# Absoluteness Results in Set Theory 

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#### Abstract

We investigate the consistency strength of various absoluteness principles. Following S. Friedman, we show that $\boldsymbol{\Sigma}_{\mathbf{3}}^{1}$-absoluteness for arbitrary set-forcing has the consistency strength of a reflecting cardinal. Following J. Bagaria, we show that $\Sigma_{1}\left(\mathrm{H}_{\omega_{2}}\right)$-absoluteness for $\omega_{1}$-preserving forcing is inconsistent and that for any partial ordering $P, \Sigma_{1}\left(\mathrm{H}_{\omega_{2}}\right)$-absoluteness for $P$ is equivalent to $\operatorname{BFA}(P)$, the bounded forcing axiom for $P$ - and hence $\Sigma_{1}\left(\mathrm{H}_{\omega_{2}}\right)$-absoluteness for ccc forcing is equiconsistent with ZFC. Then, following S. Shelah and M. Goldstern, we show that BPFA, the forcing axiom for the class of proper posets, is equiconsistent with the existence of a reflecting cardinal. We review that for any partial ordering $P, \Sigma_{1}\left(\mathrm{H}_{\omega_{2}}\right)$-absoluteness for $P$ implies $\boldsymbol{\Sigma}_{3}^{1}$-absoluteness for $P$ and finally, following S. Friedman, we turn back to investigate the consistency strength of $\boldsymbol{\Sigma}_{\mathbf{3}}^{\mathbf{1}}$-absoluteness for various classes of forcings: We show that $\boldsymbol{\Sigma}_{\mathbf{3}}^{1}$-absoluteness for proper (or even semiproper) forcing is equiconsistent with $Z F C$, that $\boldsymbol{\Sigma}_{\mathbf{3}}^{1}$-absoluteness for $\omega_{1}$-preserving forcing is equiconsistent with the existence of a reflecting cardinal, that $\boldsymbol{\Sigma}_{\mathbf{3}}^{\mathbf{1}}$-absoluteness for $\omega_{1}$-preserving class forcing is inconsistent, that, under the additional assumption that $\omega_{1}$ is inaccessible to reals, $\boldsymbol{\Sigma}_{\mathbf{3}}^{\mathbf{1}}$-absoluteness for proper forcing has the consistency strength of a reflecting cardinal and finally that $\boldsymbol{\Sigma}_{\mathbf{3}}^{\mathbf{1}}$-absoluteness for stationary-preserving forcing has the consistency strength of a reflecting cardinal.


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## 1 Introduction: General Set Theory

In this section, we give (mostly standard) definitions to fix our notation and we list auxiliary results which will be relied upon in the following. We omit many of the proofs which can be found in [17], [19] or [7].

### 1.1 Models, absoluteness and formula hierarchies

Notation: We let $\mathfrak{M}, \mathfrak{N}$ denote models $\mathfrak{M}=(M, \in, \ldots), \mathfrak{N}=(N, \in, \ldots)$ with universes $M$ and $N$ respectively. If $\mathfrak{M}=(M, \in)$, we sometimes write $M$ to denote the model $(M, \in)$. For a model $\mathfrak{M}, \mathcal{L}_{\mathfrak{M}}$ denotes the language of $\mathfrak{M}$, i.e. if $\mathfrak{M}=\left(M, \in,\left(P_{i}\right)_{i \in I}\right), \mathcal{L}_{\mathfrak{M}}=\left(\in,\left(P_{i}\right)_{i \in I}\right)$.

For $X \subseteq M$, we let $\mathcal{L}_{\mathfrak{M}}(X)$ denote the language $\mathcal{L}_{\mathfrak{M}}$ enriched with constant symbols $c_{x}$ for every $x \in X$.

Definition 1.1 A closed formula $\varphi \in \mathcal{L}_{\mathfrak{M}} \cap \mathcal{L}_{\mathfrak{N}}(M \cap N)$ is said to be absolute between $\mathfrak{N}$ and $\mathfrak{M}$ if $\mathfrak{N} \vDash \varphi \leftrightarrow \mathfrak{M} \vDash \varphi$, it is said to be upwards absolute between $\mathfrak{N}$ and $\mathfrak{M}$ if $N \subseteq M$ and $\mathfrak{N} \models \varphi \rightarrow \mathfrak{M} \vDash \varphi$, if $N \supseteq M$ and $\mathfrak{N} \vDash \varphi \rightarrow \mathfrak{M} \vDash \varphi, \varphi$ is said to be downwards absolute between $\mathfrak{N}$ and $\mathfrak{M}$.

## Definition 1.2 (The Lévy Hierarchy)

- A formula is $\Sigma_{n}(X)$ if it is of the form

$$
\exists x_{1} \forall x_{2} \exists x_{3} \ldots Q x_{n} \psi\left(x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{m}\right)
$$

where for all $i, y_{i} \in X$ and all the quantifiers in $\psi$ are bounded and $Q$ stands for $\exists$ in case $n$ is odd and for $\forall$ if $n$ is even.

- A formula is $\Pi_{n}(X)$ if it is the negation of a $\Sigma_{n}(X)$ formula.
- A property is $\Sigma_{n}(X)$ or $\Pi_{n}(X)$ if it is expressible by some $\Sigma_{n}(X)$ or $\Pi_{n}(X)$ statement respectively.
- A property is $\Delta_{n}(X)$ if it is both $\Sigma_{n}(X)$ and $\Pi_{n}(X)$.

Definition $1.3 \mathfrak{N}$ is said to be an elementary submodel of $\mathfrak{M}, \mathfrak{N} \prec \mathfrak{M}$, if $N \subseteq M, \mathcal{L}_{\mathfrak{N}} \subseteq \mathcal{L}_{\mathfrak{M}}$ and for every closed $\varphi \in \mathcal{L}_{\mathfrak{N}}(N), \mathfrak{N} \equiv \varphi \leftrightarrow \mathfrak{M}=\varphi$. $\mathfrak{N}$ is said to be a $\Sigma_{n}$-elementary submodel of $\mathfrak{M}, \mathfrak{N} \prec_{n} \mathfrak{M}$, if $N \subseteq M, \mathcal{L}_{\mathfrak{N}} \subseteq \mathcal{L}_{\mathfrak{M}}$ and for every closed $\varphi \in \mathcal{L}_{\mathfrak{N}}(N) \cap \Sigma_{n}(N), \mathfrak{N}|=\varphi \leftrightarrow \mathfrak{M}|=\varphi$.

Lemma 1.4 For any structure $\mathfrak{M}=(M, R, \ldots)$ with a binary relation $R$, " $R$ is well-founded on $M$ " is a $\Delta_{1}(\{M, R\})$ statement.

Proof:

- $\Sigma_{1}: \exists f: M \rightarrow$ Ord $\forall x, y \in M(x R y \rightarrow f(x)<f(y))$.
- $\Pi_{1}: \neg \exists X(X \subseteq M \wedge \forall x \in X \exists y \in X y R x)$.

Lemma 1.5" $y=\operatorname{tcl}(A)$ ", the transitive closure of $A$, is a $\Delta_{1}(\{y, A\})$ statement.

Proof:

- $\Sigma_{1}: A \subseteq y \wedge y$ transitive $\wedge \forall a \in y \exists f \exists n \in \omega \operatorname{dom}(f)=n+1 \wedge f(0)=$ $\emptyset \wedge f(n) \in A \wedge \forall k<n f(k) \in f(k+1) \wedge a \in \operatorname{range}(f)$.
- $\Pi_{1}: A \subseteq y \wedge y$ transitive $\wedge \forall x(A \subseteq x \wedge x$ transitive $\rightarrow y \subseteq x)$.

Fact 1.6 (Mostowski-Sheperdon Collapsing Lemma) ([7], I, Th.7.2) Let $E$ be a binary relation on $X$ such that $E$ is extensional and well-founded. Then there is a unique transitive set $M$ and a unique map $\pi$ such that:

$$
\pi:(X, E) \cong(M, \in)
$$

$M$ is called the transitive collapse of $(X, E), M=\operatorname{tcoll}(X, E)$.
Lemma 1.7 Assume $E$ is an extensional and well-founded binary relation on $X=\operatorname{dom}(E) \cup \operatorname{range}(E)$. Then $" y=\operatorname{tcoll}(X, E)$ " is a $\Delta_{1}(\{y, X, E\})$ statement.

Proof:

- $\Sigma_{1}: \exists f: X \rightarrow y \forall a \in X f(a)=\{f(b): b E a\}$.
- $\Pi_{1}: \forall f \forall D \subseteq X((\forall a \in D \forall b \in X b E a \rightarrow b \in D) \wedge f: D \rightarrow y \wedge \forall a \in$ $D f(a)=\{f(b): b E a\} \wedge f " D$ transitive $) \rightarrow((D=X \wedge f " D=$ $y) \vee(\exists d \in X \backslash D \forall z \in X(z E d \rightarrow z \in D) \wedge \exists e \in y e=\{f(c): c \in$ $D \wedge c E d\})$ ).


## Definition 1.8 (The hierarchy of projective formulas)

- A formula is $\Sigma_{n}^{1}\left(y_{1}, \ldots, y_{m}\right)$ if it is of the form

$$
\exists x_{1} \subseteq \omega \forall x_{2} \subseteq \omega \exists x_{3} \subseteq \omega \ldots Q x_{n} \subseteq \omega \psi\left(x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{m}\right)
$$

where for all $i, y_{i} \subseteq \omega$ and all the quantifiers in $\psi$ range over $\omega$ and $Q$ stands for $\exists$ in case $n$ is odd and for $\forall$ if $n$ is even.

- A formula is $\boldsymbol{\Sigma}_{\mathbf{n}}^{1}$ if it is $\Sigma_{n}^{1}\left(y_{1}, \ldots y_{m}\right)$ for some $y_{1}, \ldots, y_{m} \subseteq \omega$.
- A formula is $\Pi_{n}^{1}\left(y_{1}, \ldots, y_{m}\right)$ or $\boldsymbol{\Pi}_{\mathbf{n}}^{1}$ if it is the negation of a $\Sigma_{n}^{1}\left(y_{1}, \ldots, y_{m}\right)$ or a $\boldsymbol{\Sigma}_{\mathbf{n}}^{1}$ formula respectively.
- A formula is arithmetical (in a) if it is $\boldsymbol{\Sigma}_{\mathbf{0}}^{\mathbf{1}}\left(\boldsymbol{\Sigma}_{\mathbf{0}}^{\mathbf{1}}(a)\right)$.
- A formula is projective if it is $\boldsymbol{\Sigma}_{\mathbf{n}}^{\mathbf{1}}$ or $\boldsymbol{\Pi}_{\mathbf{n}}^{\mathbf{1}}$ for some $n \in \omega$.
- A relation on $\omega^{k} \times\left(\omega^{\omega}\right)^{l}$ is $\Sigma_{n}^{1}\left(y_{1}, \ldots, y_{m}\right), \Pi_{n}^{1}\left(y_{1}, \ldots, y_{m}\right), \boldsymbol{\Sigma}_{\mathbf{n}}^{1}, \boldsymbol{\Pi}_{\mathbf{n}}^{1}$, arithmetical or projective if it is expressible by some formula in the respective formula class.
- A relation on $\omega^{k} \times\left(\omega^{\omega}\right)^{l}$ is $\Delta_{n}^{1}\left(y_{1}, \ldots, y_{m}\right)$ or $\boldsymbol{\Delta}_{\mathbf{n}}^{\mathbf{1}}$ if it is both $\Sigma_{n}^{1}\left(y_{1}, \ldots, y_{m}\right)$ and $\Pi_{n}^{1}\left(y_{1}, \ldots, y_{m}\right)$-or $\boldsymbol{\Sigma}_{\mathbf{n}}^{1}$ and $\boldsymbol{\Pi}_{\mathbf{n}}^{1}$ respectively.

Fact 1.9 ([18], pp 152) $A \subseteq \omega^{k} \times\left(\omega^{\omega}\right)^{l}$ is arithmetical iff $A$ is definable in $(\omega, \in,+, \cdot, \exp ,<, 0,1), A \subseteq \omega^{k} \times\left(\omega^{\omega}\right)^{l}$ is arithmetical in $a \subseteq \omega$ iff $A$ is definable in $(\omega, \in,+, \cdot, \exp ,<, 0,1, a)$.

Lemma 1.10 There exists an arithmetical bijection $\Gamma: \omega \times \omega \rightarrow \omega$.
Proof: Let $A_{\alpha \beta}:=\left\{(\gamma, \delta) \in \omega \times \omega:(\gamma, \delta)<_{2}(\alpha, \beta)\right\}$ and $\Gamma(\alpha, \beta):=\left|A_{\alpha \beta}\right|$, with $(\gamma, \delta)<_{2}(\alpha, \beta) \leftrightarrow \max \{\gamma, \delta\}<\max \{\alpha, \beta\} \vee(\max \{\gamma, \delta\}=\max \{\alpha, \beta\} \wedge$ $((\delta=\beta \wedge \gamma<\alpha) \vee(\delta<\beta)))$. It is easy to find an arithmetical formula that defines $\Gamma$.

Lemma 1.11 There exists an arithmetical enumeration of $\omega^{<\omega}$.
Proof: For example, let $f: \omega^{<\omega} \rightarrow \omega,\left\langle x_{0}, x_{1}, \ldots, x_{n}\right\rangle \mapsto 2^{x_{0}+1} * 3^{x_{1}+1} * \ldots *$ $p_{n}{ }^{x_{n}+1}$ where $\left(p_{i}\right)_{i \in \omega}$ is the increasing enumeration of prime numbers, which is arithmetical.

Fact 1.12 (Mostowski's Absoluteness Theorem) ([17], Th.25.4) If $P$ is a $\boldsymbol{\Sigma}_{1}^{1}$ relation then $P$ is absolute for every transitive model that is adequate for $P$, i.e. every transitive model that satisfies some finite $Z F C^{*} \subset$ $Z F C$ and contains the parameters occuring in a $\boldsymbol{\Sigma}_{1}^{1}$-formula describing $P$.

### 1.2 Kripke-Platek Set Theory / Admissible Sets

Throughout this paper, we usually work in $Z F C$, Zermelo-Fraenkel Set Theory with the Axiom of Choice. In this section we are going to define some subtheories of $Z F C$ which will be of use later on. $Z F$ denotes ZermeloFraenkel Set Theory without the Axiom of Choice, $Z F^{-}$denotes $Z F$ with the Power Set Axiom deleted. In the following, for theories $S$ and $T$, let $S \subseteq T$ denote the statement that every axiom of $S$ is a theorem of $T$.

## Definition 1.13 (Basic Set Theory)

$B S \subseteq Z F^{-}$is the theory with the following axioms:

- Extensionality: $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow(x=y))$
- Induction Schema: $\forall \vec{a}(\forall x((\forall y \in x) \varphi(y, \vec{a}) \rightarrow \varphi(x, \vec{a})) \rightarrow \forall x \varphi(x, \vec{a}))$ for any formula $\varphi(x, \vec{a})$
- Pairing: $\forall x \forall y \exists z \forall w((w \in z) \leftrightarrow(w=x \vee w=y))$
- Union: $\forall x \exists y \forall z((z \in y) \leftrightarrow(\exists u \in x)(z \in u))$
- Infinity: $\exists x(x \in \mathbf{O r d} \wedge(x \neq \emptyset) \wedge(\forall y \in x)(\exists z \in x) y \in z)$
- Cartesian Product: $\forall x \forall y \exists z \forall u((u \in z) \leftrightarrow(\exists a \in x)(\exists b \in y) u=(a, b))$
- $\Sigma_{0}$-Comprehension Schema: $\forall \vec{a} \forall x \exists y \forall z((z \in y) \leftrightarrow(z \in x \wedge \varphi(z, \vec{a})))$ for any $\Sigma_{0}$-formula $\varphi(x, \vec{a})$

Definition 1.14 (Amenable Sets) $A$ transitive set $M$ is said to be amenable iff it is a model of BS, i.e. satisfies the following conditions:

- Pairing: $(\forall x, y, \in M)(\{x, y\} \in M)$
- Union: $(\forall x \in M)(\bigcup x \in M)$
- Infinity: $\omega \in M$
- Cartesian Product: $(\forall x, y \in M)(x \times y \in M)$
- $\Sigma_{0}$-Comprehension: if $R \subseteq M$ is $\Sigma_{0}(M)$, then $(\forall x \in M)(R \cap x \in M)$

Definition 1.15 (Kripke-Platek Set Theory) $K P \subseteq Z F^{-}$is the theory that consists of the axioms of $B S$ together with the $\Sigma_{0}$ Collection Schema:

- $\forall \vec{a}(\forall x \exists y \varphi(y, x, \vec{a}) \rightarrow \forall u \exists v(\forall x \in u)(\exists y \in v) \varphi(y, x, \vec{a}))$
for any $\Sigma_{0}$-formula $\varphi(x, \vec{a})$
Definition 1.16 (Admissible Sets) $A$ transitive set $M$ is said to be admissible iff it is a model of KP, i.e. iff it is amenable and satisfies that for any $\Sigma_{0}(M)$ relation $R \subseteq M \times M$, if $(\forall x \in M)(\exists y \in M) R(y, x)$, then for any $u \in M$ there is $v \in M$ such that $(\forall x \in u)(\exists y \in v) R(y, x)$.

The following two facts show that the axioms of $K P$ imply (seemingly) stronger principles:

Fact 1.17 ( $\Sigma_{1}$-Collection) ([7], I, Lemma 11.3)
For any $\varphi(y, x, \vec{a}) \in \Sigma_{1}$,

$$
K P \vdash \forall \vec{a}(\forall x \exists y \varphi(y, x, \vec{a}) \rightarrow \forall u \exists v(\forall x \in u)(\exists y \in v) \varphi(y, x, \vec{a})) .
$$

Fact 1.18 ( $\Delta_{1}$-Comprehension) ([7], I, Lemma 11.4)
For any $\varphi(z, \vec{a}) \in \Delta_{1}^{K P}$ (i.e. $\varphi(z, \vec{a}) \in \Sigma_{1}$ and there exists $\psi(z, \vec{a}) \in \Pi_{1}$ such that $K P \vdash \forall z \forall \vec{a}(\varphi(z, \vec{a}) \leftrightarrow \psi(z, \vec{a})))$,

$$
K P \vdash \forall \vec{a} \forall x \exists y \forall z(z \in y \leftrightarrow(z \in x \wedge \varphi(z, \vec{a}))) .
$$

Lemma 1.19 Let $M$ be a transitive set. Then there is a sentence $\phi$ s.t. $\mathfrak{M}|=\phi \leftrightarrow \mathfrak{M}|=K P \leftrightarrow \mathfrak{M}$ is admissible.

Proof: We will show that there is a finite number of theorems of $K P$, which, true in $\mathfrak{M}$, imply that $\mathfrak{M} \models K P$; in particular, we have to "get rid" of every axiom schema in $K P$ : Obviously, the induction schema holds in every transitive set. We will show that, using a universal $\Sigma_{1}$ formula, the schemata of $\Sigma_{1}$-Collection and $\Delta_{1}$-Comprehension are implied by a theorem of $K P$ :

Let $\varphi(n, x, \vec{a}) \equiv \exists T$ transitive $\wedge \operatorname{Sat}(T, n, x, \vec{a})$ with Sat being the satisfaction relation, i.e. $\operatorname{Sat}(T, n, x, \vec{a}) \leftrightarrow T \models \psi_{n}(x, \vec{a})$ with $n=\sharp \psi_{n}$. Sat is a $\Delta_{1}^{K P}$-property (for details, see [7], pp 31). It follows that $\varphi(n, x, \vec{a})$ is a $\Sigma_{1}$-property; if $\psi_{n}$ is $\Delta_{1}^{K P}$, then $\varphi(n, x, \vec{a})$ is a $\Delta_{1}^{K P}$ property, since $\neg \varphi(n, x, \vec{a}) \leftrightarrow \exists T$ transitive $\wedge x \in T \wedge \operatorname{Sat}\left(T, F_{\neg}(n), x, \vec{a}\right)$ where $F_{\neg}$ is a $\Sigma_{0}$-function satisfying $m=F_{\neg}(n) \leftrightarrow \psi_{m}=\neg \psi_{n}$ (for details, see [7], pp 31).

Hence $\Sigma_{1}$-Collection and $\Delta_{1}$-Comprehension are implied by the following theorems of $K P$ (which follow from instances of the above schemata):

- $\forall n \in \omega\left(\Sigma_{1}(n) \rightarrow \forall \vec{a}(\forall x \exists y \varphi(n, y, x, \vec{a}) \rightarrow\right.$

$$
\forall u \exists v(\forall x \in u)(\exists y \in v) \varphi(n, y, x, \vec{a})))
$$

- $\forall n \in \omega\left(\Delta_{1}(n) \rightarrow \forall \vec{a} \forall x \exists y \forall z(z \in y \leftrightarrow z \in x \wedge \varphi(n, z, \vec{a}))\right)$
where $\Sigma_{1}(x)$ and $\Delta_{1}(x)$ are predicates with the properties
- $\Sigma_{1}(n) \leftrightarrow \psi_{n} \in \Sigma_{1} ; \Delta_{1}(n) \leftrightarrow \psi_{n} \in \Delta_{1}^{K P}$.

Details can be found in [7], pp 42.

### 1.3 Infinite Combinatorics

Definition 1.20 A collection $\mathcal{A}$ of sets is a $\Delta$-System iff there exists $r$ such that $\forall a, b \in \mathcal{A} a \cap b=r$. Such $r$ is called the root of $\mathcal{A}$.

