcal{L}_\varphi} \phi_\in(\lambda, A \cap V_\kappa)$$

holds in N.

Fix a subtheory T_0 of T in N that has cardinality less than λ in N. The fact that λ is a cardinal in N now allows us to construct a model of T_0 with domain $V_{\theta+1}^V$ in N that interprets b as λ and all constant symbols of the form d_γ that appear in sentences in T_0 as ordinals less than λ. □

Since κ is a ZFC_n-Ψ-outward compactness cardinal for $\mathcal{L}_\varphi$, the above claim shows that the theory T is consistent. This allows us to find a transitive set M, an ordinal $\zeta < \lambda \in M$ and an elementary embedding $j : V_{\theta+1} \longrightarrow M$ with $j(\kappa) > \lambda$ and

$$\langle M, \in\rangle \models_{\mathcal{L}_\varphi} \phi_\in(\lambda, j(A \cap V_\kappa)) \ \wedge \ \Psi'(\lambda, j(\kappa), j(\eta), j(V_\theta)).$$

We then know that

$$j(A \cap V_\kappa) \cap V_\lambda \ = \ A \cap M \cap V_\lambda.$$

Moreover, since elementarity implies that $j(V_\theta) = M \cap V_{j(\theta)}$, we can conclude that

$$\langle M \cap V_\lambda, \in\rangle \models_{\mathcal{L}_\varphi} \Psi^*(\lambda, j(\kappa), j(\eta))$$

and this shows that $\Psi(V_{j(\theta)}, M \cap V_{j(\theta)}, \lambda, j(\kappa), j(\eta))$ holds. These computations show that the cardinal κ is Ψ-strong for the class A. □

We can now combine the above results to characterize the principles discussed in Section 5 through compactness properties of abstract logics.

Corollary 6.11. *The following schemes are equivalent over* ZFC*:*

(1) Vopěnka's Principle.
(2) For every natural number n and every abstract logic $\mathcal{L}$, there exists a ZFC_n-Ψ_{ext}-outward compactness cardinal for $\mathcal{L}$.
(3) For every natural number n and every abstract logic $\mathcal{L}$ with simple formulas, there exists a ZFC_n-Ψ_{sc}-outward compactness cardinal for $\mathcal{L}$.

Proof. First, assume that (3) holds. Then Theorem 6.10 implies that for every formula $\varphi(v)$ in the language of set theory, there exists a cardinal that is Ψ_{sc}-large for the class $\{x \mid \varphi(x)\}$. An application of Lemma 5.4 then shows that (1) holds. This completes the proof of the corollary, because the implication from (1) to (2) is already given by Proposition 6.7 and the implication from (2) to (3) holds trivially. □

Corollary 6.12. *The following schemes are equivalent over* ZFC*:*

(1) Ord *is Woodin.*
(2) For every natural number n and every abstract logic $\mathcal{L}$ with simple formulas, there exists a ZFC_n-Ψ_{str}-outward compactness cardinal for $\mathcal{L}$.

Proof. If we assume that (2) holds, then Theorem 6.10 implies that for every formula $\varphi(v)$ in the language of set theory, there exists a cardinal that is Ψ_{str}-large for the class $\{x \mid \varphi(x)\}$, and then Lemma 5.5 shows that (1) holds. In the other direction, assume that (1) holds. Then Lemma 5.5 shows that for every class A, there is a proper class of cardinals that are Ψ_{str}-large for A. In this situation, Theorem 6.9 shows that (2) holds. □

7. Open questions

We close this paper by listing some questions that are raised by the results of this paper. First, as already mentioned in Section 2.4, results of Bagaria and Magidor in [3] and [4] entail that the combinatorics of Ψ_{stc}-large cardinals are fundamentally different from those of the other large cardinal notions studied in Section 2. For example, it is consistent that the least Ψ_{stc}-large cardinal is singular and therefore all embeddings witnessing the Ψ_{stc}-largeness of the given cardinal have critical points that are strictly smaller than this cardinal. It is therefore natural to ask if such characteristics of the given large cardinal notion can be read off from the formula inducing it:

Question 7.1. Is there a natural characterization of the collection of all formulas Ψ with the property that it is provable that if there is a Ψ-large cardinal, then the least such cardinal is regular?

In a similar direction, the above results also raise the question whether they are somehow connected to the *Identity Crisis* phenomenon, first studied by Magidor in [14]. More precisely, a combination of Corollary 3.6(4) with results in [3] and [14] shows that, assuming the consistency of sufficiently strong large cardinal axioms, the axioms of ZFC do not decide whether the first Ψ_{stc}-large cardinal is the first measurable cardinal. In contrast, Corollary 3.6 also shows that if Ψ_{ext}-, Ψ_{sc}- or Ψ_{str}-large cardinals exist, then the first cardinal of this type is bigger than the first measurable cardinal.

Question 7.2. Is there a natural characterization of the collection of all formulas Ψ with the property that the existence of a Ψ-large cardinals provably implies that the least cardinal of this type is bigger than the first measurable cardinal?

Next, we consider the question whether other canonical formulas induce large cardinal notions that also consistently exhibit unusual behavior. Remember that a cardinal κ is *globally superstrong* (see [9]) if for every $\lambda > \kappa$, there exists a transitive class M and an elementary embedding $j : V \longrightarrow M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $V_{j(\kappa)} \subseteq M$. There is an obvious candidate for a formula Ψ such that Ψ-largeness corresponds to global superstrongness: let $\Psi_{gss}(v_0, \ldots, v_4)$ denote the canonical formula in the language of set theory with the property that $\Psi_{gss}(N, M, \mu, \nu, \rho)$ holds if and only if the tuple $\langle N, M, \mu, \nu, \rho\rangle$ is suitable and $N \cap V_\nu \subseteq M$. Then, Ψ_{gss} is an $\mathcal{L}^2$-measure of closeness. Moreover, it is easy to see that every globally superstrong cardinal is Ψ_{gss}-large. But, it is not clear if it is possible to prove a variation of Lemma 2.9 for this formula.

Question 7.3. Is it provable that a cardinal κ is Ψ_{gss}-large if and only if there is a globally superstrong cardinal less than or equal to κ? Is it consistent that the least Ψ_{gss}-large cardinal is singular?

Finally, motivated by the observations made in Section 4, we ask if there is a canonical outward compactness characterization for strong compactness:

Question 7.4. Is there an $\mathcal{L}^2$-measure of closeness Ψ that naturally induces a large cardinal property below extendibility and has the property that ZFC proves that a cardinal κ is Ψ-large if and only if there is a strongly compact cardinal less than or equal to κ?

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