# $\mathbf{\Delta}^1_1$ subsets of ${}^\kappa\kappa$

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*presenting joint work with Philipp Lücke*

November 2, 2014

# Basic Motivation

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Can set-theoretically interesting objects, which are (in some intuitive sense) complicated, have simple definitions over ${}^{\kappa}\kappa$ - or, equivalently, over $H(\kappa^+)$, while certain properties of canonical inner models fail to hold?

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If $\kappa = \omega$, classical results show that the $\Sigma_1$-definability of such objects over $H(\omega_1)$ implies strong L-like properties.

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## Summary Preview

If $\kappa = \omega$, classical results show that the $\Sigma_1$-definability of such objects over $H(\omega_1)$ implies strong L-like properties. However if $\kappa$ is uncountable with $\kappa^{<\kappa} = \kappa$, it is consistent for such objects to be $\Delta_1$-definable over $H(\kappa^+)$ while certain inner model properties fail.

# Classical Results

**Theorem (Gödel, 1930ies)**

*In* **L**, *there is a (lightface)* $\Sigma_1$*-definable wellorder of* $H(\kappa^+)$ *for every infinite cardinal* $\kappa$.

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## Theorem (Mansfield, 1975)

*The existence of a* $\boldsymbol{\Sigma}_1$*-definable wellorder of* $H(\omega_1)$ *is equivalent to the statement that there is a real* $x$ *such that all reals are contained in* $\mathbf{L}[x]$*. In particular, if there is a* $\boldsymbol{\Sigma}_1$*-definable wellordering of* $H(\omega_1)$*,* CH *holds.*

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**Theorem (Martin - Steel, 1985)**

*If there are infinitely many Woodin cardinals, then Projective Determinacy holds. The latter implies that there is no definable wellorder of* $H(\omega_1)$.

# The GCH setting

**Theorem (Friedman - Holy, 2011)**

*If $\kappa$ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and $2^\kappa = \kappa^+$, then there is a small, cofinality-preserving forcing that introduces a $\boldsymbol{\Sigma}_1$-definable wellordering of $H(\kappa^+)$ and preserves $2^\kappa = \kappa^+$.*

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**Theorem (Asperó - Holy - Lücke, 2013)**

*The assumption $2^\kappa = \kappa^+$ can be dropped in the above theorem, replacing preservation of $2^\kappa = \kappa^+$ by preservation of the value of $2^\kappa$.*

# $\boldsymbol{\Sigma}_1$ and non-GCH?

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We will answer this question negatively. To motivate our approach, we want to show how one can (quite easily) introduce a $\boldsymbol{\Sigma}_2$-definable wellordering of $H(\kappa^+)$ when $\kappa$ is uncountable and $\kappa^{<\kappa} = \kappa$, using a very well-behaved notion of forcing.

# Almost Disjoint Coding

Given some suitable enumeration $\langle s_\alpha \mid \alpha < \kappa \rangle$ of ${}^{<\kappa}\kappa$, forcing with Solovay's almost disjoint coding forcing (or rather, its generalization to $\kappa$) makes a given set $A \subseteq {}^\kappa\kappa$ $\mathbf{\Sigma}^0_2$-definable over ${}^\kappa\kappa$ - it adds a function $t\colon \kappa \to 2$ such that in the generic extension, for every $x \in {}^\kappa\kappa$,

$$x \in A \iff \exists\beta < \kappa\ t(\alpha) = 1 \text{ for all } \beta < \alpha < \kappa \text{ with } s_\alpha \subseteq x.$$

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Moreover this forcing is $<\kappa$-closed, $\kappa^+$-cc and a subset of $H(\kappa^+)$.

Using this, we could pick any wellordering $<$ of $H(\kappa^+)$ and make it $\mathbf{\Delta}_1$-definable over $H(\kappa^+)$ of a $P$-generic extension. But forcing with $P$ adds new subsets of $\kappa$, so $<$ is not a wellordering of $H(\kappa^+)$ anymore.