Fact 1.21 ( $\Delta$-System Lemma) ([19], II, Theorem 1.6) Let $\kappa \geq \aleph_{0}$ and $\theta$ regular such that $\theta>\kappa$ and $\forall \alpha<\theta|\alpha|^{<\kappa}<\theta$. Let $|\mathcal{A}| \geq \theta$ such that $\forall x \in \mathcal{A} x \subset \kappa \wedge|x|<\kappa$. Then there is $\mathcal{B} \subseteq \mathcal{A}$ such that $|\mathcal{B}|=\theta$ and $\mathcal{B}$ is a $\Delta$-System.

Definition 1.22 Let $\kappa$ be a regular cardinal, $\alpha>\kappa, \alpha \in$ Ord. A family $\mathcal{A}=\left(a_{\xi}\right)_{\xi<\alpha}$ of unbounded subsets of $\kappa$, s.t. for $\xi \neq \xi^{\prime}<\alpha,\left|a_{\xi} \cap a_{\xi^{\prime}}\right|<\kappa$, is called an almost disjoint family on $\kappa$ (of size $|\alpha|$ ).

Lemma 1.23 If $\left(s_{i}: i<\alpha\right)$ is a family of distinct reals, we can use that family to construct a family $\left(r_{i}: i<\alpha\right)$ of almost disjoint reals. In particular it follows that there exists an almost disjoint family on $\omega$ of size $2^{\aleph_{0}}$.

Proof: Let $f: 2^{<\omega} \rightarrow \omega$ be a fixed bijection. For a real $s_{i}$, let $r_{i}:=\{f(t): t$ is an initial segment of the indicator function of $\left.s_{i}\right\}$.

Fact 1.24 ([17], Lemma 9.23) Let $\kappa$ be a regular cardinal. Then there exists an almost disjoint family on $\kappa$ of size $\kappa^{+}$.

Definition 1.25 An ordinal function $f$ on $S$ is regressive if

$$
\forall \alpha \in S(\alpha>0 \rightarrow f(\alpha)<\alpha)
$$

Fact 1.26 (Fodor's Lemma) ([17], Theorem 8.7)
If $f$ is a regressive function on a stationary set $S \subseteq \kappa \in \mathbf{C a r d}$, then there is a stationary set $T \subseteq S$ and some $\gamma<\kappa$ such that $f(\alpha)=\gamma$ for all $\alpha \in T$.

## Definition 1.27 (Singular Cardinal Hypothesis)

The Singular Cardinal Hypothesis (SCH) is the following statement:
For every singular cardinal $\kappa$, if $2^{\mathrm{cf} \kappa}<\kappa$, then $\kappa^{\mathrm{cf} \kappa}=\kappa^{+}$.
Fact 1.28 ([17], Theorem 5.22) Assume that SCH holds:

- If $\kappa$ is a singular cardinal, then
$-2^{\kappa}=2^{<\kappa}$ if the continuum function is eventually const. below $\kappa$,
$-2^{\kappa}=\left(2^{<\kappa}\right)^{+}$otherwise.
- If $\kappa$ and $\lambda$ are infinite cardinals, then
- if $\kappa \leq 2^{\lambda}$, then $\kappa^{\lambda}=2^{\lambda}$,
- if $2^{\lambda}<\kappa$ and $\lambda<\mathrm{cf} \kappa$, then $\kappa^{\lambda}=\kappa$,
- if $2^{\lambda}<\kappa$ and $\operatorname{cf} \kappa \leq \lambda$, then $\kappa^{\lambda}=\kappa^{+}$.


### 1.4 Hereditarily countable sets and reals

Definition 1.29 For all infinite cardinals $\kappa$, let $H_{\kappa}:=\{x:|\operatorname{tcl}(x)|<\kappa\}$.
Definition 1.30 We say that $r \in 2^{\omega}$ codes $x \in \mathrm{H}_{\omega_{1}}$ iff $E_{r}:=\{(m, n) \in$ $\omega \times \omega: \Gamma(m, n) \in r\}$ is well-founded, extensional and the Mostowski Collapse $\operatorname{coll}\left(E_{r}\right)$ of $E_{r}$ equals $\operatorname{tcl}(\{x\})$. We say that r codes $x$ via $g: \operatorname{tcl}(\{x\}) \rightarrow \omega$ if the above holds and $g$ is the inverse of the collapsing map.

By lemma 1.4, lemma 1.5 and lemma 1.7, we obtain the following:
Lemma 1.31 For $r \in 2^{\omega}, x \in \mathrm{H}_{\omega_{1}}, " r$ codes $x "$ is a $\Delta_{1}$ property.
Lemma 1.32 $\forall x \in \mathrm{H}_{\omega_{1}} \exists r \in 2^{\omega} r$ codes $x$.

Proof: Choose some $g: \operatorname{tcl}(\{x\}) \xrightarrow{1-1} \omega$, then choose $E$ such that $g:\langle\operatorname{tcl}(\{x\}), \in\rangle \cong\langle\omega, E\rangle$ and let $r:=\Gamma^{\prime \prime} E$. Obviously, such $r$ codes $x$.

## Definition 1.33

- Let $r_{0} \stackrel{r}{\cong} r_{1}$ iff $E_{r} \subset \omega \times \omega$ is the graph of a partial function $f_{r}: \omega \rightarrow \omega$ such that $\forall m, n \in \operatorname{dom}\left(f_{r}\right)\left(m E_{r_{0}} n \leftrightarrow f_{r}(m) E_{r_{1}} f_{r}(n)\right)$.
- If there exists a unique maximal element $m$ in $E_{r}$, let $\operatorname{top}(r):=m$. If $r$ codes $x$ via $g$, $\operatorname{top}(r)=g(x)$.
- field $(r):=\operatorname{dom}\left(E_{r}\right) \cup \operatorname{range}\left(E_{r}\right)$
- For $m \in \operatorname{field}(r)$, define the transitive closure of $m$ with respect to $r$ as $\operatorname{tcl}_{r}(m):=\left\{n: \exists k \in \omega \exists s \in \omega^{k+1} s(0)=n \wedge s(k)=m \wedge \forall l<\right.$ $\left.k s(l) E_{r} s(l+1)\right\}$.
- $r \sqcup s:=\left\{\Gamma\left(2^{n}, 2^{m}\right): n E_{r} m\right\} \cup\left\{\Gamma\left(3^{n}, 3^{m}\right): n E_{s} m\right\}$
- Let $r_{0} \stackrel{r}{\hookrightarrow} r_{1}$ iff $E_{r_{1}}$ is extensional and $E_{r} \subset \omega \times \omega$ is the graph of a partial function $f_{r}: \omega \rightarrow \omega$ such that
$-\forall m, n \in \operatorname{dom}\left(f_{r}\right)\left(m E_{r_{0}} n \rightarrow f_{r}(m) E_{r_{1}} f_{r}(n)\right)$,
$-\forall 2^{m}, 2^{n} \in \operatorname{dom}\left(f_{r}\right)\left(f_{r}\left(2^{m}\right) E_{r_{1}} f_{r}\left(2^{n}\right) \rightarrow 2^{m} E_{r_{0}} 2^{n}\right)$,
$-\forall 3^{m}, 3^{n} \in \operatorname{dom}\left(f_{r}\right)\left(f_{r}\left(3^{m}\right) E_{r_{1}} f_{r}\left(3^{n}\right) \rightarrow 3^{m} E_{r_{0}} 3^{n}\right)$.

Observation: Note that all the notions introduced in definition 1.33 are arithmetical (for $\operatorname{tcl}_{r}(m)$, use lemma 1.11).

Lemma 1.34 Let $\varphi\left(x_{0}, \ldots, x_{k}\right)$ be a $\boldsymbol{\Delta}_{\mathbf{0}}$ statement in $\mathcal{L}(\in),\left(p_{0}, \ldots, p_{k}\right) \in$ $\mathrm{H}_{\omega_{1}}$. Then there exists a formula $\psi\left(x_{0}, \ldots, x_{k}\right)$ which is $\Delta_{1}^{1}\left(x_{0}, \ldots, x_{k}\right)$ such that for any reals $r_{0}, \ldots, r_{k}$ coding $p_{0}, \ldots, p_{k}$ (such reals do exist),

$$
\left\langle\mathrm{H}_{\omega_{1}}, \in\right\rangle \models \varphi\left(p_{0}, \ldots, p_{k}\right) \quad \longleftrightarrow \quad \psi\left(r_{0}, \ldots, r_{k}\right)
$$

Proof:

- " $p_{0} \in p_{1}$ " is $\Delta_{1}^{1}\left(r_{0}, r_{1}\right)$ :
$-\Sigma_{1}^{1}: \exists r r_{0} \stackrel{r}{\cong} r_{1} \wedge \operatorname{field}\left(r_{0}\right) \subseteq \operatorname{dom}\left(f_{r}\right) \wedge f_{r}\left(\operatorname{top}\left(r_{0}\right)\right) E_{r_{1}} \operatorname{top}\left(r_{1}\right)$
$-\Pi_{1}^{1}: \forall s \forall t\left(r_{0} \sqcup r_{1} \stackrel{s}{\hookrightarrow} t \wedge \operatorname{dom}\left(f_{s}\right)=\operatorname{field}\left(r_{0} \sqcup r_{1}\right) \wedge \operatorname{range}\left(f_{s}\right)=\right.$ field $(t)) \rightarrow f_{s}\left(2^{\operatorname{top}\left(r_{0}\right)}\right) E_{t} f_{s}\left(3^{\operatorname{top}\left(r_{1}\right)}\right)$
- " $p_{0}=p_{1}$ " is $\Delta_{1}^{1}\left(r_{0}, r_{1}\right)$ :

$$
-\Sigma_{1}^{1}: \exists r r_{0} \stackrel{r}{\cong} r_{1} \wedge \operatorname{field}\left(r_{0}\right) \subseteq \operatorname{dom}\left(f_{r}\right) \wedge f_{r}\left(\operatorname{top}\left(r_{0}\right)\right)=\operatorname{top}\left(r_{1}\right)
$$

$$
\begin{aligned}
- & \Pi_{1}^{1}: \forall s \forall t\left(r_{0} \sqcup r_{1} \stackrel{s}{\hookrightarrow} t \wedge \operatorname{dom}\left(f_{s}\right)=\operatorname{field}\left(r_{0} \sqcup r_{1}\right) \wedge \operatorname{range}\left(f_{s}\right)=\right. \\
& \quad \text { field }(t)) \rightarrow f_{s}\left(2^{\text {top }\left(r_{0}\right)}\right)=f_{s}\left(3^{\operatorname{top}\left(r_{1}\right)}\right)
\end{aligned}
$$

For bounded quantifiers, we replace variables $x_{i}$ ranging over $\operatorname{tcl}\left(\left\{p_{i}\right\}\right)$ by variables $y_{i}$ ranging over $\omega$ and express " $x_{0} \in p_{0} ", " x_{1} \in p_{1}$ ", " $x_{0} \in x_{1}$ " and $" x_{0}=x_{1}$ " using the above replacing $\operatorname{top}\left(r_{i}\right)$ by $y_{i}$ and field $\left(r_{0}\right)$ by $\operatorname{tcl}_{r_{0}}\left(y_{0}\right)$.

By replacing atomic formulas in this way and then shifting all quantifiers to the front we can convert $\varphi\left(p_{0}, \ldots, p_{k}\right)$ to a $\Delta_{1}^{1}\left(r_{0}, \ldots, r_{k}\right)$ statement $\psi\left(r_{0}, \ldots, r_{k}\right)$ such that

$$
\left\langle\mathrm{H}_{\omega_{1}}, \in\right\rangle \models \varphi\left(p_{0}, \ldots, p_{k}\right) \quad \longleftrightarrow \quad \psi\left(r_{0}, \ldots, r_{k}\right) .
$$

Lemma 1.35 Let $\varphi\left(x_{0}, \ldots, x_{k}\right)$ be a statement in $\mathcal{L}(\in),\left(p_{0}, \ldots, p_{k}\right) \in \mathrm{H}_{\omega_{1}}$. Then there exists a formula $\psi\left(x_{0}, \ldots, x_{k}\right)$ such that for any reals $r_{0}, \ldots, r_{k}$ coding $p_{0}, \ldots, p_{k}$ the following hold:

1. $\left\langle\mathrm{H}_{\omega_{1}}, \in\right\rangle \models \varphi\left(p_{0}, \ldots, p_{k}\right) \quad \longleftrightarrow \quad \psi\left(r_{0}, \ldots, r_{k}\right)$.
2. If $\varphi\left(p_{0}, \ldots, p_{k}\right)$ is $\boldsymbol{\Sigma}_{\mathbf{n}}, \psi\left(r_{0}, \ldots, r_{k}\right)$ is $\boldsymbol{\Sigma}_{\mathbf{n}+\mathbf{1}}^{\mathbf{1}}$.

If $\varphi\left(p_{0}, \ldots, p_{k}\right)$ is $\boldsymbol{\Pi}_{\mathbf{n}}, \psi\left(r_{0}, \ldots, r_{k}\right)$ is $\boldsymbol{\Pi}_{\mathbf{n}+\mathbf{1}}^{1}$.
Proof: The proof uses induction on formula complexity.
For $\boldsymbol{\Delta}_{\mathbf{0}}$ statements, the above has already been shown in lemma 1.34.
Assume $\varphi$ is $\boldsymbol{\Sigma}_{\mathbf{n}+\boldsymbol{1}}$ and we can already convert $\boldsymbol{\Pi}_{\mathbf{n}}$ statements. For simplicity, assume $\varphi(p) \equiv \exists x \Theta(x, p)$ and $\Theta$ is $\Pi_{n}(x, p)$. We need to express

$$
\begin{equation*}
\exists x \in \mathrm{H}_{\omega_{1}} \quad \mathrm{H}_{\omega_{1}} \models \Theta(x, p) \tag{1}
\end{equation*}
$$

By our induction hypothesis, $\mathrm{H}_{\omega_{1}} \models \Theta(x, p)$ can be converted to a $\boldsymbol{\Pi}_{\mathbf{n}+\mathbf{1}}^{1}$ formula $\tilde{\Theta}$. Let $r_{p}$ be a real coding $p$. Then (1) is equivalent to the following:

$$
\begin{equation*}
\exists r\left(r \text { codes an extensional, well-founded relation } \wedge \tilde{\Theta}\left(r, r_{p}\right)\right) \tag{2}
\end{equation*}
$$

Furthermore " $r \subseteq \omega$ codes a well-founded relation" is equivalent to the statement " $\nexists r^{\prime} \subseteq \omega$ s.t. $r^{\prime}$ codes an infinite descending branch through $r$ ", which is a $\Pi_{1}^{1}(r)$ property. So (2) as a whole is a $\boldsymbol{\Sigma}_{\mathbf{n}+\mathbf{2}}^{1}$ formula.

Now assume $\varphi$ is $\boldsymbol{\Pi}_{\mathbf{n}+\boldsymbol{1}}$ and we can already convert $\boldsymbol{\Sigma}_{\mathbf{n}}$ statements. By the same argument, it follows that we can find an appropriate $\psi \in \boldsymbol{\Pi}_{\mathbf{n}+\mathbf{2}}^{1}$.

Lemma 1.36 If $A \subseteq \omega$ is $\boldsymbol{\Sigma}_{2}^{1}$ then it is $\Sigma_{1}{ }^{\mathrm{H}_{\omega_{1}}}$ with parameters in $\mathcal{P}(\omega)$.
Proof: If $A$ is a $\boldsymbol{\Sigma}_{2}^{1}$ set of reals, then for some $\boldsymbol{\Pi}_{1}^{1}$ relation $P$, $A=\{x: \exists y \subseteq \omega P(x, y)\}$. By fact $1.12, " x \in A$ iff $\exists$ countable transitive model $M \ni x$ adequate for $P \exists y \in M$ such that $M \models P(x, y)$ " which gives us a $\Sigma_{1}{ }^{{ }_{\omega_{1}}}$ definition of $A$.

The next result is an immediate consequence of the above:
Corollary $1.37 A \subseteq \omega$ is $\boldsymbol{\Sigma}_{\mathbf{n}+\mathbf{1}}^{\mathbf{1}}$ iff it is $\Sigma_{n}{ }^{\mathrm{H}_{\omega_{1}}} .(n \geq 1)$

### 1.5 Constructibility

### 1.5.1 Constructibility

Throughout this section, we work in $\mathbf{L}$.
Fact 1.38 ([7], pp 71) There exists a $\Delta_{1}$ relation $<_{\mathbf{L}}$ that well-orders $\mathbf{L}$ s.t.

- for every ordinal $\alpha,<_{\mathbf{L}} \cap\left(L_{\alpha}\right)^{2}$ well-orders $L_{\alpha}$,
- $\left(x<_{\mathbf{L}} y \wedge y \in L_{\alpha}\right) \rightarrow x \in L_{\alpha}$,
- $\left(x \in L_{\alpha} \wedge y \notin L_{\alpha}\right) \rightarrow x<_{\mathbf{L}} y$.

Fact 1.39 ([7], pp 63) The function $\alpha \mapsto L_{\alpha}$ is $\Delta_{1}$.
Fact 1.40 ([7], II, Lemma 1.1) $\forall \alpha \geq \omega,\left|L_{\alpha}\right|=|\alpha|$.
Fact 1.41 ([7], II, Lemma 5.5) $\forall \alpha \geq \omega, \mathcal{P}\left(L_{\alpha}\right)^{\mathbf{L}} \subseteq L_{\left(\alpha^{+}\right)^{\mathbf{L}}}$.
Lemma 1.42 If $\kappa$ is a regular, uncountable $\mathbf{L}$-cardinal, then $H_{\kappa}^{\mathbf{L}}=L_{\kappa}$.
Proof: Work in L: If $x \in L_{\kappa}$, then $x \in L_{\alpha}$ for some ordinal $\alpha<\kappa$, hence $\operatorname{tcl}(x) \subseteq L_{\alpha}$, so $|\operatorname{tcl}(x)| \leq\left|L_{\alpha}\right|=|\alpha|<\kappa$ by fact 1.40 , so $x \in H_{\kappa}$ and thus we have shown $L_{\kappa} \subseteq H_{\kappa}$. Assume $L_{\kappa} \neq H_{\kappa}$ and fix $A \in H_{\kappa} \backslash L_{\kappa}$ such that $A \cap\left(H_{\kappa} \backslash L_{\kappa}\right)=\emptyset$, which we can do because $H_{\kappa}$ and $L_{\kappa}$ are both transitive and $\in$ is well-founded on $H_{\kappa} . A \subseteq H_{\kappa} \cap L_{\kappa}=L_{\kappa}$, so because $\kappa$ is regular and $|A|<\kappa, A \subseteq L_{\alpha}$ for some ordinal $\alpha<\kappa$, hence by fact 1.41, $A \in L_{\alpha^{+}}$. As $\alpha^{+} \leq \kappa, A \in L_{\kappa}$, a contradiction.

Corollary 1.43 If $\kappa \geq \omega, \kappa \in \operatorname{Card}^{\mathbf{L}}$, then $H_{\kappa}^{\mathbf{L}}=L_{\kappa}$.
Lemma 1.44 Let $(\kappa \in \mathbf{C a r d})^{\mathbf{L}}$ and $\xi<\kappa$.
Then the following are equivalent:

1. $\mathbf{L} \vDash \xi \in \mathbf{C a r d}$
2. $\forall \alpha<\kappa\left(\xi \in L_{\alpha} \rightarrow L_{\alpha}=\xi \in \mathbf{C a r d}\right)$

Proof: Assume $\mathbf{L} \models \xi \notin \mathbf{C a r d}$, i.e. $\mathbf{L} \models \exists \beta<\xi \exists f: \beta \xrightarrow{\text { onto }} \xi$. Let $\beta$ and $f$ be witnesses for that statement. Fact 1.41 implies $f \in L_{\kappa}$, thus there is $\alpha<\kappa$ s.t. $\xi, f \in L_{\alpha}$ yielding $L_{\alpha} \models \xi \notin$ Card.

### 1.5.2 Relative Constructibility

Definition $1.45 L_{0}[A]=\emptyset, L_{\alpha+1}[A]=\operatorname{Def}^{A}\left(L_{\alpha}[A]\right)$, where $\operatorname{Def}^{A}(X)$ denotes the set of all subsets of $X$ which are definable (with parameters) in the structure $(X, \in, A \cap X)$ with a unary predicate interpreted as $A \cap X$, $L_{\gamma}[A]=\bigcup_{\alpha<\gamma} L_{\alpha}[A]$ for limits $\gamma, \mathbf{L}[A]=\bigcup_{\alpha \in \mathbf{O r d}} L_{\alpha}[A]$.

Throughout this section, we work in $\mathbf{L}[A]$ for some $A \subseteq \mathbf{L}$.
Fact 1.46 ([7], Exercise 2L)
There is a $\Delta_{1}(A)$ relation $<_{\mathbf{L}[\mathbf{A}]}$ that well-orders $\mathbf{L}[\mathbf{A}]$ such that

- for every ordinal $\alpha,<_{\mathbf{L}[\mathbf{A}]} \cap\left(L_{\alpha}[A]\right)^{2}$ well-orders $L_{\alpha}[A]$,
- $\left(x<_{\mathbf{L}[\mathbf{A}]} y \wedge y \in L_{\alpha}[A]\right) \rightarrow x \in L_{\alpha}[A]$,
- $\left(x \in L_{\alpha}[A] \wedge y \notin L_{\alpha}[A]\right) \rightarrow x<_{\mathbf{L}[\mathbf{A}]} y$.

Fact 1.47 ([7], Exercise 2A) $\forall \alpha \geq \omega,\left|L_{\alpha}[A]\right|=|\alpha|$.
Fact 1.48 ([17], pp 192) There is $\varphi \in \mathcal{L}(\in, A)$, where $A$ is a unary predicate, such that for any transitive model $\mathfrak{M}=(M, \in, A), \mathfrak{M} \vDash \varphi$ if and only if for some limit ordinal $\alpha, M=L_{\alpha}[A]$.