# $\mathbf{\Sigma}_2$-definable Wellorderings

## Observation

If $\kappa$ is an uncountable cardinal with $\kappa^{<\kappa} = \kappa$, then there is a $<\kappa$-closed, $\kappa^+$-cc partial order $P \subseteq H(\kappa^+)$ that introduces a $\mathbf{\Sigma}_2$-definable wellordering of $H(\kappa^+)$.

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Proof-Sketch: Pick any wellordering $<$ of $H(\kappa^+)$. Apply the almost disjoint coding forcing (denote it by $P$) to make $<$ $\boldsymbol{\Delta}_1$-definable over $H(\kappa^+)$. $P$ is $\kappa^+$-cc and $P \subseteq H(\kappa^+)$.

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$$x <^* y \iff \exists \dot{x} \forall \dot{y} \left[ (\dot{x}^G = x \wedge \dot{y}^G = y) \rightarrow \dot{x} < \dot{y} \right],$$

where $G$ is the $P$-generic filter.

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where $G$ is the $P$-generic filter. Using $\mathbf{\Sigma}_1$-definability of $P$ and $G$ over the new $H(\kappa^+)$, $<^*$ is a $\mathbf{\Sigma}_2$-definable wellordering of the new $H(\kappa^+)$. $\square$

# $\mathbf{\Sigma}_1$?

If $2^\kappa = \kappa^+$, it is possible to pull a small trick and spare one quantifier in the above (by coding all initial segments of $<$, which in that case have size at most $\kappa$ and are thus elements of $H(\kappa^+)$). Otherwise however, the above suggests that one cannot hope for a wellordering of the $H(\kappa^+)$ of the ground model to *induce* a $\mathbf{\Sigma}_1$-definable wellordering of the $H(\kappa^+)$ of some generic extension, at least not *directly* via names.

# Our Theorem

By different means, we obtained the following.

**Theorem**

*If $\kappa$ is an uncountable cardinal with $\kappa^{<\kappa} = \kappa$, then there is a partial order $P$ which is $<\kappa$-closed, $\kappa^+$-cc and a subset of $H(\kappa^+)$, which introduces a $\boldsymbol{\Sigma}_1$-definable wellordering of $H(\kappa^+)$.*

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The basic idea of our solution is to build a forcing $P$ that, in the course of an iteration, adds a wellordering of $H(\kappa^+)$ of the $P$-generic extension while simultaneously making (larger and larger fragments of) this wellordering nicely definable.

Let $\lambda = 2^\kappa$. We inductively construct a sequence $\langle P_\gamma \mid \gamma \leq \lambda \rangle$ of partial orders such that $P_\delta$ is a complete subforcing of $P_\gamma$ whenever $\delta \leq \gamma \leq \lambda$ (i.e. an iteration of length $\lambda$) and let $P = P_\lambda$.

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The coding forcing $C(A)$ is capable of coding some $A \subseteq \lambda$ by a generically added subset of $\kappa$ in a $\Sigma_1$-way over $H(\kappa^+)$ s.t. if $B \supseteq A$ then $C(A)$ is a complete subforcing of $C(B)$ (we need this to obtain the complete subforcing property above).

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# Club Coding

joint work with David Asperó and Philipp Lücke

# The Coding Problem

We need a forcing that codes a given $A \subseteq \lambda = 2^\kappa$ by a generically added subset of $\kappa$. This could be achieved using the Almost Disjoint Coding forcing. However to obtain the desired property that $P_{\gamma_0}$ is a complete subforcing of $P_{\gamma_1}$ whenever $\gamma_0 < \gamma_1$, we need our coding forcing $C$ to have the following property:

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# Our solution

We thus choose $C(A)$ to be a variation of the Almost Disjoint Coding forcing for $A$ (that could in fact rather be seen as a generalization of the Canonical Function Coding by Asperó and Friedman to a non-GCH context), that combines the classic forcing with iterated club shooting and has the desired property that $A \subseteq B$ implies that $C(A)$ is a complete subforcing of $C(B)$. In particular, $C(A)$ will make $A$ $\mathbf{\Sigma}_1$-definable, but not Borel. Thus the argument from the previous slide does not apply here.