Notation: In the following, we often write $\mathbf{L}[A]$ or $L_{\alpha}[A]$ to actually denote the models $(L[A], \in, A \cap \mathbf{L}[A])$ or $\left(L_{\alpha}[A], \in, A \cap L_{\alpha}[A]\right)$ respectively.

Lemma 1.49 If $\mathfrak{M}$ is an elementary submodel of $L_{\gamma}[A], \gamma$ a limit ordinal, then the transitive collapse $\overline{\mathfrak{M}}$ of $\mathfrak{M}$ equals (let $\pi$ denote the collapsing map) $L_{\beta}[B]$ with $B=\pi^{\prime \prime}(A \cap M)$ for some limit ordinal $\beta$.

Proof: Let $\varphi$ be such that the property in fact 1.48 holds. Then $L_{\gamma}[A] \models \varphi$, hence $\mathfrak{M} \mid=\varphi$, i.e. $\overline{\mathfrak{M}}=L_{\beta}[B]$ with $B=\pi^{\prime \prime}(A \cap M)$.

Lemma 1.50 If $\mathfrak{M}$ is a countable elementary submodel of $L_{\gamma}[A], \gamma \geq \omega_{1}$ a limit ordinal, then $\omega_{1} \cap M=\alpha$ for some $\alpha<\omega_{1}$.

Proof: Let $\beta<\omega_{1}, \beta \in M$. Let $f$ be the $<_{L_{L}}$-least mapping $f: \omega \xrightarrow{\text { onto }} \beta$. By elementarity, $f \in M$. Since $\omega \subset M$, $f^{\prime \prime}(\omega)=\beta \subset M$.

Corollary 1.51
If $\mathfrak{M}$ is a countable elementary submodel of $L_{\omega_{1}}[A]$, then $M$ is transitive.
Lemma 1.52 If $\mathfrak{M}$ is a countable elementary submodel of $L_{\gamma}[A], \gamma \geq \omega_{1} a$ limit ordinal, $A \subseteq \omega_{1}$, then the transitive collapse $\overline{\mathfrak{M}}$ of $\mathfrak{M}$ equals $L_{\beta}[A \cap \alpha]$ for some $\alpha, \beta<\omega_{1}$, in fact $\alpha=\omega_{1}^{\overline{\mathfrak{M}}}$ (if $\gamma>\omega_{1}$ ) and $\beta=\operatorname{Ord}(\overline{\mathfrak{M}})$.

Proof: By lemma 1.49, $\overline{\mathfrak{M}}=L_{\beta}\left[\pi^{\prime \prime}(A \cap M)\right]$. By lemma $1.50, A \cap M=$ $\left(A \cap \omega_{1}\right) \cap M=A \cap\left(M \cap \omega_{1}\right)=A \cap \alpha$ for some $\alpha<\omega_{1}$. Since $\overline{\mathfrak{M}}$ is transitive, assuming $\gamma>\omega_{1}, \alpha=\omega_{1}^{\bar{M}}$.

Lemma 1.53 For $A \subseteq \omega_{1}, \mathbf{L}[A] \models \mathrm{GCH}$.
Proof: If $x \subseteq \omega, x \in \mathbf{L}[A]$, then $x$ is an element of some countable, elementary submodel $\mathfrak{M}$ of $L_{\omega_{1}}[A]$ and of the transitive collapse $\overline{\mathfrak{M}}$ of $\mathfrak{M}$. By lemma 1.49, $\overline{\mathfrak{M}}=L_{\beta}[A \cap \alpha]$ with $\alpha, \beta<\omega_{1}$. It follows that $2^{\aleph_{0}} \leq$ $\left|\bigcup_{\alpha, \beta<\omega_{1}} L_{\beta}[A \cap \alpha]\right|=\aleph_{1}$, hence $2^{\aleph_{0}}=\aleph_{1}$.

If $x \subseteq \omega_{\gamma}, \gamma \geq 1, x \in \mathbf{L}[A]$, then $x$ is an element of some elementary submodel $\mathfrak{M}$ of $L_{\omega_{\gamma}}[A]$ of size $\aleph_{\gamma}$ with $\omega_{\gamma} \subseteq M$. It follows that $x$ is also an element of the transitive collapse $\overline{\mathfrak{M}}$ of $\mathfrak{M}$. Like above, it now follows that $2^{\aleph_{\gamma}}=\aleph_{\gamma}{ }^{+}$.

### 1.6 Absoluteness

Definition 1.54 Suppose $\mathcal{P}$ is a definable class of posets in $\mathbf{V}$. Then $\Sigma_{n}\left(H_{\omega_{1}}\right)$ absoluteness for $\mathcal{P}$-forcing is the statement that $H_{\omega_{1}} \mathbf{V} \prec \Sigma_{n} H_{\omega_{1}} \mathbf{V}^{P}$ for any $P \in \mathcal{P}$. We abbreviate this as $\operatorname{Abs}\left(\Sigma_{n}\left(H_{\omega_{1}}\right), \mathcal{P}\right)$.
Analogous definitions apply to larger cardinals.
Definition 1.55 $\mathbf{\Sigma}_{\mathbf{n}}^{\mathbf{1}}$ absoluteness for $\mathcal{P}$-forcing means that for any real $r \in$ $\mathbf{V}$ and any $\varphi \in \Sigma_{n}^{1}(r), \mathbf{V} \models \varphi(r) \leftrightarrow \mathbf{V}^{P} \models \varphi(r)$ for any $P \in \mathcal{P}$. $\Sigma_{n}^{1}$ absoluteness for $\mathcal{P}$-forcing is the same without parameter.

By lemma 1.31, lemma 1.35 and corollary 1.37 the following is immediate:
Corollary $1.56 \boldsymbol{\Sigma}_{\mathbf{n}+\mathbf{1}}^{1}$ absoluteness for $\mathcal{P}$-forcing $\longleftrightarrow \operatorname{Abs}\left(\Sigma_{n}\left(\mathrm{H}_{\omega_{1}}\right), \mathcal{P}\right)$.
Fact 1.57 (Lévy-Shoenfield Absoluteness Theorem) ([16], Thr. 36) Let $a \subseteq \omega, \varphi \in \Sigma_{1}(a)$ and $\theta=\omega_{1}^{\mathbf{L}[a]}$. Then

$$
\varphi(a) \rightarrow \mathbf{L}_{\theta}[a] \models \varphi(a)
$$

Applying lemma 1.36 and upwards absoluteness of $\Sigma_{1}$ statements yields the following:

Corollary 1.58 If $\varphi \in \Sigma_{2}^{1}(a)$ then

$$
\varphi(a) \leftrightarrow \mathfrak{M} \models \varphi(a)
$$

for every transitive $M \supseteq \mathbf{L}_{\theta}[a]$ with $\theta=\omega_{1}^{\mathbf{L}[a]}$.
Corollary 1.58 can be weakened to the following (for a brief review of class forcing, see section 2.1.6):

## Corollary 1.59

- $\boldsymbol{\Sigma}_{2}^{1}$ absoluteness for set/class forcing holds.
- $\operatorname{Abs}\left(\Sigma_{1}\left(\mathrm{H}_{\omega_{1}}\right)\right.$, set/class forcing $)$ holds.

Lemma 1.60 For any poset $P$, the following holds:

$$
\operatorname{Abs}\left(\Sigma_{1}\left(H_{\kappa}\right), P\right) \leftrightarrow \text { for any } \varphi \in \Sigma_{1}\left(H_{\kappa}\right), \quad\left(\varphi \leftrightarrow \Vdash_{P} \varphi\right)
$$

Proof: Let $\varphi \in \Sigma_{1}\left(H_{\kappa}\right)$. First assume that $\varphi \leftrightarrow \Vdash_{P} \varphi$ and $\Vdash_{P} H_{\kappa} \vDash \varphi$. Then, because of upward absoluteness, $\Vdash_{P} \varphi$ and hence, by our assumption, $\varphi$ holds. By lemma 1.64 below, it follows that $H_{\kappa} \models \varphi$.

The second direction of the proof works almost the same.

### 1.7 Large Cardinals

### 1.7.1 Inaccessible Cardinals

Definition $1.61 \kappa \in$ Card is weakly inaccessible if $\kappa$ is a regular limit cardinal, (strongly) inaccessible if $\kappa$ is regular and $\forall \lambda<\kappa 2^{\lambda}<\kappa$, i.e. $\kappa$ is a regular strong limit cardinal.

Definition $1.62 \omega_{1}$ is inaccessible to reals if for every $x \subseteq \omega$, $\omega_{1}$ is inaccessible in $\mathbf{L}[x]$.

Lemma $1.63 \omega_{1}$ is inaccessible to reals iff for every $x \subseteq \omega, \omega_{1}^{\mathbf{L}[x]}<\omega_{1}$.
Proof: First assume $\omega_{1}$ is inaccessible to reals and let $x \subseteq \omega$. Then $\omega_{1}$ is inaccessible in $\mathbf{L}[x]$, hence $\omega_{1}>\omega_{1}^{\mathbf{L}[x]}$.

For the other direction, assume that for some $y \subseteq \omega, \omega_{1}$ is not inaccessible in $\mathbf{L}[y]$, i.e. $\exists \lambda<\omega_{1}, \lambda \in \operatorname{Card}^{\mathbf{L}[y]} \wedge\left(2^{\lambda}\right)^{\mathbf{L}[y]} \geq \omega_{1}$. Since $\lambda<\omega_{1}$, there exists $f: \omega \xrightarrow{\text { onto }} \lambda$, since $f \in \mathrm{H}_{\omega_{1}}$, we can code $f$ by a real $r$. But then $\mathbf{L}[y][r] \models \lambda<\omega_{1}^{\mathbf{L}[y][r]}$, hence $\omega_{1}^{\mathbf{L}[y][r]}=\left(\lambda^{+} \mathbf{L}^{\mathbf{L}[y][r]} \geq\left(\lambda^{+}\right)^{\mathbf{L}[y]}=\left(2^{\lambda}\right)^{\mathbf{L}[y]} \geq \omega_{1}\right.$. Since $\mathbf{L}[y][r]=\mathbf{L}[c]$ for some $c \subseteq \omega$, this proves the lemma.

### 1.7.2 Reflecting cardinals

Lemma $1.64 \forall \kappa \geq \omega_{1} H_{\kappa} \prec_{\Sigma_{1}}$ V.
Proof: Let $\varphi \in \Sigma_{1}\left(H_{\kappa}\right)$. If $H_{\kappa} \models \varphi$ then by upwards absoluteness of $\Sigma_{1}$ statements, $\mathbf{V} \models \varphi$. Conversely, let $\mathbf{V} \models \varphi$. Then by a Löwenheim-Skolem argument, there exists a well-founded, extensional model of $\varphi$ containing $\operatorname{tcl}(p)$ for each parameter $p$ occuring in $\varphi$, of size $<\kappa$. Let $\mathfrak{M}$ be the transitive collapse of that model, then $\mathfrak{M} \models \varphi$. Because $M \subseteq H_{\kappa}$ and $\varphi$ is upwards absolute, $H_{\kappa} \models \varphi$.

Corollary $1.65 \forall \kappa \geq \omega_{1} \forall \varphi \in \Sigma_{2}\left(H_{\kappa}\right)\left(H_{\kappa} \models \varphi \rightarrow \mathbf{V} \models \varphi\right)$.
Proof: Assume $\varphi \in \Sigma_{2}\left(H_{\kappa}\right), \varphi \equiv \exists x \forall y \psi(x, y, p), \psi$ is $\Delta_{0}(x, y, p)$ and $H_{\kappa}=$ $\varphi$. Fix a witness $x \in H_{\kappa}$ for $\varphi$. Then $H_{\kappa} \vDash \forall y \psi(x, y, p)$. By lemma 1.64, $\mathbf{V} \models \forall y \psi(x, y, p)$, hence $\mathbf{V} \models \varphi$.

So we have just shown that, for any $\kappa \geq \omega_{1}, \Sigma_{1}\left(H_{\kappa}\right)$ statements are absolute between $H_{\kappa}$ and $\mathbf{V}$ and $\Sigma_{2}\left(H_{\kappa}\right)$ statements are upwards absolute between $H_{\kappa}$ and $\mathbf{V}$. We can also obtain a singular cardinal $\kappa$ such that $\Sigma_{2}\left(H_{\kappa}\right)$ statements are absolute between $H_{\kappa}$ and $\mathbf{V}$ :

Lemma $1.66 \exists \kappa H_{\kappa} \prec_{\Sigma_{2}} \mathbf{V}$.

Proof: Assume $\varphi \equiv \exists x \forall y \psi(x, y)$ where $\psi$ is $\Delta_{0}(x, y)$ and $\mathbf{V} \models \varphi$. Then there exists a cardinal $\lambda_{\varphi}$ such that $\exists x \in H_{\lambda_{\varphi}} \forall y \psi(x, y)$. Let $\lambda_{0}:=$ $\bigcup_{\varphi \in \Sigma_{2}(\emptyset)} \lambda_{\varphi}$. Then for each $\varphi \in \Sigma_{2}(\emptyset)$, by downwards absoluteness of $\Pi_{1}$ formulas, $H_{\lambda_{0}} \models \varphi \leftrightarrow \mathbf{V} \models \varphi$.

Now, similar to above, for each $\varphi \in \Sigma_{2}\left(H_{\lambda_{0}}\right), \varphi \equiv \exists x \forall y \psi(x, y, p)$ with $p \in$ $H_{\lambda_{0}}$ and $\mathbf{V} \models \varphi$, we can find a cardinal $\lambda_{\varphi}$ such that $\exists x \in H_{\lambda_{\varphi}} \forall y \psi(x, y, p)$. Let $\lambda_{1}:=\bigcup_{\varphi \in \Sigma_{2}\left(H_{\lambda_{0}}\right)} \lambda_{\varphi}$. Then for each $\varphi \in \Sigma_{2}\left(H_{\lambda_{0}}\right), H_{\lambda_{1}} \models \varphi \leftrightarrow \mathbf{V} \models \varphi$.

Going on like this, we get an $\omega$-chain of cardinals $\left\langle\lambda_{0}, \lambda_{1}, \ldots\right\rangle$ such that

$$
\forall i<\omega \forall \varphi \in \Sigma_{2}\left(H_{\lambda_{i-1}}\right)\left(H_{\lambda_{i}} \models \varphi \leftrightarrow \mathbf{V} \models \varphi\right) . \quad\left(\text { let } \lambda_{-1}:=\emptyset\right)
$$

Let $\lambda:=\bigcup_{i<\omega} \lambda_{i}$. Then $H_{\lambda} \prec \Sigma_{2} \mathbf{V}$ :

Let $\varphi \in \Sigma_{2}\left(H_{\lambda}\right), \varphi \equiv \exists x \forall y \psi(x, y, p)$ with $p \in H_{\lambda}$ and $\psi \in \Delta_{0}(x, y, p)$. Assume $\mathbf{V} \models \varphi$. $\exists i<\omega p \in H_{\lambda_{i-1}}$, so, by our above construction, $\exists x \in$ $H_{\lambda_{i}} \mathbf{V} \models \forall y \psi(x, y, p)$. So, by downwards absoluteness of $\Pi_{1}$ statements, $H_{\lambda} \models \exists x \forall y \psi(x, y, p)$ i.e. $H_{\lambda} \models \varphi$.

Definition 1.67 $A$ regular cardinal $\kappa$ is reflecting if for every $a \in H_{\kappa}$ and every first order formula $\varphi(x)$, if for some (regular) cardinal $\lambda, H_{\lambda} \models \varphi(a)$, then there exists a cardinal $\delta<\kappa$ such that $H_{\delta} \models \varphi(a)$.

Observation: The above definition remains equivalent whether regular is inserted (where it is indicated in brackets) or not. This follows from the proof of theorem 1.69 below - there we show that, using the seemingly weaker definition of a reflecting cardinal (with regular inserted), if $\kappa$ is reflecting, then $H_{\kappa} \prec_{\Sigma_{2}} \mathbf{V}$ and, using the seemingly stronger definition of a reflecting cardinal (leaving out that particular regularity assumption), if $H_{\kappa} \prec \Sigma_{2} \mathbf{V}$, then $\kappa$ is reflecting.

Lemma $1.68 \kappa$ reflecting $\rightarrow \kappa$ inaccessible.
Proof: Let $x \in H_{\kappa}$. Then for some (regular) cardinal $\lambda, H_{\lambda} \vDash \exists y y=$ $|\mathcal{P}(x)|$. Because $\kappa$ is reflecting, there exists a cardinal $\delta<\kappa$ such that $H_{\delta}\left|\exists y y=|\mathcal{P}(x)|\right.$ and because $\mathcal{P}(x)^{H_{\delta}}=\mathcal{P}(x), \exists y \in H_{\kappa} y=|\mathcal{P}(x)|$ and hence $\forall \alpha<\kappa \quad 2^{\alpha}<\kappa$.

Theorem 1.69 Let $\kappa$ be a regular cardinal. Then

$$
\kappa \text { is reflecting } \longleftrightarrow V_{\kappa} \prec_{\Sigma_{2}} \mathbf{V}
$$

Proof: First assume $\kappa$ is reflecting. Then by lemma 1.68, $\kappa$ is inaccessible. This implies that $H_{\kappa}=V_{\kappa}$ and thus by lemma $1.65, \Sigma_{2}$ statements are upwards absolute between $V_{\kappa}$ and $\mathbf{V}$. Now if $\mathbf{V} \models \varphi$ with $\varphi \in \Sigma_{2}\left(H_{\kappa}\right)$, then we can find a regular cardinal $\lambda$ such that $H_{\lambda} \models \varphi$. Because $\kappa$ is reflecting, there is $\delta<\kappa$ such that $H_{\delta} \models \varphi$, hence $H_{\kappa} \models \varphi$.

Now suppose $V_{\kappa} \prec \Sigma_{2} \mathbf{V}$ and $\kappa$ regular. Again, $\kappa$ must be inaccessible, as $" \exists y \exists \alpha \in \operatorname{Ord} \exists f: \alpha \xrightarrow{\text { onto }} y \wedge y=\mathcal{P}(x) "$ is a $\Sigma_{2}(x)$ statement, hence for each $x \in V_{\kappa},|\mathcal{P}(x)| \in V_{\kappa}$. Again, $H_{\kappa}=V_{\kappa}$ follows. Secondly suppose that for some cardinal $\lambda, H_{\lambda} \models \varphi$ with $\varphi$ having parameters in $H_{\kappa}$.

$$
y=H_{\delta} \longleftrightarrow \forall x(|\operatorname{tcl}(x)|<\delta \leftrightarrow x \in y)
$$

Because $|\operatorname{tcl}(x)|<\delta$ is a $\Delta_{1}(x, \delta)$ statement, $y=H_{\delta}$ is a $\Pi_{1}(y, \delta)$ statement and hence, by lemma $1.64, H_{\delta}$ is absolute between $H_{\kappa}$ and $\mathbf{V}$ (for $\delta<\kappa$ ). $\exists \delta H_{\delta} \models \varphi$ is a $\Sigma_{2}$ statement, so because $H_{\kappa}=V_{\kappa} \prec \Sigma_{2} \mathbf{V}$, it follows that $H_{\kappa} \models\left(\exists \delta H_{\delta} \models \varphi\right)$ and also $\exists \delta<\kappa H_{\kappa} \models\left(H_{\delta} \models \varphi\right)$, so we can finally conclude $\exists \delta<\kappa H_{\delta} \models \varphi$, yielding $\kappa$ to be a reflecting cardinal.

Corollary 1.70 If $H_{\kappa} \prec \Sigma_{2} \mathbf{V}$ ( $\kappa$ might be singular), then for every $a \in H_{\kappa}$ and every first order formula $\varphi(x)$, if for some cardinal $\lambda, H_{\lambda}=\varphi(a)$, then there exists a cardinal $\delta<\kappa$ such that $H_{\delta} \models \varphi(a)$.

Observation: Note that $\exists \delta \delta$ regular $\wedge H_{\delta} \models \varphi$ is a $\Sigma_{2}$ statement as well. Hence we can (equivalently) define reflecting cardinals as follows:

A regular cardinal $\kappa$ is reflecting if for every $a \in H_{\kappa}$ and every first order formula $\varphi(x)$, if for some regular cardinal $\lambda, H_{\lambda} \models \varphi(a)$, then there exists a regular cardinal $\delta<\kappa$ such that $H_{\delta} \models \varphi(a)$.

Lemma 1.71 Let $\kappa$ be a regular cardinal. Then $\kappa$ is reflecting iff for any first order formula $\varphi(x)$, for any $a \in H_{\kappa}$ and for unboundedly many $\lambda \geq \kappa$,

$$
H_{\lambda} \models \varphi(a) \rightarrow \exists \delta<\kappa H_{\delta} \models \varphi(a) .
$$

Proof: First note that if $\lambda^{\prime}<\lambda$, since $\left(H_{\lambda^{\prime}}\right)^{H_{\lambda}}=H_{\lambda^{\prime}}$, the following holds:

$$
\begin{equation*}
H_{\lambda^{\prime}} \models \varphi(a) \leftrightarrow H_{\lambda} \models " H_{\lambda^{\prime}} \models \varphi(a) " \tag{3}
\end{equation*}
$$

Now assume that the assumptions from our lemma hold, we will show that $\kappa$ is reflecting: Let $\varphi(x)$ be any first order formula and let $a \in H_{\kappa}$. Let $\lambda^{\prime} \geq \kappa$ such that $H_{\lambda^{\prime}} \models \varphi(a)$ and let $\lambda \geq \lambda^{\prime}$ such that the above implication holds for $\lambda$ (i.e. formulas with parameters in $H_{\kappa}$ which hold in $H_{\lambda}$ are reflected to some $H_{\delta}$ with $\delta<\kappa$ ).