# Club Coding

## Definition

Given $A \subseteq {}^{\kappa}\kappa$, we let $C(A)$ be the partial order whose conditions are tuples

$$p = (s_p, t_p, \langle c_x^p \,|\, x \in a_p \rangle)$$

such that the following hold for some successor ordinal $\gamma_p < \kappa$.

1. $s_p \colon \gamma_p \to {}^{<\kappa}\kappa$, $t_p \colon \gamma_p \to 2$ and $a_p \in [A]^{<\kappa}$.
2. If $x \in a_p$, then $c_x^p$ is a closed subset of $\gamma_p$ and

$$s_p(\alpha) \subseteq x \to t_p(\alpha) = 1$$

   for all $\alpha \in c_x^p$.

We let $q \leq p$ if $s_p = s_q \restriction \gamma_p$, $t_p = t_q \restriction \gamma_p$, $a_p \subseteq a_q$ and $c_x^p = c_x^q \cap \gamma_p$ for every $x \in a_p$.

# Club Coding

## Lemma

*Assume $G$ is $C(A)$-generic, $s = \bigcup_{p \in G} s_p$ and $t = \bigcup_{p \in G} t_p$. Then $s \colon \kappa \to {}^{<\kappa}\kappa$, $t \colon \kappa \to 2$ and $A$ is equal to the set of all $x \in ({}^{\kappa}\kappa)^{V[G]}$ such that*

$$\forall \alpha \in C \ [s(\alpha) \subseteq x \to t(\alpha) = 1]$$

*holds for some club subset $C$ of $\kappa$ in $V[G]$.*

*Moreover, $C(A)$ is $<\kappa$-closed, $\kappa^+$-cc, a subset of $H(\kappa^+)$ and whenever $A \subseteq B \subseteq {}^{\kappa}\kappa$, then $C(A)$ is a complete subforcing of $C(B)$.*

# Simplifying the parameter

joint work with Philipp Lücke

If $\kappa = \lambda^+$ and $\lambda^{<\lambda} = \lambda$, one can improve our earlier result to a $\Sigma_1$-definition for a wellorder that only uses a parameter from the ground model, basically by coding, during the above construction, the parameter into the stationarity pattern of a ground model $\kappa$-sequence of disjoint stationary subsets of $\kappa$ on $\mathrm{cof}(\lambda)$.

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**Theorem**

*If $\kappa$ is a regular uncountable $\mathbf{L}$-cardinal, then there is a cofinality-preserving forcing extension of $\mathbf{L}$ with a $\Sigma_1(\kappa)$-definable wellorder of $H(\kappa^+)$ and $2^\kappa > \kappa^+$.*

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Strong large cardinal assumptions imply that for $H(\omega_2)$, a defining parameter for a $\boldsymbol{\Sigma}_1$-definable wellordering cannot even be *simple*.

**Theorem (A Corollary of results by Woodin)**

*Assume that there are infinitely many Woodin cardinals with a measurable cardinal above. If there is a wellordering of $H(\omega_2)$ that is $\Sigma_1$-definable over $H(\omega_2)$ with parameter $z \subseteq \omega_1$, then $z \notin \mathbf{L}(\mathbb{R})$.*

# Other Consistency Results?

We hope to be able to show that $\mathbf{\Delta}^1_1$-definability of certain interesting subsets of ${}^\kappa\kappa$ is compatible with the negation of other **L**-like properties, such with large cardinal strength, by mixing the forcing presented in this talk with large cardinal collapses.

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**Open Question**

Is it consistent that the perfect set property holds for all $\kappa$-Borel subsets of ${}^\kappa\kappa$ while it fails for a $\mathbf{\Delta}^1_1$ set?

Thank you.