By (3), $H_{\lambda} \models " H_{\lambda^{\prime}} \models \varphi(a) "$, so $H_{\lambda} \models \exists \chi " H_{\chi} \models \varphi(a) "$. This sentence is reflected to some $\delta<\kappa$ by assumption, i.e. $\exists \delta<\kappa H_{\delta} \models " \exists \chi H_{\chi} \models \varphi(a)$ ", henceforth $\exists \delta<\kappa \exists \chi<\delta H_{\delta} \models " H_{\chi} \models \varphi(a)$ "; again by (3), we conclude that $\exists \chi<\kappa H_{\chi}=\varphi(a)$ which shows that $\kappa$ is a reflecting cardinal.

### 1.7.3 Mahlo Cardinals

Definition 1.72 An inaccessible cardinal $\kappa$ is called a Mahlo cardinal if the set of all regular cardinals below $\kappa$ is stationary.

Lemma 1.73 If $\kappa$ is inaccessible, then the set of all strong limit cardinals below $\kappa$ is closed unbounded. Therefore, if $\kappa$ is Mahlo, then the set of all inaccessibles below $\kappa$ is stationary and $\kappa$ is the $\kappa$-th inaccessible cardinal.

Proof: Let $\alpha<\kappa$. By iterating the power set operation, we can obtain a strong limit cardinal above $\alpha$. Since $\kappa$ is inaccessible, it will be below $\kappa$, yielding unboundedness. Closedness is obvious.

Lemma 1.74 If both a reflecting and a Mahlo cardinal exist, then the least Mahlo is strictly below the least reflecting, which is not Mahlo itself. The existence of a Mahlo cardinal implies the consistency of a stationary class of reflecting cardinals.

Proof: Let $\theta$ be the least Mahlo and $\kappa$ the least reflecting cardinal. If $\theta>\kappa, H_{\theta^{+}} \models \exists \lambda$ Mahlo; this is reflected by some $H_{\eta}, \eta<\kappa$. As being Mahlo is absolute between $H_{\eta}$ and $\mathbf{V}$, there is a Mahlo cardinal below $\kappa$, a contradiction.

Now assume that $\kappa$, the least reflecting cardinal, is Mahlo. For any $\xi<\kappa$, let $f(\xi)$ be least such that for all $p \in H_{\xi^{+}}$and all formulas $\varphi(x)$, if there is a regular $\delta$ such that $H_{\delta} \models \varphi(p)$, then there is such $\delta$ below $f(\xi)$. Since $\kappa$ is reflecting and regular, $f$ maps $\kappa$ into $\kappa$. Since $\kappa$ is Mahlo, there must be a regular closure point of $f$ below $\kappa$. Any such closure point is a reflecting cardinal below $\kappa$, a contradiction.

If $\theta$ is any Mahlo cardinal, since $H_{\theta}$ is a model of $Z F C$, we can construct $f$ in $H_{\theta}$ similar to above - for any $\xi$, let $f(\xi)$ be least such that for all $p \in H_{\xi^{+}}$ and all formulas $\varphi(x)$, if there is a regular $\delta$ such that $H_{\delta} \models \varphi(p)$, then there
is such $\delta$ below $f(\xi)$. The set of closure points of $f$ is a club in $\theta$, hence, since $\theta$ is Mahlo, there is a stationary set of regular closure points of $f$ below $\theta$, each of those is a reflecting cardinal in $H_{\theta}$.

Corollary 1.75 If $\theta$ is Mahlo in $\mathbf{L}$, then $\mathbf{L}_{\theta} \models \exists \kappa$ reflecting.

### 1.7.4 $0^{\sharp}$

$0^{\sharp}$ is a set of natural numbers defined as

$$
0^{\sharp}=\left\{\sharp \varphi: L_{\aleph_{\omega}} \models \varphi\left(\aleph_{1}, \ldots, \aleph_{n}\right)\right\} .
$$

Fact 1.76 (Silver) ([17], pp 311) $0^{\sharp}$ exists if and only if

1. If $\kappa$ and $\lambda$ are uncountable cardinals and $\kappa<\lambda$, then $\left(L_{\kappa}, \in\right) \prec\left(L_{\lambda}, \in\right)$.
2. There is a unique class-sized club of ordinals I containing all uncountable cardinals such that for every uncountable cardinal $\kappa$ :
(a) $|I \cap \kappa|=\kappa$,
(b) $I \cap \kappa$ is a set of indiscernibles for $\left(L_{\kappa}, \in\right)$ and
(c) every $a \in L_{\kappa}$ is definable in $\left(L_{\kappa}, \in\right)$ from $I \cap \kappa$.

The elements of $I$ are called Silver Indiscernibles.
Applying the reflection principle in $\mathbf{L}$ yields the following:
Corollary 1.77 If $0^{\sharp}$ exists, then for every uncountable cardinal $\kappa$,

$$
\left(L_{\kappa}, \in\right) \prec(\mathbf{L}, \in) .
$$

This obviously implies the following:
Corollary 1.78 If $0^{\sharp}$ exists, then $\mathbf{V} \neq \mathbf{L}$.
Fact 1.79 [Jensen's Covering Theorem] ([17], Theorem 18.30) If $0^{\sharp}$ does not exist, then for every uncountable set $X \subset$ Ord, there exists $Y \in \mathbf{L}$ such that $Y \supseteq X$ and $|Y|=|X|$.

Lemma 1.80 If $0^{\sharp}$ does not exist, then for every $\lambda \geq \aleph_{2}$, if $\lambda$ is a regular cardinal in $\mathbf{L}$, then $\mathrm{cf} \lambda=|\lambda|$. Consequently, every singular cardinal is a singular cardinal in $\mathbf{L}$.

Proof: Let $\lambda$ be a limit ordinal such that $\lambda \geq \aleph_{2}$ and $\lambda$ is a regular cardinal in $\mathbf{L}$. Let $X$ be an unbounded subset of $\lambda$ such that $|X|=\operatorname{cf} \lambda$. By the Covering Theorem, there exists a constructible $Y$ such that $X \subseteq Y \subseteq \lambda$ and $|Y|=|X| * \aleph_{1}$. Since $Y$ is unbounded in $\lambda$ and $\lambda$ is a regular $\mathbf{L}$-cardinal, $|Y|=|\lambda|$. Hence $|\lambda|=\operatorname{cf} \lambda * \aleph_{1}$; since $\lambda \geq \aleph_{2}$, this yields $|\lambda|=\operatorname{cf} \lambda$.

## Lemma 1.81

If $0^{\sharp}$ does not exist, then, for every singular cardinal $\kappa,\left(\kappa^{+}\right)^{\mathbf{L}}=\kappa^{+}$.
Proof: Let $\kappa$ be a singular cardinal and let $\lambda:=\left(\kappa^{+}\right)^{\mathbf{L}}$. We want to show that $\lambda=\kappa^{+}$: If not, $|\lambda|=\kappa$ and since $\kappa$ is singular, $\operatorname{cf} \lambda<\kappa$. However, this means $\operatorname{cf} \lambda<|\lambda|$, contradicting the above lemma.

Lemma 1.82 If $0^{\sharp}$ does not exist, then the Singular Cardinal Hypothesis holds.

Proof: Let $\kappa$ be such that $2^{\text {cf } \kappa}<\kappa$ and let $A:=[\kappa]^{\text {cf } \kappa}$. We show that $|A| \leq \kappa^{+}$, which is sufficient. By the Covering Theorem (fact 1.79), for every $x \in A$, there exists a constructible $y \subseteq \kappa$ such that $x \subseteq y$ and $|y|=\lambda$ with $\lambda=\aleph_{1} \cdot \operatorname{cf} \kappa$, thus $A \subseteq \bigcup\left\{[y]^{\text {cf } \kappa}: y \in C\right\}$ with $C:=\{y \subseteq \kappa:|y|=$ $\lambda \wedge y \in \mathbf{L}\}$. If $y \in C$, then $\left|[y]^{\text {cf } \kappa}\right|=\lambda^{\text {cf } \kappa}=\left(\aleph_{1} \cdot \operatorname{cf} \kappa\right)^{\text {cf } \kappa}=2^{\text {cf } \kappa}<\kappa$. Since $|C| \leq\left|\mathcal{P}^{\mathbf{L}}(\kappa)\right|=\left|\left(\kappa^{+}\right)^{\mathbf{L}}\right| \leq \kappa^{+}$, it follows that $|A| \leq \kappa^{+}$.

### 1.7.5 $R^{\sharp}$

For $R \subseteq \omega, R^{\sharp}$ is a set of natural numbers defined as

$$
R^{\sharp}=\left\{\sharp \varphi:\left(L_{\aleph_{\omega}}[R], \in, R\right) \models \varphi\left(\aleph_{1}, \ldots, \aleph_{n}\right)\right\}
$$

All results from the previous section can be relativized, working over $\mathbf{L}[R]$ instead of $\mathbf{L}$, to obtain results analogous to above for $R^{\sharp}$ instead of $0^{\sharp}$.

## 2 Introduction: Forcing

### 2.1 Definitions and Facts

### 2.1.1 Properties of forcings

Definition 2.1 A partial order $P$ is $\kappa$-closed if for every $\lambda<\kappa$, every descending sequence $\left\langle p_{\alpha}: \alpha<\lambda\right\rangle$ has a lower bound. $P$ is $\sigma$-closed if it is $\omega_{1}$-closed.

Definition 2.2 A partial order $P$ is $\kappa$-distributive if the intersection of $<\kappa$ open dense sets of $P$ is open dense.

Fact 2.3 ([17], Lemma 15.8) If $P$ is $\kappa$-closed, then $P$ is $\kappa$-distributive.
Fact 2.4 ([17], Theorem 15.6) Let $\kappa$ be an infinite cardinal, assume that $P$ is $\kappa$-distributive and let $G$ be generic for $P$ over $\mathbf{V}$. Then if $f \in \mathbf{V}[G]$ is a function from $\lambda$ into $\mathbf{V}$ and $\lambda<\kappa$, then $f \in \mathbf{V}$. In particular, such $\lambda$ has no new subsets in $\mathbf{V}[G]$.

Definition 2.5 A partial order $P$ satisfies the $\kappa$-chain-condition or is $\kappa$ cc if every antichain in $P$ is of size $<\kappa$. $P$ satisfies the countable chain condition or is ccc if it is $\omega_{1}-c c$.

Fact 2.6 ([17], Theorem 7.15) If $P$ is $\kappa$-cc and $\kappa$ singular, then $\exists \lambda<\kappa P$ is $\lambda$-cc, i.e. the least $\lambda$ s.t. $P$ satisfies the $\lambda$-cc is a regular cardinal.

Definition 2.7 A partial ordering $P$ has property (K) if every uncountable subset of $P$ contains an uncountable pairwise compatible set.

Remark: Obviously, $P$ has property $(\mathrm{K}) \rightarrow P$ ccc.
Definition 2.8 A partial order $P$ is $\kappa$-centered if there is $f: P \rightarrow \kappa$ such that $\forall \xi<\kappa \forall W \in\left[f^{-1}(\xi)\right]^{<\kappa} \exists \gamma \forall \delta \in W \gamma \leq \delta$.

Definition 2.9 A partial order $P$ is $\kappa$-linked if there is $f: P \rightarrow \kappa$ such that $f(p)=f(q) \rightarrow p \| q$.

Remark: Obviously, $P \kappa$-centered $\rightarrow P \kappa$-linked $\rightarrow P \kappa^{+}$-cc.
Definition 2.10 A partial order $P$ is proper if for every uncountable cardinal $\lambda$, every stationary subset of $[\lambda]^{\omega}$ remains stationary in the generic extension.

Fact 2.11 ([17], Lemma 31.3) $P \sigma$-closed $\rightarrow P$ proper.
Fact 2.12 ([17], Lemma 31.2) $P$ ссс $\rightarrow P$ proper.

Definition 2.13 Let $P \in H_{\lambda}$ be a p.o., $\lambda>\omega$ and $\mathfrak{M} \prec\left(H_{\lambda}, \in\right)$ s.t. $P \in M$. $q \in P$ is $(M, P)$-generic if for every $I \in M$ which is predense (or equivalently, for every maximal antichain $I \in M), I \cap M$ is predense below $q$.

Fact 2.14 ([21], III, Lemma 2.6; [17], Lemma 31.6) Let $\lambda$ be a regular uncountable cardinal such that $\mathcal{P}(P) \in H_{\lambda}$ and let $M \prec\left(H_{\lambda}, \in\right)$ such that $P \in M$. Then the following are equivalent:

1. $q$ is $(M, P)$-generic,
2. If $\dot{\tau}$ is a name for an ordinal and $\dot{\tau} \in M$, then $q \Vdash \dot{\tau} \in M$,
3. For every $P$-name $\dot{\tau} \in M$, if $\Vdash \dot{\tau} \in \mathbf{V}$, then $q \Vdash \dot{\tau} \in M$,
4. $q \Vdash \dot{G} \cap M$ is a filter on $P$ generic over $M$, where $\dot{G}=\{(\check{p}, p): p \in P\}$ is the standard name for the generic filter.

Fact 2.15 ([6], Lemma 2.5) The following are equivalent:

1. $P$ is proper.
2. There is a regular cardinal $\lambda$ such that $P \in H_{\lambda}$ and there is a club $C$ of countable elementary submodels $\mathfrak{M} \prec\left(H_{\lambda}, \in\right)$ containing $P$ s.t.

$$
\forall p \in M \exists q \leq p(q \text { is }(M, P) \text {-generic }) .
$$

3. For all regular cardinals $\lambda$ such that $P \in H_{\lambda}$, there is a club $C$ of countable elementary submodels $\mathfrak{M} \prec\left(H_{\lambda}, \in\right)$ containing $P$ s.t.

$$
\forall p \in M \exists q \leq p(q \text { is }(M, P) \text {-generic })
$$

Definition 2.16 Let $P \in H_{\lambda}$ be a p.o., $\lambda>\omega$ and $\mathfrak{M} \prec\left(H_{\lambda}, \in\right)$ s.t. $P \in M$. $q \in P$ is $(M, P)$-semigeneric if for every name $\dot{\alpha} \in M$ s.t. $\Vdash " \dot{\alpha}$ is a countable ordinal", $q \Vdash \dot{\alpha} \in M$.

By fact 2.14, 2, the following is immediate:
Corollary $2.17 q(M, P)$-generic $\rightarrow q(M, P)$-semigeneric.
Definition 2.18 $P$ is semiproper if for all $\lambda>2^{|P|}$ such that $P \in H_{\lambda}$, there is a club $C$ of countable elementary submodels $\mathfrak{M} \prec\left(H_{\lambda}, \in\right)$ such that $P \in M$ and $\forall p \in M \exists q \leq p q(M, P)$-semigeneric.

By fact 2.15 and corollary 2.17 , the following is immediate:
Corollary 2.19 $P$ proper $\rightarrow P$ semiproper.
Fact 2.20 ([17], Theorem 34.4)
$P$ semiproper $\rightarrow P$ preserves stationary subsets of $\omega_{1}$.

### 2.1.2 Embeddings

Definition 2.21 Let $P$ and $Q$ be partial orders and $i: P \rightarrow Q$. is a complete embedding iff the following hold:

1. $\forall p, p^{\prime} \in P\left(p^{\prime} \leq p \rightarrow i\left(p^{\prime}\right) \leq i(p)\right)$
2. $\forall p_{1}, p_{2} \in P\left(p_{1} \perp p_{2} \leftrightarrow i\left(p_{1}\right) \perp i\left(p_{2}\right)\right)$
3. $\forall q \in Q \exists p \in P \forall p^{\prime} \in P\left(p^{\prime} \leq p \rightarrow i\left(p^{\prime}\right) \| q\right)$

In 3, we call $p$ a reduction of $q$ to $P$.
Fact 2.22 ([19], VII, Theorem 7.5) Suppose $i, P$ and $Q$ are in $\mathbf{V}, i: P \rightarrow Q$ and $i$ is a complete embedding. Let $H$ be $Q$-generic over $\mathbf{V}$. Then $i^{-1}(H)$ is $P$-generic over $\mathbf{V}$ and $\mathbf{V}\left[i^{-1}(H)\right] \subseteq \mathbf{V}[H]$.

Definition 2.23 Let $P$ and $Q$ be partial orders and $i: P \rightarrow Q$. $i$ is a dense embedding iff it is a complete embedding and $i^{\prime \prime}(P)$ is dense in $Q$.

Definition 2.24 If $P$ is a separative partial order, we let $\mathcal{B}(P)$ denote the unique (up to isomorphism) complete boolean algebra containing $P$ as a dense subset.

Definition 2.25 Let $P$ and $Q$ be partial orders. $P \sim Q$ iff $\mathcal{B}(P) \cong \mathcal{B}(Q)$.

### 2.1.3 Product Forcing

Definition 2.26 (Product of Forcings) Let $P$ and $Q$ be two notions of forcing. The product $P \times Q$ is the coordinatewise partially ordered set product of $P$ and $Q$ :

$$
\left(p_{1}, q_{1}\right) \leq\left(p_{2}, q_{2}\right) \text { iff } p_{1} \leq p_{2} \wedge q_{1} \leq q_{2}
$$

If $G$ is a generic filter on $P \times Q$, let

$$
G_{1}:=\{p \in P: \exists q(p, q) \in G\}, \quad G_{2}:=\{q \in Q: \exists p(p, q) \in G\}
$$

Fact 2.27 ([17], Lemma 15.9) Let $P$ and $Q$ be two notions of forcing in $\mathbf{V}$. Then the following are equivalent:

1. $G \subseteq P \times Q$ is generic over $\mathbf{V}$.
2. $G=G_{1} \times G_{2} \wedge G_{1} \subseteq P$ is generic over $\mathbf{V} \wedge G_{2} \subseteq Q$ is generic over $\mathbf{V}\left[G_{1}\right]$.

Moreover, $\mathbf{V}[G]=\mathbf{V}\left[G_{1}\right]\left[G_{2}\right]$ and if $G_{1} \subseteq P$ is generic over $\mathbf{V}$ and $G_{2} \subseteq Q$ is generic over $\mathbf{V}\left[G_{1}\right]$, then $G_{1} \subseteq P$ is generic over $\mathbf{V}\left[G_{2}\right]$ and $\mathbf{V}\left[G_{1}\right]\left[G_{2}\right]=$ $\mathbf{V}\left[G_{2}\right]\left[G_{1}\right]$.

### 2.1.4 Iterated Forcing

The basic observation is that a two-step iteration can be represented by a single forcing extension. Let $P$ be a notion of forcing in $\mathbf{V}$ and let $\dot{Q} \in \mathbf{V}^{P}$ be a $P$-name for a p.o. in the generic extension.

## Definition 2.28 (Composition of Forcings)

(i) $P * \dot{Q}:=\left\{(p, \dot{q}): p \in P \wedge \Vdash_{P} \dot{q} \in \dot{Q}\right\}$
(ii) $\left(p_{1}, \dot{q_{1}}\right) \leq\left(p_{2}, \dot{q_{2}}\right)$ iff $p_{1} \leq p_{2} \wedge p_{1} \Vdash \dot{q}_{1} \leq \dot{q}_{2}$

In (i), we identify $\left(p, \dot{q_{0}}\right)$ and $\left(p, \dot{q_{1}}\right)$ iff $p \Vdash \dot{q}_{0}=\dot{q}_{1}$.
Fact 2.29 ([17], Theorem 16.2)
(i) Let $G$ be a $\mathbf{V}$-generic filter on $P$, let $Q=\dot{Q}^{G}$ and let $H$ be a $\mathbf{V}[G]$ generic filter on $Q$. Then $G * H:=\left\{(p, \dot{q}) \in P * \dot{Q}: p \in G \wedge \dot{q}^{G} \in H\right\}$ is a $\mathbf{V}$-generic filter on $P * \dot{Q}$ and $\mathbf{V}[G * H]=\mathbf{V}[G][H]$.
(ii) Let $K$ be a $\mathbf{V}$-generic filter on $P * \dot{Q}$. Then $G:=\{p \in P: \exists \dot{q}(p, \dot{q}) \in$ $K\}$ and $H:=\left\{\dot{q}^{G}: \exists p(p, \dot{q}) \in K\right\}$ are respectively $a \mathbf{V}$-generic filter on $P$ and $a \mathbf{V}[G]$ - generic filter on $Q:=\dot{Q}^{G}$ and $K=G * H$.

Definition 2.30 (Finite Support Iteration) Let $\alpha \geq 1$. A forcing notion $P_{\alpha}$ is an iteration of length $\alpha$ with finite support if it is a set of $\alpha$ sequences with the following properties:

1. If $\alpha=1$, then for some forcing notion $Q_{0}$,

- $P_{1}$ is the set of all 1-sequences $\langle p(0)\rangle$ where $p(0) \in Q_{0}$,
- $\langle p(0)\rangle \leq_{1}\langle q(0)\rangle \leftrightarrow p(0) \leq_{Q_{0}} q(0)$.

2. If $\alpha=\beta+1$, then $P_{\beta}=P_{\alpha} \upharpoonright \beta=\left\{p \upharpoonright \beta: p \in P_{\alpha}\right\}$ is an iteration of length $\beta$ and there is some forcing notion $\dot{Q}_{\beta} \in \mathbf{V}^{P_{\beta}}$ s.t.

- $p \in P_{\alpha} \leftrightarrow p \upharpoonright \beta \in P_{\beta} \wedge \Vdash_{\beta} p(\beta) \in \dot{Q_{\beta}}$,
- $p \leq_{\alpha} q \leftrightarrow p \upharpoonright \beta \leq_{\beta} q \upharpoonright \beta \wedge p \upharpoonright \beta \vdash_{\beta} p(\beta) \leq_{\dot{Q}_{\beta}} q(\beta)$.

3. If $\alpha$ is a limit ordinal, then for every $\beta<\alpha, P_{\beta}=P_{\alpha} \upharpoonright \beta$ is an iteration of length $\beta$ and

- $p \in P_{\alpha} \leftrightarrow \forall \beta<\alpha p \upharpoonright \beta \in P_{\beta}$ and for all but finitely many $\beta<\alpha$, $\vdash^{-} p(\beta)=\mathbf{1}$,
- $p \leq_{\alpha} q \leftrightarrow \forall \beta<\alpha p \upharpoonright \beta \leq_{\beta} q \upharpoonright \beta$.
where $\leq_{\alpha}$ abbreviates $\leq_{P_{\alpha}}$ and $\Vdash_{\alpha}$ abbreviates $\Vdash_{P_{\alpha}}$.

The finite set $\left\{\beta<\alpha: \neg \Vdash_{\beta} p(\beta)=\mathbf{1}\right\}$ is the so-called support of $p \in$ $P \alpha$. An iteration with finite support of length $\alpha$ is uniquely determined by the sequence $\left\langle\dot{Q}_{\beta}: \beta<\alpha\right\rangle$. Thus we call $P_{\alpha}$ the finite support iteration of $\left\langle\dot{Q}_{\beta}: \beta<\alpha\right\rangle$. For each $\beta<\alpha, P_{\beta+1}$ is isomorphic to $P_{\beta} * \dot{Q}_{\beta}$. When $\alpha$ is a limit ordinal, $\left(P_{\alpha}, \leq_{\alpha}\right)$ is the so-called direct limit of the $P_{\beta}, \beta<\alpha$.

Definition 2.31 (General Iteration) Let $\alpha \geq 1$. A forcing notion $P_{\alpha}$ is an iteration of length $\alpha$ if it is a set of $\alpha$-sequences with the following properties:

1. If $\alpha=1$, then for some forcing notion $Q_{0}$,

- $P_{1}$ is the set of all 1 -sequences $\langle p(0)\rangle$ where $p(0) \in Q_{0}$,
- $\langle p(0)\rangle \leq_{1}\langle q(0)\rangle \leftrightarrow p(0) \leq_{Q_{0}} q(0)$.

2. If $\alpha=\beta+1$, then $P_{\beta}=P_{\alpha} \upharpoonright \beta=\left\{p \upharpoonright \beta: p \in P_{\alpha}\right\}$ is an iteration of length $\beta$ and there is some forcing notion $\dot{Q}_{\beta} \in \mathbf{V}^{P_{\beta}}$ s.t.

- $p \in P_{\alpha} \leftrightarrow p \upharpoonright \beta \in P_{\beta} \wedge \Vdash_{\beta} p(\beta) \in \dot{Q_{\beta}}$,
- $p \leq_{\alpha} q \leftrightarrow p \upharpoonright \beta \leq_{\beta} q \upharpoonright \beta \wedge p \upharpoonright \beta \Vdash_{\beta} p(\beta) \leq_{Q_{\beta}} q(\beta)$.

3. If $\alpha$ is a limit ordinal, then for every $\beta<\alpha, P_{\beta}=P_{\alpha} \upharpoonright \beta$ is an iteration of length $\beta$ and

- the $\alpha$-sequence $\langle\mathbf{1}, \ldots, \mathbf{1}, \ldots\rangle$ is in $P_{\alpha}$,
- If $p \in P_{\alpha}, \beta<\alpha$ and $q \in P_{\beta}$ s.t. $q \leq_{\beta} p \upharpoonright \beta$, then $r \in P_{\alpha}$ where $\forall \xi<\beta r(\xi)=q(\xi)$ and $\beta \leq \xi<\alpha \rightarrow r(\xi)=p(\xi)$,
- $p \leq_{\alpha} q \leftrightarrow \forall \beta<\alpha p \upharpoonright \beta \leq_{\beta} q \upharpoonright \beta$.

Clearly, an iteration with finite support is an iteration.
Property 3 guarantees that if $\left.P_{\beta}=P_{\alpha}\right\rceil \beta$, then $\mathbf{V}^{P_{\beta}} \subseteq \mathbf{V}^{P_{\alpha}}$, i.e. whenever $G$ is a generic filter on $P_{\alpha}$, then there is some $H \in \mathbf{V}[G]$ that is a generic filter on $P_{\beta}$ :

Lemma 2.32 Let $h: Q \rightarrow P$ be such that

- $q_{1} \leq q_{2} \rightarrow h\left(q_{1}\right) \leq h\left(q_{2}\right)$,
- $\forall q \in Q \forall p \leq h(q) \exists q^{\prime} \leq q h\left(q^{\prime}\right) \leq p$.

Then $\mathbf{V}^{P} \subseteq \mathbf{V}^{Q}$.
Proof: We show that if $D \subseteq P$ is open dense, then $h^{-1}(D)$ is dense in $Q$, hence if $G$ is generic on $Q$, then $\{p \in P: \exists q \in Q p \geq h(q)\}$ is generic on $P$ :

Let $q \in Q$ and choose $p \leq h(q), p \in D$. By our assumption, there is $q^{\prime} \in h^{-1}\left(P_{p}\right) \subseteq h^{-1}(D)$ s.t. $q^{\prime} \leq q$, hence $h^{-1}(D)$ is dense in $Q$.

Lemma 2.33 If $P_{\alpha}$ is an iteration and $P_{\beta}=P_{\alpha} \upharpoonright \beta$, then $\mathbf{V}^{P_{\beta}} \subseteq \mathbf{V}^{P_{\alpha}}$.
Proof: Let $h: \mathbf{V}^{P_{\beta}} \rightarrow \mathbf{V}^{P_{\alpha}}, p \mapsto p \upharpoonright \beta$. We will show that $h$ satisfies the conditions from lemma 2.32, hence $\mathbf{V}^{P_{\beta}} \subseteq \mathbf{V}^{P_{\alpha}}$ :

- $q_{1} \leq q_{2} \rightarrow h\left(q_{1}\right) \leq h\left(q_{2}\right)$ obviously holds.
- $\forall q \in Q \forall p \leq h(q) \exists q^{\prime} \in Q q^{\prime} \| q \wedge h\left(q^{\prime}\right) \leq p:$

Let $q \in P_{\alpha}$ and $p \leq q \upharpoonright \beta$. By property 3 of definition 2.31, let $r \in P_{\alpha}$ s.t. $(\xi<\beta \rightarrow r(\xi)=p(\xi)) \wedge(\beta \leq \xi<\alpha \rightarrow r(\xi)=q(\xi))$. Then $r \| q$, because $r \leq q$ and $h(r)=p$. So, let $q^{\prime}=r$ in the above condition.

A general iteration depends not only on the $\dot{Q}_{\beta}$, but also on the limit stages of the iteration. Let $P_{\alpha}$ be an iteration of length $\alpha$ where $\alpha$ is a limit ordinal.

- $P_{\alpha}$ is a direct limit if for every $\alpha$-sequence $p$,

$$
p \in P_{\alpha} \leftrightarrow \exists \beta<\alpha p \upharpoonright \beta \in P_{\beta} \wedge \forall \xi \geq \beta \Vdash_{\xi} p(\xi)=\mathbf{1} .
$$

- $P_{\alpha}$ is an inverse limit if for every $\alpha$-sequence $p$,

$$
p \in P_{\alpha} \leftrightarrow \forall \beta<\alpha p \upharpoonright \beta \in P_{\beta} .
$$

Finite support iterations are exactly those that use only direct limits. In general, let $s(p)$, the support of $p \in P_{\alpha}$, be $\left\{\beta<\alpha: \neg \Vdash_{\beta} p(\beta)=\mathbf{1}\right\}$. If $I$ is an ideal on $\alpha$ containing all finite sets, then an iteration with $I$-support is an iteration that satisfies for every limit ordinal $\gamma \leq \alpha$ :

$$
p \in P_{\gamma} \leftrightarrow \forall \beta<\gamma p \upharpoonright \beta \in P_{\beta} \wedge s(p) \in I .
$$

Definition 2.34 [Countable Support Iteration] A countable support iteration is an iteration with I-support s.t. I is the ideal of at most countable sets. If $P_{\alpha}$ is a countable support iteration of length $\alpha$, then for every limit ordinal $\gamma \leq \alpha$, if $\operatorname{cf} \gamma=\omega$, then $P_{\gamma}$ is an inverse limit, if $\operatorname{cf} \gamma>\omega$, then $P_{\gamma}$ is a direct limit.

Fact 2.35 (Shelah) ([17], Theorem 31.15) If $P_{\alpha}$ is a countable support iteration of $\left\{\dot{Q}_{\beta}: \beta<\alpha\right\}$ such that every $\dot{Q}_{\beta}$ is a proper forcing notion in $\mathbf{V}^{P_{\alpha} \backslash \beta}$, then $P_{\alpha}$ is proper.

Fact 2.36 ([17], Theorem 16.30) Let $\kappa$ be a regular uncountable cardinal and let $\alpha$ be a limit ordinal. Let $P_{\alpha}$ be an iteration of length $\alpha$ s.t. $\forall \beta<$ $\alpha P_{\beta}=P_{\alpha} \upharpoonright \beta$ is $\kappa$-cc. If $P_{\alpha}$ is a direct limit and either $\operatorname{cf}(\alpha) \neq \kappa$ or $\operatorname{cf}(\alpha)=\kappa$ and for a stationary set of $\beta<\alpha, P_{\beta}$ is a direct limit, then $P_{\alpha}$ is $\kappa$-cc.

Definition 2.37 [Revised Countable Support Iteration] Let $\alpha \geq 1$. A forcing notion $P_{\alpha}$ is a RCS (revised countable support) iteration of $\left\{\dot{Q}_{\beta}: \beta<\alpha\right\}$, if for every limit ordinal $\eta \leq \alpha, P_{\eta}$ is an iteration consisting of all $\eta$ sequences $p$ that satisfy:

$$
\left(\forall q \leq p \exists \gamma<\eta \exists r \leq{ }_{\gamma} q \upharpoonright \gamma\right)\left(r \Vdash_{\gamma} \operatorname{cf} \eta=\omega \vee \forall \beta \geq \gamma p \upharpoonright[\gamma, \beta) \Vdash_{P_{\gamma, \beta}} p(\beta)=\mathbf{1}\right)
$$

where $q$ ranges over elements of the inverse limit of the $\dot{Q}_{\beta}$ and $P_{\gamma, \beta}$ is the restriction of the inverse limit to the interval $[\gamma, \beta)$.

Fact 2.38 (Shelah) ([17], Theorem 37.4) If $P_{\alpha}$ is an RCS iteration of $\left\{\dot{Q_{\alpha}}: \beta<\alpha\right\}$ s.t. every $\dot{Q_{\alpha}}$ is a semiproper forcing notion in $\mathbf{V}^{P_{\alpha} \upharpoonright \beta}$, then $P_{\alpha}$ is semiproper.

Definition 2.39 (Quotient Forcing) Let $\left\langle P_{\gamma}, \dot{Q_{\gamma}}: \gamma<\alpha\right\rangle$ be an iteration and assume $G_{\beta}$ is generic for $P_{\beta}$ over $\mathbf{V}$. We define $P_{\alpha} / G_{\beta}$ to be a $P_{\beta}$-name for a forcing notion s.t.

$$
\Vdash_{\beta} P_{\alpha} / G_{\beta}=\left\{p \in \check{P}_{\alpha}: p \upharpoonright \beta \in \check{G}_{\beta}\right\}
$$

For $p, q \in P_{\alpha} / G_{\beta}$, we let $p \leq_{P_{\alpha} / G_{\beta}} q$ iff $p \leq_{\alpha} q$.
Fact 2.40 ([13], Fact 4.5) The map $i: P_{\alpha} \rightarrow P_{\beta} *\left(P_{\alpha} / G_{\beta}\right)$ defined by $i(p)=$ $\langle p \upharpoonright \beta, \check{p}\rangle$ is a dense embedding. Hence, forcing with $P_{\alpha}$ amounts to the same as first forcing with $P_{\beta}$ and then with the quotient forcing $P_{\alpha} / G_{\beta}$.

Fact 2.41 ([13], Theorem 4.6) Let $\left\langle P_{\gamma}, \dot{Q_{\gamma}}: \gamma<\alpha\right\rangle$ be a countable support iteration of proper forcings and let $\gamma+\beta=\alpha$. Then there exists a $P_{\gamma}$-name $\left\langle\bar{P}_{\chi}, \bar{Q}_{\chi}: \chi<\beta\right\rangle$ of a countable support iteration of proper forcings of length $\beta$ such that $\Vdash_{\gamma} \forall \chi<\beta \bar{P}_{\chi} \sim P_{\chi} / G_{\gamma}$. In particular, $\Vdash_{\gamma} P_{\alpha} / G_{\gamma}$ is proper.

### 2.1.5 Forcing and sets of hereditarily bounded size

Lemma 2.42 If $P \in H_{\chi}$, $\chi$ regular and $\Vdash_{P} \dot{x} \in H_{\chi}$, then there exists $\dot{\tilde{x}} \in H_{\chi}$ such that $\Vdash_{P} \dot{x}=\dot{\tilde{x}}$.

Proof: By induction on $\operatorname{rank}(\dot{x})$ : The above is true for $\operatorname{rank}(\dot{x})=0$, so assume $\operatorname{rank}(\dot{x})=r$ and the claim holds for all $\dot{y}$ with $\operatorname{rank}(\dot{y})<r$. Assume $\vdash_{P} \dot{x} \in H_{\chi}, \dot{x}=\left\{\left(\dot{y}_{i}, p_{i}\right): i \in I\right\} . \forall i \in I \operatorname{rank}\left(\dot{y}_{i}\right)<r$, hence we can replace $\dot{y}_{i}$ by $\dot{z}_{i} \in H_{\chi}$ in $\dot{x}$ s.t. $\Vdash_{P} \dot{y}_{i}=\dot{z}_{i}$ to obtain $\dot{\bar{x}}$ s.t. $\Vdash_{P} \dot{\bar{x}}=\dot{x}$.

For each $p \in P$, let $A_{p}:=\{\dot{y}:(\dot{y}, p) \in \dot{\bar{x}}\}$. Fix $p \in P$. For each $\dot{y} \in A_{p}, p \Vdash \dot{y} \in \dot{\bar{x}}$. As $\Vdash_{P} \dot{\bar{x}} \in H_{\chi}$, there exists $B_{p} \subseteq A_{p}$ of size $<\chi$ s.t. $\forall \dot{y} \in A_{p} \exists \dot{z} \in B_{p} p \Vdash \dot{y}=\dot{z}$. So, for each $p \in P$, we can replace $A_{p} \times\{p\}$ by $B_{p} \times\{p\}$ in $\dot{\bar{x}}$ (as subsets) to obtain a name $\dot{\tilde{x}}$ s.t. $\Vdash_{P} \dot{\tilde{x}}=\dot{\bar{x}}$ and $|\dot{\tilde{x}}|<\chi$ (here we use $|P|<\chi$ and $\chi$ regular). Hence $\dot{\tilde{x}} \in H_{\chi}$ and $\Vdash_{P} \dot{\tilde{x}}=x$.

Lemma 2.43 If $P, \dot{x} \in H_{\chi}$, then $\Vdash_{P} \dot{x} \in H_{\chi}$.
Proof: By induction on $\operatorname{rank}(\dot{x})$ : The above is true for $\operatorname{rank}(\dot{x})=0$, so assume $\operatorname{rank}(\dot{x})=r$ and the claim holds for all $\dot{y}$ s.t. $\operatorname{rank}(\dot{y})<r$. Assume $\dot{x} \in H_{\chi}, \dot{x}=\left\{\left(\dot{y}_{i}, p_{i}\right): i \in I\right\} . \forall i \in I \dot{y}_{i} \in H_{\chi}$ follows, hence, by induction hypothesis, $\Vdash_{P} \dot{y}_{i} \in H_{\chi}$. Because $\dot{x} \in H_{\chi},|\dot{x}|<\chi$. As $|P|<\chi, \chi$ remains a cardinal in any $P$-generic extension, hence $\Vdash_{P}|\dot{x}|<\chi$, yielding $\vdash_{P} \dot{x} \in H_{\chi}$.

Corollary 2.44 If $P, \dot{y} \in H_{\chi}$, $\chi$ regular and $\varphi(\cdot)$ is a formula, then:

$$
\Vdash_{P} H_{\chi} \models \varphi(\dot{y}) \leftrightarrow H_{\chi} \models \Vdash_{P} \varphi(\dot{y})
$$

Proof: By induction on formula complexity. It is obvious for quantifier-free formulas and if the claim holds for $\psi$, it is obvious (by the definition of $\Vdash$ ) that it also holds for $\varphi \equiv \neg \psi$. So assume $\varphi \equiv \exists x \psi$ and that the claim holds for $\psi$ :

$$
\begin{gathered}
\Vdash_{P} H_{\chi} \models \exists x \psi(x) \stackrel{\text { using lemma } 2.42 \text { and lemma } 2.43}{\longleftrightarrow} \\
\exists \dot{x} \in H_{\chi} \Vdash_{P} H_{\chi} \models \psi(\dot{x}) \stackrel{\text { using our induction hypothesis }}{\longleftrightarrow} \\
\exists \dot{x} \in H_{\chi} H_{\chi} \models \Vdash_{P} \psi(\dot{x}) \longleftrightarrow H_{\chi} \models \exists \dot{x} \Vdash_{P} \psi(\dot{x}) \stackrel{\text { by definition of } \Vdash}{\longleftrightarrow} \\
H_{\chi} \models \Vdash_{P} \exists x \psi(x) \square
\end{gathered}
$$

### 2.1.6 Class Forcing

We now give a very brief review of class forcing. For a more detailed introduction on Class Forcing, see [10], ch. 2.2:

Let $A \subseteq M$. We say that $(M, A)$ is a model of $Z F C\left(Z F C^{*} \subseteq Z F C\right)$ if $M$ is a model of $Z F C\left(Z F C^{*}\right)$ and the scheme of replacement holds in $M$ for formulas which mention $A$ as a predicate. We also require that $(M, A)$ satisfies $\mathbf{V}=\mathbf{L}[A]$, i.e. $\mathbf{V}=\bigcup_{\alpha \in \mathbf{O r d}} \mathbf{L}\left[A \cap V_{\alpha}\right]$, we call such $M$ a ground model. If $(M, A)$ is a model of $Z F C$ it is easy to find $A^{*} \subseteq M$ such that $\left(M, A^{*}\right)$ is a ground model with the same definable predicates (for details, see [10]).

A partial ordering $P$ is a class forcing for $M$ if for some ground model $(M, A), P$ is definable (with parameters) over $(M, A) . G \subseteq P$ is $P$-generic over $(M, A)$ iff:

- $p, q \in G \rightarrow p \| q$.
- $p \geq q \in G \rightarrow p \in G$.
- $D \subseteq P$ dense and $(M, A)$-definable (with parameters) $\rightarrow G \cap D \neq \emptyset$.

A $P$-generic extension over $(M, A)$ is a model of the form $(M[G], A, G)$ with a $P$-generic $G$ as unary predicate.

Lemma 2.45 ([10], Lemma 2.16)

- $M \subseteq M[G], M[G]$ is transitive, $\operatorname{Ord}(M[G])=\mathbf{O r d}(M)$.
- $\alpha \in \operatorname{Ord}(M) \rightarrow G \cap V_{\alpha}{ }^{M[G]} \in M[G]$.
- If $M \subseteq N,(N, G)$ amenable and $N \models Z F$, then $M[G] \subseteq N$ and $M$ is definable over $(N, A)$, in fact $M=\mathbf{L}[A]^{N}$.


### 2.2 Some notions of forcing

In this section, we will define and analyze the forcing notions that we will use throughout this paper.

### 2.2.1 The Lévy Collapse

Definition 2.46 For $S \subseteq$ Ord, $\gamma$ regular, define $\operatorname{coll}(\gamma, S):=\{p: p$ is a function $\wedge \operatorname{dom}(p) \subseteq S \times \gamma \wedge|p|<\gamma \wedge \forall(\xi, \zeta) \in \operatorname{dom}(p) p(\xi, \zeta) \in \xi\}$ ordered by $p \leq q$ iff $p \supseteq q$.

Let $G$ be a generic set of conditions for $\operatorname{coll}(\gamma, S)$. For each $\xi \in S$ and each $\chi \in \xi$, the set of conditions $q$ where for some $\zeta \in \gamma, p(\xi, \zeta)=\chi$ is dense. Thus $F:=\bigcup G$ is a function such that for each $\xi \in S$, the function $\zeta \mapsto F(\xi, \zeta)$ is a surjection from $\gamma$ onto $\xi$. Hence $\Vdash^{\operatorname{coll}(\gamma, S)} \boldsymbol{\forall \xi \in \check { S } | \xi | \leq \check { \gamma } \text { . }}$
$\operatorname{coll}(\gamma,\{\kappa\})$ is called the Lévy Collapse of $\kappa$ onto $\gamma$, it collapses $\kappa$ to $\gamma$. $\operatorname{coll}(\gamma, \kappa)$ is called the gentle Lévy Collapse of $\kappa$ onto $\gamma^{+}$, it collapses every $\lambda<\kappa$ to $\gamma$. If $\kappa \geq \gamma^{+}$, then in the forcing extension $\kappa=\gamma^{+}$holds.

## Lemma 2.47

1. $\operatorname{coll}(\gamma, S)$ is $\gamma$-closed
2. $\kappa^{<\gamma}=\kappa \rightarrow \operatorname{coll}(\gamma,\{\kappa\})$ is $\kappa^{+}-c c$
3. $\kappa$ regular, $\kappa>\gamma \wedge \forall \xi<\kappa \quad \xi^{<\gamma}<\kappa \rightarrow \operatorname{coll}(\gamma, \kappa)$ is $\kappa-c c$
4. $\operatorname{coll}(\gamma, S)$ is weakly homogeneous.

Proof:

1. Because $\gamma$ is regular, the union of less than $\gamma$-many functions of size $<\gamma$ has size $<\gamma$.
2. $\operatorname{coll}(\gamma,\{\kappa\}) \subseteq[\{\kappa\} \times \gamma \times \kappa]^{<\gamma} \cong \kappa^{<\gamma}=\kappa$, so obviously every antichain in $\operatorname{coll}(\gamma,\{\kappa\})$ has to have size $<\kappa^{+}$.
3. Assume to the contrary that $A$ is an antichain of $\operatorname{coll}(\gamma, \kappa)$ of size $\kappa$. Applying the $\Delta$-System Lemma (1.21) to $\{\operatorname{dom}(p): p \in A\}$, which has cardinality $\kappa$ (counting multiple occurences), we can find a root $r \in[\kappa \times \gamma]^{<\gamma}$ and $A^{\prime} \subseteq A$ of size $\kappa$ such that for any two distinct $p, q \in A^{\prime}, \operatorname{dom}(p) \cap \operatorname{dom}(q)=r$. As $\gamma<\kappa$ and $\kappa$ is regular, for $p \in A^{\prime}$, there are, for some $\xi<\kappa$, at most $\xi^{<\gamma}<\kappa$ possibilities for $p \upharpoonright r$. Thus there is $B \subseteq A^{\prime}$ such that $|B|=\kappa$ and $\forall p, q \in B p \upharpoonright r=q \upharpoonright r$, but as the domains of elements of $B$ are disjoint outside of $r$, all elements of $B$ are compatible, contradicting the assumption of $B$ being an antichain.
4. Given $p, q \in \operatorname{coll}(\gamma, S)$, we can find a bijection $f: \gamma \rightarrow \gamma$ such that $\forall\langle\alpha, \xi\rangle \in \operatorname{dom} p \forall\langle\beta, \zeta\rangle \in \operatorname{dom} q f(\xi) \neq \zeta$, because both $\operatorname{dom} p$ and $\operatorname{dom} q$ have cardinality $<\gamma$. This induces an automorphism $e$ of $\operatorname{coll}(\gamma, S)$, $r \mapsto e(r):(\langle\alpha, f(\xi)\rangle \in \operatorname{dom} e(r)$ iff $\langle\alpha, \xi\rangle \in \operatorname{dom} r) \wedge e(r)(\alpha, f(\xi)):=$ $r(\alpha, \xi)$. Since $\operatorname{dom} e(p) \cap \operatorname{dom} q=\emptyset, e(p) \| q$.

Lemma 2.48 Let $S=R \cup T$ and $R \cap T=\emptyset$. Then $i:\langle p, q\rangle \mapsto p \cup q$ is an isomorphism from $\operatorname{coll}(\lambda, R) \times \operatorname{coll}(\lambda, T)$ onto $\operatorname{Coll}(\lambda, S)$. Furthermore $\operatorname{coll}(\lambda, T)=\operatorname{coll}(\lambda, T)^{\mathbf{V}[G]}$ for any $G$ that is $\operatorname{coll}(\lambda, R)$-generic. Moreover, the second claim also holds if $R \cap T \neq \emptyset$.

Proof: The isomorphism property is obvious. The second fact holds since by $\lambda$-closedness of $\operatorname{coll}(\lambda, R)$, no new subsets of $T \times \lambda \times \bigcup T$ of size less than $\lambda$ are added (see fact 2.4).

### 2.2.2 Adding a closed unbounded set

Theorem 2.49 [15] Let $A \subseteq \omega_{1}$ be stationary. Then there exists an $\omega_{1-}$ preserving generic extension $\mathbf{V}[G]$ such that $\mathbf{V}[G] \vDash \exists C \subseteq A C$ is closed and unbounded in $\omega_{1}$.

Proof: Let $P:=\{p \subseteq A: p$ closed $\}$, ordered by $p \leq q$ iff $p$ is an end-extension of $q$. Let $G$ be a generic filter on $P$. Then $\bigcup G \subseteq A$ is club in $\omega_{1}$ : For any $\alpha<\omega_{1}, D_{\alpha}:=\{p \in P: \sup (p)>\alpha\}$ is dense in $P$, because for any $q \in P$, because $A$ is stationary and thus unbounded in $\omega_{1}$, there is $a \in A$ such that $a>\sup (q) \wedge a>\alpha$, so $q \cup\{a\}$ end-extends $q$ and $q \cup\{a\} \in D_{\alpha}$. So for any $\alpha<\omega_{1}$, there is $p \in G$ such that $\exists \beta \in p \beta>\alpha$ and hence $\exists \beta \in \bigcup G \beta>\alpha$, yielding $\bigcup G$ to be unbounded in $\omega_{1}$. To see that $\bigcup G$ is closed, suppose $\alpha$ is a limit point of $\bigcup G$. Then $\exists q \in G \sup (q)>\alpha$. Because the elements of $G$ are end-extending each other, $\alpha \cap \bigcup G \subseteq q$. Because $q$ is closed, $\alpha \in q$, hence $\alpha \in \bigcup G$.

It remains to show that $\omega_{1}^{\mathbf{V}[G]}=\omega_{1}^{\mathbf{V}}$, we will show that for any $f: \omega \rightarrow \mathbf{V}$ with $f \in \mathbf{V}[G]$, actually $f \in \mathbf{V}$. Thus $\omega_{1}$ is not collapsed by forcing with $P$.

Claim: For any $f: \omega \rightarrow \mathbf{V}, f \in \mathbf{V}[G]$ and $p \in P$ s.t. $p \Vdash " \dot{f}: \omega \rightarrow y "$ and $y \subset \mathbf{V}$, the following holds: $\exists q \leq p \forall n \in \omega \exists x q \Vdash \dot{f}(n)=x$.

Corollary: For any $f$ and $p$ as in the above claim, let $q \in P$ be such that the above condition holds. Let $g \in \mathbf{V}$ be defined as $g(n)=x \leftrightarrow q \Vdash \dot{f}(n)=x$. Then $\forall n \in \omega g(n)=f(n)$, hence $g=f$ and $f \in \mathbf{V}$, proving our theorem above.

Proof of Claim: Note that $P$ is not $\sigma$-closed (which would suffice).
Let $p \in P$ s.t. $p \Vdash " \dot{f}: \omega \rightarrow y "$. For each $\alpha<\omega_{1}$, we define a set $A_{\alpha}$ and an ordinal $h_{\alpha}$ as follows:

- $A_{0}:=\{p\}$
- $h_{\alpha}:=\sup \left(\left\{\sup (p): p \in A_{\alpha}\right\}\right)$
- $A_{\lambda}=\bigcup_{\beta<\lambda} A_{\beta}$ for limit ordinals $\lambda$
- $A_{\alpha+1}:=$ a minimal $A_{\alpha}^{\prime} \supseteq A_{\alpha}$ s.t. for any $q \in A_{\alpha}^{\prime}$ and $n<\omega$ there is $q^{*} \leq q$ in $A_{\alpha}^{\prime}$ s.t. $q^{*} \Vdash f(n)=x$ for some $x \in \mathbf{V}$ and $\sup \left(q^{*}\right)>h_{\alpha}$.

By induction, each $A_{\alpha}$ is countable. Let $B:=\left\{h_{\alpha}: \alpha<\omega_{1}\right\}$. Then $B \in \mathbf{V}$ and $B$ is unbounded in $\omega_{1}$. Let $B^{\prime}$ be the set of limit points of $B-\operatorname{not}$ necessarily contained in $B$. Then $B^{\prime} \in \mathbf{V}$ is a club in $\omega_{1}$, hence, because $A$ is stationary, we can choose $\eta \in B^{\prime} \cap A$ and $\left\langle\alpha_{i}\right\rangle_{i<\omega}$ s.t. $\bigcup\left\{h_{\alpha_{i}}: i<\omega\right\}=\eta$, $\alpha_{0}>0$ and the $\alpha_{i}$ are strictly increasing.

Now we build an $\omega$-chain of conditions $\left\langle q_{i}\right\rangle_{i<\omega}$ as follows: Let $q_{0} \in A_{\alpha_{0}}$ decide a value for $f(0)$. Let $q_{1} \leq q_{0} \in A_{\alpha_{1}}$ with $\sup \left(q_{1}\right)>h_{\alpha_{0}}$ decide a value for $f(1)$ and so forth. Then $q:=\bigcup_{i<\omega} q_{i} \cup\{\eta\}$ is closed and hence a condition in $P$ which extends $p$ and decides a value for $f(n)$ for each $n \in \omega$.

### 2.2.3 Cohen Reals

The following notion of forcing adjoins $\kappa$ real numbers, called Cohen Reals:

Let $P$ be the set of all functions $p$ such that:

- $\operatorname{dom}(p)$ is a finite subset of $\kappa \times \omega$,
- $\operatorname{range}(p) \subseteq\{0,1\}$
and let $p \leq q \leftrightarrow p \supseteq q$.

Let $G$ be a generic set of conditions and let $f:=\bigcup G$. By a genericity argument, $f$ is a function from $\kappa \times \omega$ into $\{0,1\}$. For each $\alpha<\kappa$, let $f_{\alpha}$ be the function on $\omega$ defined by $f_{\alpha}(n):=f(\alpha, n)$ and let $a_{\alpha}:=\left\{n \in \omega: f_{\alpha}(n)=1\right\}$. It follows that each $a_{\alpha}$ is a real, $a_{\alpha} \notin \mathbf{V}$ and $\alpha \neq \beta \rightarrow a_{\alpha} \neq a_{\beta}$.

Lemma 2.50 The forcing $P$ for adjoining $\kappa$ Cohen Reals has property (K).

Proof: Let $p_{\alpha}, \alpha<\omega_{1}$ be conditions in $P$, let $a_{\alpha}:=\operatorname{dom}\left(p_{\alpha}\right)$ for each $\alpha<\omega_{1}$. By the $\Delta$-system lemma (1.21), there exists $x \subseteq \omega_{1},|x|=\aleph_{1}$, s.t. $A:=\left\{a_{\alpha}: \alpha \in x\right\}$ is a $\Delta$-system, i.e. there is $r$ s.t. for any $a_{\alpha}, a_{\beta} \in A$, $a_{\alpha} \cap a_{\beta}=r$. As there are only finitely many possibilities for $p \upharpoonright r$ for $p \in P$, let $y \subseteq x,|y|=\aleph_{1}$ and choose $u$ s.t. $\forall \alpha \in y p_{\alpha} \upharpoonright r=u$. Then $\left\{p_{\alpha}: \alpha \in y\right\}$ is an uncountable subset of $\left\{p_{\alpha}: \alpha<\omega_{1}\right\}$ of pairwise compatible elements.

### 2.2.4 Sealing the $\omega_{1}$-branches of a tree

## Definition 2.51

- Let $T$ be a tree of height $\omega_{1}$. We say that $B \subseteq T$ is an $\omega_{1}$-branch iff $B$ is a maximal linearly ordered subset of $T$ and has order-type $\omega_{1}$.
- A nonempty tree $T$ is perfect if for any $t \in T$, there exist $s_{1}$, s2 s.t. $t \leq$ $s_{1}, s_{2}$ and neither $s_{1} \leq s_{2}$, nor $s_{2} \leq s_{1}$ ( $s_{1}$ and $s_{2}$ are incomparable).

Lemma 2.52 Suppose $T$ is a tree of height $\omega_{1}$ and $P$ has property (K). Then forcing with $P$ adds no new $\omega_{1}$-branches through $T$.

Proof: Assume for a contradiction that $p \Vdash \dot{B}$ is a new $\omega_{1}$-branch. Let $S:=$ $\{s \in T: \exists q \leq p q \Vdash s \in \dot{B}\}$. Obviously, for each $\alpha<\omega_{1}$, there is $s_{\alpha} \in S$ and $p_{\alpha} \leq p$ s.t. $p_{\alpha} \Vdash s_{\alpha} \in \dot{B}$ and $s_{\alpha}$ has level $\alpha$ in $T$. Since $P$ has property (K), there is $A \subseteq \omega_{1},|A|=\aleph_{1}$, s.t. $\left\{p_{\alpha}: \alpha \in A\right\}$ is pairwise compatible. But then $\left\{s_{\alpha}: \alpha \in A\right\}$ is linearly ordered, so there is an uncountable branch $B$ through $S$. Since $p \Vdash \dot{B} \notin \mathbf{V}$, it follows that $\forall s \in S \exists t, u \in S s \leq t, u \wedge t \perp u$, where $\perp$ denotes incompatibility with respect to the tree ordering $\geq$. Hence $S^{\prime}:=$ $\{s \in S: s$ is $\leq$-minimal s.t. $s \notin B\}$ is an uncountable, pairwise incomparable (and hence incompatible) set. But if $s \perp t, q_{1} \Vdash s \in \dot{B}, q_{2} \Vdash t \in \dot{B}$, then $q_{1} \perp q_{2}$, which gives us an uncountable, pairwise incompatible subset of $P$ contradicting that $P$ has property (K).

Lemma 2.53 Suppose $T$ is a tree of height $\omega_{1}$ and $2^{\aleph_{0}}>|T|$. If $P$ is $\sigma$-closed, then forcing with $P$ adds no new $\omega_{1}$-branches through $T$.

Proof: Assume for a contradiction that $p \Vdash \dot{B}$ is a new $\omega_{1}$-branch. For each $\sigma \in 2^{<\omega}$ we will find $p_{\sigma} \leq p$ and $s_{\sigma} \in T$ s.t. $p_{\sigma} \Vdash s_{\sigma} \in \dot{B}$. Also, if $\sigma \subset \tau$, then $s_{\sigma}<s_{\tau}$ and if $\sigma$ and $\tau$ are incomparable (with respect to $\subseteq$ ), then $s_{\sigma}$ and $s_{\tau}$ are incomparable. We proceed by induction on $|\sigma|$ : If $\sigma=\emptyset$, let $p_{0} \leq p$ and $s_{0}$ be arbitrary s.t. $p_{0} \Vdash s_{0} \in \dot{B}$. Given $p_{\sigma}$ and $s_{\sigma}$, let $\tau_{1}:=\sigma \frown\langle 0\rangle$ and $\tau_{2}:=\sigma^{\frown}\langle 1\rangle$. Since $p_{\sigma} \Vdash \dot{B}$ is a new $\omega_{1}$-branch, there exist incomparable $s_{\tau_{1}}, s_{\tau_{2}}>s_{\sigma}$ and there exist $p_{\tau_{1}}, p_{\tau_{2}} \leq p_{\sigma}$ s.t. $p_{\tau_{i}} \Vdash s_{\tau_{i}} \in \dot{B}$, $i=1,2$.

For $f \in 2^{\omega}$, choose $p_{f}$ s.t. $\forall n \in \omega p_{f} \leq p_{f \upharpoonright n}$, which is possible since $P$ is $\sigma$-closed by assumption. Then, for any $n \in \omega, p_{f} \Vdash s_{f \mid n} \in \dot{B}$. Since $p_{f} \Vdash \dot{B}$ is a new $\omega_{1}$-branch, there is $p_{f}^{\prime} \leq p_{f}$ and $s_{f} \in T$ s.t. $\forall n \in \omega s_{f} \geq s_{f \upharpoonright n}$
and $p_{f}^{\prime} \Vdash s_{f} \in \dot{B}$. But, by our above construction, if $f \neq g, s_{f}$ and $s_{g}$ are incomparable which implies $|T| \geq 2^{\aleph_{0}}$, contradicting our assumption.

Corollary 2.54 Let $T$ be a tree of height $\omega_{1}, \kappa>|T|$ and let $R_{1}$ be the forcing notion adding $\kappa$ Cohen Reals. In $\mathbf{V}^{R_{1}}$, let $R_{2}$ be a $\sigma$-closed forcing notion. Then every branch of $T$ in $\mathbf{V}^{R_{1} * R_{2}}$ is already in $\mathbf{V}$.

Proof: Immediate from lemma 2.50, lemma 2.52 and lemma 2.53.
Corollary 2.55 Let $T$ be a tree of height $\omega_{1}, \kappa>|T|$, let $b(T)$ denote the number of $\omega_{1}$-branches of $T$, assume $b(T)>\aleph_{1}$ and let $R_{1}$ be the forcing notion adding $\kappa$ Cohen Reals, $R_{2}:=\operatorname{coll}\left(\omega_{1},\{b(T)\}\right)^{\mathbf{V}^{R_{1}}}$. T will have $\aleph_{1}$ $\omega_{1}$-branches in $\mathbf{V}^{R_{1} * \dot{R}_{2}}$. Moreover, $R_{1} * \dot{R_{2}}$ is proper.

Proof: The first statement follows immediately from lemma 2.47, 1 and corollary 2.54 . The second statement follows immediately from fact 2.11, fact 2.12 and definition 2.10.

Definition 2.56 Let $T$ be a tree of height $\omega_{1}$ with $\aleph_{1} \omega_{1}$-branches $B_{i}, i<$ $\omega_{1}$, and assume that each node of $T$ is on some $\omega_{1}$-branch. For each $j<\omega_{1}$, let $B_{j}^{\prime}:=B_{j} \backslash \bigcup_{i<j} B_{i}$ and $x_{j}:=\min B_{j}^{\prime}$, so that the sets $B_{j}^{\prime}$ are disjoint end segments of the branches $B_{j}$ and that they form a partition of $T$. Let $A:=\left\{x_{i}: i<\omega_{1}\right\}$. The forcing $P_{T}^{\prime}$ is defined as

$$
P_{T}^{\prime}:=\left\{f:|f|<\aleph_{0}, f: A \rightarrow \omega, \forall x, y \in \operatorname{dom}(f)(x<y \rightarrow f(x) \neq f(y))\right\}
$$

where $f \leq g \leftrightarrow f \supseteq g$.
Theorem 2.57 (Baumgartner) [4] $P_{T}^{\prime}$ is ccc.
Proof: We will use the notation from definition 2.56. $A$, considered as a substructure of $T$ is a tree. Moreover, $A$ has no uncountable branches: If $B$ were such a branch, then for some $\alpha, B \subseteq B_{\alpha}$ and $B \cap(T \backslash A)=\emptyset . B \subseteq B_{\alpha}$ implies that $\left|B \cap B_{\alpha}^{\prime}\right|=\aleph_{1}$, but $B \cap B_{\alpha}^{\prime}=\left(B \cap B_{\alpha}^{\prime} \cap A\right) \cup\left(B \cap B_{\alpha}^{\prime} \cap(T \backslash A)\right)=$ $B \cap B_{\alpha}^{\prime} \cap A$ and $B_{\alpha}^{\prime} \cap A=\left\{x_{\alpha}\right\}$, a contradiction.

Assume $I=\left\{p_{\alpha}: \alpha \in \omega_{1}\right\}$ is an antichain of $P_{T}^{\prime}$. As $\left|p_{\alpha}\right|<\aleph_{0}$, there are only $\aleph_{0}$ possibilities for $\left|p_{\alpha}\right|$, hence we can without loss of generality assume that for all $\alpha<\omega_{1},\left|p_{\alpha}\right|=n$ and that $n$ is minimal in the sense that for no $n^{\prime}<n$ there is an antichain of size $\omega_{1}$ of partial functions of size $n^{\prime}$ in $P_{T}^{\prime}$. For each $\alpha<\omega_{1}$, let $a_{\alpha}:=\operatorname{dom}\left(p_{\alpha}\right)$. Now $\left\langle a_{\alpha}: \alpha<\omega_{1}\right\rangle$ is a collection of uncountably many finite sets, allowing us to apply the $\Delta$-system lemma (1.21): There exist $x \subseteq \omega_{1}$ and $r \subseteq A$ s.t. $|x|=\aleph_{1}$ and for any $\alpha, \beta \in x$, $a_{\alpha} \cap a_{\beta}=r$. As $r$ has to be finite, there are only countably many possibilities for $p_{\alpha} \upharpoonright r$ for any $p_{\alpha}: A \rightarrow \omega$, hence there are $y \subseteq x$ and $p: r \rightarrow \omega$ s.t. $|y|=\aleph_{1}$ and $\forall \alpha \in y p_{\alpha} \backslash r=p$. If $r \neq \emptyset$, then $\left\{p_{\alpha} \upharpoonright(A \backslash r): \alpha \in y\right\}$ is a pairwise
incompatible set of cardinality $\aleph_{1}$ of functions of size $n-|r|$, contradicting our assumption about minimality of $n$. Hence $r=\emptyset$ and hence for any $\alpha, \beta \in x, a_{\alpha} \cap a_{\beta}=\emptyset$. So we can without loss of generality assume that $\alpha \neq \beta \rightarrow \operatorname{dom}\left(p_{\alpha}\right) \cap \operatorname{dom}\left(p_{\beta}\right)=\emptyset$ for any $\alpha, \beta \in \omega_{1}$.

Also, by thinning out $I$ if necessary, we may assume that whenever $\alpha<\beta, p_{\alpha}(s)=p_{\beta}(t), s \neq t$ and $s$ and $t$ are comparable (which must be the case for some $s$ and $t$, as $p_{\alpha} \perp p_{\beta}$ ), then $s<t$ : Assume we have already thinned out $\left\langle p_{\alpha}: \alpha<\gamma\right\rangle$ for some $\gamma<\omega_{1}$. For simplicity of notation, we may assume that the above condition holds for $\left\langle p_{\alpha}: \alpha<\gamma\right\rangle$. We will find $\gamma^{\prime} \geq \gamma$ such that that condition holds for $\alpha, \beta \in \gamma \cup\left\{\gamma^{\prime}\right\}$, which is sufficient. Assume towards a contradiction that there is no such $\gamma^{\prime} \geq \gamma$, i.e. for any $\gamma^{\prime} \geq \gamma$ there is $\alpha<\gamma$ and there are $s, t \in A$ such that $s \neq t, p_{\alpha}(s)=p_{\gamma^{\prime}}(t)$ and $t<s$. But $\left|\bigcup_{\alpha<\gamma} \operatorname{dom}\left(p_{\alpha}\right)\right| \leq \aleph_{0}$ and, because $T$ is a tree of height $\omega_{1}$, letting $D:=\left\{t: \exists s \in \bigcup_{\alpha<\gamma} \operatorname{dom}\left(p_{\alpha}\right) t<s\right\},|D| \leq \aleph_{0}$. Note that for any $\gamma^{\prime} \geq \gamma, D \cap \operatorname{dom}\left(p_{\gamma^{\prime}}\right) \neq \emptyset$. Since $\left|\omega_{1} \backslash \gamma\right|=\aleph_{1}$ and $|D| \leq \aleph_{0}$, this contradicts our assumption that $\alpha \neq \beta \rightarrow \operatorname{dom}\left(p_{\alpha}\right) \cap \operatorname{dom}\left(p_{\beta}\right)=\emptyset$.

Let $U$ be a uniform ultrafilter on $\omega_{1}$ (i.e. $\forall x \in U|x|=\aleph_{1}$ ) and for each $\alpha \in \omega_{1}$, let $\operatorname{dom}\left(p_{\alpha}\right)=\left\{s_{0}^{\alpha}, \ldots, s_{n-1}^{\alpha}\right\}$. Now for each $\alpha<\omega_{1},\{\beta \in$ $\left.\omega_{1}: \exists i, j \quad s_{i}^{\alpha}<t_{j}^{\beta}\right\} \supseteq \kappa \backslash \alpha \in U$, so by the finite intersection property of $U$, we can find $i(\alpha)$ and $j(\alpha)$ s.t. $\left\{\beta \in \omega_{1}: s_{i(\alpha)}^{\alpha}<t_{j(\alpha)}^{\beta}\right\} \in U$. Again by the finite intersection property of $U$, there must be $i, j<n$ s.t. $E=\{\alpha: i(\alpha)=$ $i \wedge j(\alpha)=j\} \in U$. But now if $\alpha_{1}, \alpha_{2} \in E$, then there must be $\beta>\alpha_{1}, \alpha_{2}$ s.t. $s_{i}^{\alpha}<t_{j}^{\beta}$ for $\alpha=\alpha_{1}, \alpha_{2}$. Since $A$ is a tree, this implies that $s_{i}^{\alpha_{1}}$ and $s_{i}^{\alpha_{2}}$ are comparable, hence $\left\{s_{i}^{\alpha}: \alpha \in E\right\}$ may be extended to an uncountable branch through $A$, a contradiction.

Theorem 2.58 Let $T$ be a tree of height $\omega_{1}$. Assume that every node of $T$ is on some $\omega_{1}$-branch and that there are uncountably many $\omega_{1}$-branches. Then there is a proper forcing notion $P_{T}$ forcing the following:

1. $T$ has $\aleph_{1} \omega_{1}$-branches, i.e. $\exists b: \omega_{1} \times \omega_{1} \rightarrow T$ s.t. each set $B_{\alpha}^{\prime}=$ $\left\{b(\alpha, \beta): \beta<\omega_{1}\right\}$ is an end-segment of a branch of $T$ enumerated in its natural order, every $\omega_{1}$-branch is (modulo a countable set) equal to $B_{\alpha}^{\prime}$ for some $\alpha<\omega_{1}$ and the sets $B_{\alpha}^{\prime}$ form a partition of $T$.
2. There is a function $g: T \rightarrow \omega$ s.t. $\forall s<t$ in $T$, if $g(s)=g(t)$, then there is some (unique) $\alpha<\omega_{1}$ s.t. $\{s, t\} \subseteq B_{\alpha}^{\prime}$.

We call $P_{T}$ "sealing the $\omega_{1}$-branches of $T$ ".
Proof: By corollary 2.55, we may assume that $T$ has $\aleph_{1} \omega_{1}$-branches $\left\{B_{i}: i<\right.$ $\left.\omega_{1}\right\}$, as if not, we can first force with $R_{1} * \dot{R}_{2}$. Let $\left\{B_{i}^{\prime}: i<\omega_{1}\right\},\left\{x_{i}: i<\omega_{1}\right\}$, $A$ and $P_{T}^{\prime}$ be defined as they are in definition 2.56. Let $P_{T}:=P_{T}^{\prime}$. Any generic filter on $P_{T}^{\prime}$ induces a generic $f_{G}: A \rightarrow \omega$ (letting $f_{G}:=\bigcup G$ ). Let $g: T \rightarrow \omega$ be defined by $g(y)=f_{G}\left(x_{\alpha}\right)$ for all $y \in B_{\alpha}^{\prime}$. Then $g$ satisfies 2.

Finally, $P_{T}^{\prime}$ adds no $\omega_{1}$-branches through $T$, therefore 1 holds: Assume $b$ is an $\omega_{1}$-branch of $T$ in $\mathbf{V}^{P_{T}^{\prime}}$. As $\Vdash_{P_{T}^{\prime}} " \dot{b}$ is an $\omega_{1}$-branch through $T "$, $\vdash_{P_{T}^{\prime}} " \exists n \in \omega \exists a \subseteq \dot{b}|a|=\aleph_{1} \wedge \forall t \in a g(t)=n "$. But this implies that for some $\alpha<\omega_{1}, \Vdash_{P_{T}^{\prime}} \dot{b} \subseteq B_{\alpha}$, hence $\Vdash_{P_{T}^{\prime}} \dot{b}=B_{\alpha} \in \mathbf{V}$.

Corollary 2.59 $P_{T}$ seals the $\omega_{1}$-branches of $T$, i.e. if $P \in \mathbf{V}^{P_{T}}$ is a forcing notion and $\dot{b} \in \mathbf{V}^{P_{T}}$ is a $P$-name for an $\omega_{1}$-branch of $T$, then

$$
\mathbf{V}^{P_{T}} \models \Vdash_{P} \dot{b} \in \mathbf{V}^{P_{T}}
$$

Proof: The proof is the same as the proof showing that $P_{T}^{\prime}$ adds no $\omega_{1-}$ branches through $T$ in theorem 2.58 above. In fact, a stronger statement follows by that proof: $\mathbf{V}^{P_{T}} \models \Vdash_{P} \dot{b} \in \mathbf{V}$.

### 2.2.5 Almost disjoint coding

Let $\alpha$ be a regular cardinal, $\beta>\alpha \in$ Ord. Let $\mathcal{A}=\left(a_{\xi}\right)_{\xi<\beta}$ be an almost disjoint family of size $|\beta|$ on $\alpha$. Let $B \subseteq \beta$. Using $\mathcal{A}$, we can force to add a subset $A$ of $\alpha$ such that $A$ codes $B$ in the following sense:

$$
B=\left\{\xi<\beta:\left|A \cap a_{\xi}\right|<\alpha\right\}
$$

In the following, for $q \subseteq \beta$, let $\mathcal{A} \upharpoonright q$ denote $\left\{a_{\xi}: \xi \in q\right\}$.

## Definition 2.60 (Almost disjoint coding)

$$
P_{\mathcal{A}, B}:=[\alpha]^{<\alpha} \times[B]^{<\alpha} \text { ordered by }
$$

$$
(p, q) \leq\left(p^{\prime}, q^{\prime}\right) \leftrightarrow p \text { end-extends } p^{\prime}, q \supseteq q^{\prime} \text { and }\left(p \backslash p^{\prime}\right) \cap \bigcup \mathcal{A} \upharpoonright q=\emptyset
$$

$P_{\mathcal{A}, B}$ is called the almost disjoint coding of $B \operatorname{using} \mathcal{A}$.
Lemma 2.61 For $\sigma \in B, D_{\sigma}:=\left\{(p, q) \in P_{\mathcal{A}, B}: \sigma \in q\right\}$ is dense in $P_{\mathcal{A}, B}$.
Proof: Given $(p, q)$, extend it to $(p, q \cup\{\sigma\}) \leq(p, q)$ to hit $D_{\sigma}$.
Lemma 2.62 For each $\rho<\alpha, \sigma \in(\beta \backslash B)$, the set

$$
D_{\rho, \sigma}:=\left\{(p, q) \in P_{\mathcal{A}, B}: \operatorname{ot}\left(p \cap a_{\sigma}\right) \geq \rho\right\}
$$

is dense in $P_{\mathcal{A}, B}$.
Proof: Let $(p, q) \in P_{\mathcal{A}, B}$. Let $S:=a_{\sigma} \backslash \bigcup \mathcal{A} \upharpoonright q=a_{\sigma} \backslash \bigcup_{\xi \in q}\left(a_{\sigma} \cap a_{\xi}\right)$. Since $\sigma \notin B$, for $\xi \in q, q \subseteq B,\left|a_{\sigma} \cap a_{\xi}\right|<\alpha$, hence $|S|=\alpha$. Let $p^{\prime}$ be obtained from $p$ by extending $p$ by a subset of $S$ of order-type $\rho$. On the one hand it follows that ot $\left(p^{\prime} \cap a_{\sigma}\right) \geq \rho$, on the other hand, since we have avoided every $a_{\xi}, \xi \in q$ in the construction of $p^{\prime},\left(p^{\prime}, q\right) \leq(p, q)$.

Now let $G$ be generic for $P_{\mathcal{A}, B}$ and let $A:=\bigcup\{p: \exists q(p, q) \in G\}$. As $G$ meets all of the above dense sets, it follows that $A$ codes $B$ in the abovementioned sense, i.e. $B=\left\{\xi<\beta:\left|A \cap a_{\xi}\right|<\alpha\right\}$.

Lemma $2.63 P_{\mathcal{A}, B}$ is $\alpha$-closed.
Proof: Let $\rho<\alpha,\left(p_{\xi}, q_{\xi}\right)_{\xi<\rho}$ a decreasing sequence of conditions and let $p^{\prime}:=\bigcup_{\xi<\rho} p_{\xi}, q^{\prime}:=\bigcup_{\xi<\rho} q_{\xi}$. If $\gamma \in q_{\xi}, \xi<\rho$, then $p^{\prime} \cap a_{\gamma}=p_{\xi} \cap a_{\gamma}$ and hence $\left(p^{\prime} \backslash p_{\xi}\right) \cap a_{\gamma}=\emptyset$, implying $\left(p^{\prime}, q^{\prime}\right) \leq\left(p_{\xi}, q_{\xi}\right)$.

Lemma $2.64 P_{\mathcal{A}, B}$ is $\left|[\alpha]^{<\alpha}\right|$-linked. If $\left|[\alpha]^{<\alpha}\right|=\alpha, P_{\mathcal{A}, B}$ is also $\alpha$-centered.
Proof: If $\left|[\alpha]^{<\alpha}\right|=\alpha$, let $f: P_{\mathcal{A}, B} \rightarrow[\alpha]^{<\alpha} \cong \alpha$ be the projection $(p, q) \mapsto p$. For a set $W$ of size $<\alpha$ of conditions with the same first component $p$, $\left(p, \bigcup_{(p, q) \in W} q\right)$ is stronger than any condition in $W$. Thus $P_{\mathcal{A}, B}$ is $\alpha$-centered. Using the same $f, P_{\mathcal{A}, B}$ is immediately seen to be $\left|[\alpha]^{<\alpha}\right|$-linked.

Corollary 2.65 If $\alpha=\omega$ or $\alpha$ is a strong limit cardinal, $P_{\mathcal{A}, B}$ is $\alpha^{+}$-cc.

### 2.2.6 Almost disjoint coding of a function

In this variant of almost disjoint coding, we generically add a real $b$ such that for a given function $f: A \subseteq 2^{\omega} \rightarrow 2^{\omega}, b$ codes $f(r)$ for every $r \in A$.

Let $\left\langle s_{i}: i \in \omega\right\rangle$ be a recursive enumeration of $2^{<\omega}$ such that each $s \in 2^{<\omega}$ is enumerated before any of it's proper extensions. Fix a recursive partition of $\omega$ into infinitely many pieces $X_{i}, i<\omega$. For $a \in 2^{\omega}$, define

$$
\begin{gathered}
f^{a}:=\left\{j: a \upharpoonright \operatorname{length}\left(s_{j}\right)=s_{j}\right\} \\
f_{i}^{a}:=\left\{j: a \upharpoonright \operatorname{length}\left(s_{j}\right)=s_{j} \wedge \operatorname{length}\left(s_{j}\right) \in X_{i}\right\}
\end{gathered}
$$

$\left\{f^{a}: a \in 2^{\omega}\right\}$ is an almost disjoint family: Let $r \neq s \in 2^{\omega}$. Let $k_{0} \in r \triangle s \neq \emptyset$, then for all $k \geq k_{0}, s \upharpoonright k \neq r \upharpoonright k$, hence $f^{r} \cap f^{s}$ is finite. Obviously it follows that $\left\{f_{i}^{a}: a \in 2^{\omega}, i \in \omega\right\}$ is an almost disjoint family as well.

## Definition 2.66 (Almost disjoint coding of a function)

Let $A \subseteq 2^{\omega}, f: A \rightarrow 2^{\omega}$. Define $P_{f}$ as follows:

$$
\begin{gathered}
P_{f}:=2^{<\omega} \times\left[\bigcup_{r \in A}(\{r\} \times f(r))\right]^{<\omega} \text { ordered by } \\
(s, g) \leq(t, h) \leftrightarrow s \supseteq t, g \supseteq h \wedge \forall(a, i) \in h(s \backslash t) \cap f_{i}^{a}=\emptyset .
\end{gathered}
$$

For $a, b \in 2^{\omega}$, let $b \odot a:=\left\{i \in \omega: b \cap f_{i}^{a}\right.$ is finite $\}$.
Lemma 2.67
For $a \in A, i \in f(a), D:=\left\{(p, h) \in P_{f}:(a, i) \in h\right\}$ is dense in $P_{f}$.

Proof: Like lemma 2.61: For any condition $(p, h)$ and any pair $(a, i)$ as above, $(p, h \cup\{(a, i)\})$ is a condition extending $(p, h)$.

Lemma 2.68 For each $(a, i) \in A \times \omega$ such that $i \notin f(a)$ and for each $n \in \omega$, $D:=\left\{(p, h) \in P_{f}:\left(p \cap f_{i}^{a}\right) \backslash n \neq \emptyset\right\}$ is dense in $P_{f}$.

Proof: Like lemma 2.62: Let $(p, h) \in P_{f}$ be arbitrary. As $i \notin f(a),(a, i) \notin h$, so $f_{i}^{a} \backslash \bigcup_{(s, j) \in h} f_{j}^{s}$ must be infinite by almost disjointness. Picking $k>n$ in that latter set, $(p \cup\{k\}, h)$ is a condition in $D$ extending $(p, h)$.

Now let $G$ be generic for $P_{f}$ and let $b:=\bigcup\{p: \exists q(p, q) \in G\}$. As $G$ meets all of the above dense sets, it follows that for all $a \in A, f(a)=b \odot a$.

Lemma $2.69 P_{f}$ is $\aleph_{0}$-centered and thus ccc.
Proof: Like lemma 2.64: Let $f: P_{f} \rightarrow 2^{<\omega} \cong \omega$ be the projection $(p, h) \mapsto p$. If $\left(p_{i}, h_{i}\right), i \in k$, are conditions with the same first component, $\left(p_{0}, \bigcup_{i \in k} h_{i}\right)$ is also a condition and is stronger than any $\left(p_{i}, h_{i}\right), i \in k$.

### 2.2.7 Reshaping

For $A \subseteq \omega_{1}$, we define the reshaping forcing as follows:

$$
P:=\left\{p \stackrel{\text { bnd }}{\subseteq} \omega_{1}: \forall \xi \leq \sup p \mathbf{L}[p \cap \xi, A \cap \xi] \models \xi \cong \omega\right\},
$$

ordered by end-extension, where $p \stackrel{\text { bnd }}{\subseteq} \omega_{1}$ if $p$ is a bounded subset of $\omega_{1}$.
Lemma 2.70 For each $\alpha<\omega_{1}, D_{\alpha}:=\{p \in P: \sup p \geq \alpha\}$ is dense in $P$.
Proof: Let $p_{0}$ be a condition in $P$ with $\sup p_{0}=\delta$ and let $\delta<\alpha<\omega_{1}$. If $\alpha<\delta+\omega$, we can choose an arbitrary $p$ end-extending $p_{0}$ such that $\sup p=\alpha$ to hit $D_{\alpha}$. Otherwise, let $E \subseteq[\delta+1, \delta+\omega)$ code the $\in$-relation on $\alpha$ and let $p:=p_{0} \cup E \cup[\delta+\omega, \alpha)$. We show that $p \in P$ : If $\xi \leq \delta$, then $\mathbf{L}[p \cap \xi, A \cap \xi] \models \xi \cong \omega$ because $p_{0} \in P$. If $\delta<\xi<\delta+\omega$, then there is $n \in \omega$ such that $\mathbf{L}[p \cap \delta, A \cap \delta] \vDash \xi=\delta+n \cong \omega$, hence $\mathbf{L}[p \cap \xi, A \cap \xi] \vDash$ $\xi \cong \omega$. If $\delta+\omega \leq \xi \leq \alpha$, then, since $\mathbf{L}[p \cap(\delta+\omega), A \cap(\delta+\omega)] \models \alpha \cong \omega$, $\mathbf{L}[p \cap \xi, A \cap \xi] \models \xi \cong \omega$.

Obviously, $p$ hits $D_{\alpha}$ and extends $p_{0}$.
If $G$ is generic for $P$, then we obtain $H:=\bigcup G \subseteq \omega_{1}$ such that

$$
\forall \xi<\omega_{1} \mathbf{L}[H \cap \xi, A \cap \xi] \models \xi \cong \omega .
$$

This allows us to choose $A^{\prime} \subseteq \omega_{1}$ coding both $A$ and $H$ such that

$$
\forall \xi<\omega_{1} \mathbf{L}\left[A^{\prime} \cap \xi\right] \models \xi \cong \omega .
$$

We say that $A^{\prime}$ is reshaped.

### 2.3 Forcing Axioms

## Definition 2.71

- Given a partial ordering $P$ and a cardinal $\kappa$, the "Forcing Axiom for collections of $\kappa$ antichains of $P$ ", in short $\mathrm{FA}_{\kappa}(P)$, is the following statement:

For every collection $\left\{I_{\alpha}: \alpha<\kappa\right\}$ of macs of $P$, there exists a filter $G \subseteq P$ s.t. $\forall \alpha<\kappa I_{\alpha} \cap G \neq \emptyset$.

- If $\mathcal{P}$ is a class of partial orderings, $\mathrm{FA}_{\kappa}(\mathcal{P})$ is the statement that for every $P \in \mathcal{P}, \mathrm{FA}_{\kappa}(P)$ holds.
- If $P$ is ccc, we write $\mathrm{MA}_{\kappa}(P)$ for $\mathrm{FA}_{\kappa}(P)$.
- Martin's Axiom at $\kappa$, in short $\mathrm{MA}_{\kappa}$, is the statement that for every ccc poset $P, \mathrm{MA}_{\kappa}(P)$ holds.
- Martin's Axiom, in short MA, is the statement that for every $\kappa<2^{\aleph_{0}}$, $\mathrm{MA}_{\kappa}$ holds.
- Given a partial ordering $P$ and a cardinal $\kappa$, the "Bounded Forcing Axiom for collections of $\kappa$ antichains of $P$ ", in short $\mathrm{BFA}_{\kappa}(P)$, is the following statement: For every collection $\left\{I_{\alpha}: \alpha<\kappa\right\}$ of macs of $B:=\mathcal{B}(P)$, each of size at most $\kappa$, there exists a filter $G \subseteq B$ s.t. $\forall \alpha<\kappa I_{\alpha} \cap G \neq \emptyset$.
- If $\mathcal{P}$ is a class of partial orderings, $\operatorname{BFA}_{\kappa}(\mathcal{P})$ is the statement that for every $P \in \mathcal{P}, \mathrm{BFA}_{\kappa}(P)$ holds.
- The Bounded Proper Forcing Axiom BPFA is the statement that for every proper poset $P, \mathrm{BFA}_{\omega_{1}}(P)$.
- The Bounded Semiproper Forcing Axiom BSPFA is the statement that for every semiproper poset $P, \mathrm{BFA}_{\omega_{1}}(P)$.
- The Bounded Martin's Maximum BMM is the statement that for every poset $P$ that preserves stationary subsets of $\omega_{1}, \mathrm{BFA}_{\omega_{1}}(P)$.

Remarks: It is important to require maximal antichains of $\mathcal{B}(P)$ instead of $P$ for the bounded forcing axioms, because for example $\{\mathbf{1}\}$ might be the only maximal antichain of size $\leq \kappa$ of $P$ for some p.o. $P$, making $\mathrm{BFA}_{\kappa}(P)$ a vacuous statement. Furthermore, note that we can equivalently work with predense, dense or open dense sets in the above definition.

Fact 2.72 (Solovay, Tennenbaum) ([17], Theorem 16.13) Let $\kappa>\aleph_{0}$ be a regular cardinal. Then $\mathrm{MA}+2^{\aleph_{0}}=\kappa$ is equiconsistent with $Z F C$.

Lemma 2.73 $\mathrm{MA}_{2^{\aleph_{0}}}$ is false.
Proof: Let $P$ be the poset adding a single Cohen real and for each $h \in \omega_{2}$, let $D_{h}:=\{p \in P: \exists n \in \operatorname{dom}(p) p(n) \neq h(n)\}$. Assuming that $M A_{2^{\aleph_{0}}}$ holds, we can find a filter $G$ which meets $D_{h}$ for any $h \in \omega_{2}$. Let $f:=\bigcup G, f \in \omega_{2}$. Now for any $h \in{ }_{2}$, since $f \supseteq p$ for some $p \in D_{h}$, it follows that $f \neq h$; a contradiction.

## $3 \quad \Sigma_{3}^{1}$ Absoluteness for Set Forcing

Theorem 3.1 (Friedman) [9] The following are equiconsistent:

1. $\boldsymbol{\Sigma}_{\mathbf{3}}^{\mathbf{1}}$ absoluteness for set forcing
2. $\operatorname{Abs}\left(\Sigma_{2}\left(\mathrm{H}_{\omega_{1}}\right)\right.$, set forcing $)$
3. $\exists \kappa \kappa$ is reflecting

Proof: By corollary 1.56, 1 and 2 are equivalent.

2 $\rightarrow$ 3: Assume that $\operatorname{Abs}\left(\Sigma_{2}\left(\mathrm{H}_{\omega_{1}}\right)\right.$, set forcing) holds. Let $\kappa=\omega_{1}$. Because $\mathbf{V} \equiv \kappa$ regular, $\mathbf{L} \equiv \kappa$ regular. We show that $L_{\kappa} \prec \Sigma_{2} \mathbf{L}$ : Let $p \in H_{\kappa}^{\mathbf{L}}=L_{\kappa}$ (see 1.43) and $\varphi \in \Sigma_{2}\left(L_{\kappa}\right)$ such that $\mathbf{L} \models \varphi(p)$. Choose some regular $\delta$ s.t. $L_{\delta} \models \varphi(p)$ and force with $\operatorname{coll}(\omega,\{\delta\})$. The following formula expresses (see 1.44) that there exists a countable $\mathbf{L}$-cardinal $\xi$ s.t. $L_{\xi} \models \varphi(p)$ and it obviously holds in the extension:

$$
\exists \xi<\omega_{1} \forall \alpha<\omega_{1}\left(\xi \in L_{\alpha} \rightarrow L_{\alpha} \models \xi \in \mathbf{C a r d}\right) \wedge p \in L_{\xi} \wedge L_{\xi} \models \varphi(p)
$$

As for $\alpha<\omega_{1}, L_{\alpha} \in H_{\omega_{1}}$ (see 1.40) and $\left(L_{\alpha}\right)^{H_{\omega_{1}}}=L_{\alpha}$, the following formula $\psi(p)$ holds in $H_{\omega_{1}}$ of our generic extension:

$$
\exists \xi \forall \alpha\left(\xi \in L_{\alpha} \rightarrow L_{\alpha} \models \xi \in \mathbf{C a r d}\right) \wedge p \in L_{\xi} \wedge L_{\xi} \models \varphi(p)
$$

$\psi(p) \in \Sigma_{2}\left(\mathrm{H}_{\omega_{1}}\right)$, hence it also holds in $\left(H_{\kappa}\right)^{\mathbf{V}}$, and, again using absoluteness of $L_{\alpha}$, the following holds in $\mathbf{V}$ : There exists an $L$-cardinal $\xi<\kappa$ s.t. $L_{\xi} \models \varphi(p)$. But as $\xi$ is an $L$-cardinal, $L_{\xi}=H_{\xi}^{\mathbf{L}} \prec_{\Sigma_{1}} H_{\kappa}^{\mathbf{L}}=L_{\kappa}$ by (1.43) and (1.64), hence $\Sigma_{2}\left(L_{\xi}\right)$ statements are upwards absolute between $L_{\xi}$ and $L_{\kappa}$ implying $L_{\kappa} \mid=\varphi(p)$, hence $L_{\kappa} \prec_{\Sigma_{2}} \mathbf{L}$. Because $\kappa$ is a limit cardinal in $\mathbf{L}$, it follows that $L_{\kappa}=\left(H_{\kappa}\right)^{\mathbf{L}}=\left(V_{\kappa}\right)^{\mathbf{L}} \prec_{\Sigma_{2}} \mathbf{L}$, hence $\mathbf{L} \models \kappa$ is reflecting.
$3 \rightarrow \mathbf{1}: \quad$ Assume that $\kappa$ is reflecting. Then by theorem $1.69, \kappa$ is inaccessible. Let $G$ be generic for $\operatorname{coll}(\omega, \kappa)$ over $\mathbf{V}$, then $\mathbf{V}[G] \vDash \kappa=\omega_{1}$. We show that $\mathbf{V}[G] \models \boldsymbol{\Sigma}_{\mathbf{3}}^{\mathbf{1}}$ absoluteness for set forcing: Upward absoluteness is given by corollary 1.59. So let $\varphi(r)$ be some $\Sigma_{3}^{1}(r)$ statement with $r \in \mathbf{V}[G]$ and let $Q \in \mathbf{V}[G]$ be a p.o. such that $\mathbf{V}[G] \vDash \vdash_{Q} \varphi(r)$. We show that $\mathbf{V}[G] \models \varphi(r)$ : Choose a nice name $\dot{r}$ for $r \subseteq \omega$, i.e. a name of the form $\bigcup_{n \in \omega}\{\check{n}\} \times A_{n}$, where, for each $n, A_{n}$ is an antichain of $\operatorname{coll}(\omega, \kappa)$ and note that the value of $\dot{r}$ is already decided in an initial segment $\operatorname{coll}(\omega, \beta), \beta<\kappa$ of $\operatorname{coll}(\omega, \kappa)$ : each $A_{n}$ has cardinality $<\kappa$ (see $2.47,3$ ) and, because $\kappa$ is regular, we can choose $\beta<\kappa$ regular such that $\forall n \in \omega \forall q \in A_{n} \operatorname{dom}(q) \subseteq \beta \times \omega$. By lemma 2.48, $G_{0}:=G \cap \operatorname{coll}(\omega, \beta)$ is $\operatorname{coll}(\omega, \beta)$-generic over V. Furthermore,

$$
\mathbf{V}\left[G_{0}\right] \models \exists p \Vdash_{\operatorname{coll}(\omega, \kappa \backslash \beta)_{p} * \dot{Q}} \varphi(\check{r})
$$

Hence $\mathbf{V}\left[G_{0}\right] \models \exists P \Vdash_{P} \varphi(\check{r})$, therefore in $\mathbf{V}$ the following holds:

$$
\exists \dot{P} \exists q \in G_{0} \Vdash_{\operatorname{coll}(\omega, \beta)_{q}}\left(\dot{P} \text { is a partial order } \wedge \Vdash_{\dot{P}} \varphi(\dot{r})\right) .
$$

As $q$ plays no role in the following, we may assume $q=1$.
Because $\kappa$ is reflecting in $\mathbf{V}$ and $\operatorname{coll}(\omega, \beta) \in H_{\kappa}$, there exist $\alpha<\kappa$ regular and $\dot{P} \in H_{\alpha}$ such that the following holds:

$$
\begin{equation*}
H_{\alpha} \models \Vdash_{\operatorname{coll}(\omega, \beta)}\left(\dot{P} \text { is a p.o. } \wedge \Vdash_{\dot{P}} \varphi(\dot{r})\right) . \tag{4}
\end{equation*}
$$

We can also demand $\kappa$ to reflect $\exists y y=|\mathcal{P}(P)|$, hence assume that $|\mathcal{P}(P)|<\alpha$. By corollary 2.44, (4) is equivalent to the following:

$$
\begin{equation*}
\Vdash_{\operatorname{coll}(\omega, \beta)} H_{\alpha} \models\left(\dot{P} \text { is a p.o. } \wedge \Vdash_{\dot{P}} \varphi(\dot{r})\right) . \tag{5}
\end{equation*}
$$

Applying corollary 2.42 to the second conjunct of (5) yields:

$$
\begin{equation*}
\Vdash_{\operatorname{coll}(\omega, \beta)}\left(\dot{P} \text { is a p.o. } \wedge \Vdash_{\dot{P}} H_{\alpha} \models \varphi(\dot{r})\right) . \tag{6}
\end{equation*}
$$

By upward absoluteness, $\Vdash_{\text {coll }(\omega, \beta) * \dot{P}} \varphi(\dot{r})$.
Let $P:=\dot{P}^{G_{0}}$. By lemma 2.43, $|\mathcal{P}(P)|<\alpha$ in $\mathbf{V}^{G_{0}}$. By lemma 2.48, $G_{1}:=G \cap \operatorname{coll}(\omega,\{\alpha\})$ is $\operatorname{coll}(\omega,\{\alpha\})$-generic over $\mathbf{V}[G \cap \operatorname{coll}(\omega, \alpha)]$, and, because $G_{0} \subseteq G \cap \operatorname{coll}(\omega, \alpha)$, also over $\mathbf{V}\left[G_{0}\right]$. Because $|\mathcal{P}(P)|=\omega$ in $\mathbf{V}\left[G_{0}\right]\left[G_{1}\right]$, there exists a generic $G_{P} \in \mathbf{V}\left[G_{0}\right]\left[G_{1}\right]$ for $P$ over $\mathbf{V}\left[G_{0}\right]$.

As $\mathbf{V}\left[G_{0}\right] \models \Vdash_{P} \varphi(\check{r}), \mathbf{V}\left[G_{0}\right]\left[G_{P}\right] \models \varphi(r)$.
$\mathbf{V}\left[G_{0}\right]\left[G_{P}\right] \subseteq \mathbf{V}\left[G_{0}\right]\left[G_{1}\right] \subseteq \mathbf{V}[G]$, so, by upward absoluteness, $\mathbf{V}[G] \models \varphi(r)$ and hence $\mathbf{V}[G]$ is a model for $\boldsymbol{\Sigma}_{3}^{1}$-absoluteness for set forcing.

Corollary 3.2 $\Sigma_{3}^{1}$ absoluteness for set forcing is equiconsistent with ZFC.
Proof: This proof follows easily from the proof of the second direction of the previous theorem. First, by lemma 1.66, choose any cardinal $\kappa$ such that $H_{\kappa} \prec_{\Sigma_{2}} \mathbf{V}$. Let $G$ be generic for $\operatorname{coll}(\omega, \kappa)$ over $\mathbf{V}$. Then $\mathbf{V}[G]$ is a model of $\Sigma_{3}^{1}$ absoluteness for set forcing: Let $\varphi$ be some $\Sigma_{3}^{1}$ statement and let $Q \in \mathbf{V}[G]$ be a p.o. such that $\mathbf{V}[G] \models \Vdash_{Q} \varphi$. Hence $\mathbf{V} \models \exists P \Vdash_{P} \varphi$. By our above choice of $\kappa$ and by corollary 1.70,

$$
\exists \alpha<\kappa \text { regular } \exists P \in H_{\alpha}|\mathcal{P}(P)|<\alpha \wedge \Vdash_{P} \varphi(r)
$$

$G_{1}:=G \cap \operatorname{coll}(\omega,\{\alpha\})$ is $\operatorname{coll}(\omega,\{\alpha\})$-generic over $\mathbf{V}$. Because $|\mathcal{P}(P)|=\omega$ in $\mathbf{V}\left[G_{1}\right]$, there exists a generic $G_{P} \in \mathbf{V}\left[G_{1}\right]$ for $P$ over $\mathbf{V}$. As $\mathbf{V} \models \Vdash_{P} \varphi$, $\mathbf{V}\left[G_{P}\right] \models \varphi . \mathbf{V}\left[G_{P}\right] \subseteq \mathbf{V}\left[G_{1}\right] \subseteq \mathbf{V}[G]$, so, by corollary $1.58, \mathbf{V}[G] \models \varphi$ and hence $\mathbf{V}[G]$ is a model for $\Sigma_{3}^{1}$-absoluteness for set forcing.

## $4 \quad \Sigma_{1}\left(\mathrm{H}_{\omega_{2}}\right)$ Absoluteness Results

### 4.1 Inconsistency Results

Theorem 4.1 $\operatorname{Abs}\left(\Sigma_{1}\left(\mathrm{H}_{\omega_{2}}\right)\right.$, set forcing) is false.
Proof: Let $P$ be a poset collapsing $\omega_{1}$ and let $G$ be generic for $P$. Let

$$
\varphi \equiv \exists f: \omega \xrightarrow{\text { onto }} \omega_{1}^{\mathrm{V}} .
$$

Then $\mathrm{H}_{\omega_{2}} \models \neg \varphi$ and $H_{\omega_{2}}^{\mathrm{V}[G]} \models \varphi$. As $\varphi \in \Sigma_{1}\left(\mathrm{H}_{\omega_{2}}\right)$, the theorem follows.
Theorem 4.2 $\operatorname{Abs}\left(\Sigma_{1}\left(\mathrm{H}_{\omega_{2}}\right), \omega_{1}\right.$-preserving) is false.
Proof: Assume $\operatorname{Abs}\left(\Sigma_{1}\left(\mathrm{H}_{\omega_{2}}\right), \omega_{1}\right.$-preserving) holds. Let $S$ be a stationary and co-stationary subset of $\omega_{1}$ and let $P$ be the forcing notion that adds a closed unbounded subset of $S$ while preserving $\omega_{1}$ defined in theorem 2.49 and let $G$ be generic for $P$.

$$
\begin{gathered}
C \operatorname{club} \leftrightarrow\left(\forall \alpha<\omega_{1} \exists \beta \in C\right)(\beta>\alpha \wedge \sup (C \cap \alpha) \in C) \\
y=\sup (B) \leftrightarrow(\forall \beta \in B y \geq \beta) \wedge(\forall \beta<y \exists \gamma \in B \gamma>\beta)
\end{gathered}
$$

Both of the above are $\Delta_{0}\left(\mathrm{H}_{\omega_{2}}\right)$ statements, so " $\exists C C \subseteq S \wedge C$ club" is a $\Sigma_{1}\left(\mathrm{H}_{\omega_{2}}\right)$ sentence which holds in $H_{\omega_{2}}^{\mathrm{V}[G]}$. Hence it holds in $\mathrm{H}_{\omega_{2}}$, contradicting co-stationarity of $S$.

### 4.2 BFA and $\Sigma_{1}$-absoluteness

Lemma 4.3 [14] Let $P$ be a poset and assume $\Vdash_{P}$ " $\mathfrak{M}$ is a structure with universe $\kappa$ with $\kappa$-many relations ( $\dot{R}_{i}: i<\kappa$ )" and BFA $A_{\kappa}(P)$ holds. If $G$ is a filter, let

- $\dot{R}_{i}{ }^{G}:=\left\{\left(x_{1}, \ldots, x_{n}\right) \in \kappa^{n}: \exists p \in G p \Vdash \dot{\mathfrak{M}} \models \dot{R}_{i}\left(\check{x_{1}}, \ldots, \check{x_{n}}\right)\right\}$,
- $\mathfrak{M}^{*}:=\left(\kappa,\left(\dot{R}_{i}^{G}\right)_{i<\kappa}\right)$.

Then there exists a so-called "sufficiently generic" filter $G \subseteq P$ such that:
Whenever $\varphi$ is a sentence such that $\Vdash_{P} \dot{\mathfrak{M}} \models \varphi$, then $\mathfrak{M}^{*} \models \varphi$.
Proof: Let $\chi$ be a sufficiently large cardinal and let $\mathfrak{N}$ be an elementary submodel of $\left(H_{\chi}, \in,(\beta)_{\beta<\kappa},(P, \leq),\left(\dot{R}_{i}\right)_{i<\kappa}\right)$ of size $\kappa$. By $B F A_{\kappa}(P)$, we can find a filter $G \subseteq P$ such that the following hold:

- $\forall \dot{\alpha} \in N\left(\left(\vdash_{P} \dot{\alpha} \in \kappa\right) \rightarrow(\exists \beta \in \kappa \exists p \in G p \Vdash \dot{\alpha}=\check{\beta})\right)$
- $\forall \alpha_{1}, \ldots, \alpha_{n} \in \kappa$, for all $\varphi\left(x_{1}, \ldots, x_{n}\right)$, there exists $p \in G$ such that either $p \Vdash \mathfrak{M} \models \varphi\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ or $p \Vdash \mathfrak{M} \vDash \neg \varphi\left(\alpha_{1}, \ldots, \alpha_{n}\right)$.

We call such $G$ a "sufficiently generic" filter. The lemma now follows from claim 1 and claim 2 below:

Claim 1: For every quantifier-free $\varphi\left(x_{1}, \ldots, x_{n}\right)$ and $\beta_{1}, \ldots, \beta_{n} \in \kappa$,

$$
\exists p \in G p \Vdash \dot{\mathfrak{M}} \models \varphi\left(\check{\beta}_{1}, \ldots, \check{\beta}_{n}\right) \leftrightarrow \mathfrak{M}^{*} \models \varphi\left(\beta_{1}, \ldots, \beta_{n}\right) .
$$

Proof: By induction on formula complexity:

- If $\varphi \equiv R_{i}\left(\beta_{1}, \ldots, \beta_{n}\right)$, then the claim holds by our definition of $\dot{R}_{i}{ }^{G}$.
- If $\varphi \equiv \neg \psi$ and the claim holds for $\psi$, then

$$
\exists p \in G p \Vdash \dot{\mathfrak{M}} \models \varphi \leftrightarrow \nexists q \in G q \Vdash \dot{\mathfrak{M}} \models \psi \leftrightarrow \neg \mathfrak{M}^{*} \models \psi \leftrightarrow \mathfrak{M}^{*} \models \varphi
$$

- If $\varphi \equiv \psi_{1} \wedge \psi_{2}$ and the claim holds for $\psi_{1}$ and $\psi_{2}$, this is trivial.

Claim 2: For every formula $\varphi\left(x_{1}, \ldots, x_{n}\right)$ and every $\dot{\alpha_{1}}, \ldots, \dot{\alpha_{n}} \in N$, if $\vdash_{P} \dot{\mathfrak{M}} \models \varphi\left(\dot{\alpha_{1}}, \ldots, \dot{\alpha_{n}}\right)$, then $\mathfrak{M}^{*} \models \varphi\left(\dot{\alpha}_{1}^{G}, \ldots, \dot{\alpha_{n}}{ }^{G}\right)$, where for $\dot{\alpha}$ s.t. $\Vdash_{P} \dot{\alpha} \in \kappa, \dot{\alpha}^{G}=\beta \leftrightarrow \exists p \in G p \Vdash \dot{\alpha}=\check{\beta}$, which proves lemma 4.3.

Proof: First note that if $\Vdash_{P} \dot{\mathfrak{M}} \models \varphi\left(\dot{\alpha_{1}}, \ldots, \dot{\alpha_{n}}\right)$, then $\forall i \Vdash_{P} \dot{\alpha_{i}} \in \kappa$, so because $G$ is sufficiently generic, there exist $\beta_{1}, \ldots, \beta_{n} \in \kappa$ such that $\forall i \exists p \in G p \Vdash \dot{\alpha}_{i}=\check{\beta}_{i}$ and hence $\left(\Vdash_{P} \dot{\mathfrak{M}} \vDash \varphi\left(\dot{\alpha_{1}}, \ldots, \dot{\alpha_{n}}\right)\right) \rightarrow(\exists p \in G p \Vdash$ $\dot{\mathfrak{M}} \equiv \varphi\left(\check{\beta}_{1}, \ldots, \check{\beta_{n}}\right)$ ). Also note that $\forall i \dot{\alpha}_{i}{ }^{G}=\beta_{i}$. We show, by induction on formula complexity, that if $\exists p \in G p \Vdash \dot{\mathfrak{M}} \vDash \varphi\left(\check{\beta}_{1}, \ldots, \check{\beta}_{n}\right)$, then $\mathfrak{M}^{*}=\varphi\left(\beta_{1}, \ldots, \beta_{n}\right)$, which is sufficient:

- If $\varphi$ is quantifier-free, this follows from claim 1.
- If $\varphi \equiv \exists x \psi$ and claim 2 holds for $\psi,\left(\Vdash_{P} \dot{\mathfrak{M}} \models \exists x \psi\left(x, \dot{\alpha_{1}}, \ldots, \dot{\alpha_{n}}\right)\right) \rightarrow$ $\left(\exists \dot{b} \in N \Vdash_{P} \dot{\mathfrak{M}} \mid=\psi\left(\dot{b}, \dot{\alpha_{1}}, \ldots, \dot{\alpha_{n}}\right)\right)$, so $\mathfrak{M}^{*} \vDash \psi\left(\dot{b}^{G}, \beta_{1}, \ldots, \beta_{n}\right)$, hence $\mathfrak{M}^{*} \models \exists x \psi$.
- If $\varphi \equiv \forall x \psi$ and claim 2 holds for $\psi,\left(\Vdash_{P} \dot{\mathfrak{M}} \models \forall x \psi\left(x, \dot{\alpha_{1}}, \ldots, \dot{\alpha_{n}}\right)\right) \rightarrow$ $\left(\forall \dot{b} \in N\left(\Vdash_{P} \dot{b} \in \kappa \rightarrow \Vdash_{P} \dot{\mathfrak{M}} \equiv \psi\left(\dot{b}, \dot{\alpha_{1}}, \ldots, \dot{\alpha_{n}}\right)\right)\right.$, hence for such $\dot{b}, \mathfrak{M}^{*} \models \psi\left(\dot{b}^{G}, \beta_{1}, \ldots, \beta_{n}\right)$. Because $\check{\beta}^{G}=\beta$ and $\forall \beta \in \kappa \check{\beta} \in N$, $\mathfrak{M}^{*} \models \forall x \psi$.

Corollary 4.4 $\mathrm{BFA}_{\kappa}(P) \rightarrow \neg \Vdash_{P} \kappa$ is collapsed.
Proof: Assume for a contradiction that $\vdash_{P} \kappa$ is collapsed, i.e. there exist names $\dot{\mathfrak{M}}$ and $\dot{f}$ s.t. $\Vdash_{P} \dot{f}: \kappa \rightarrow \kappa$ is a function, $\Vdash_{P} \dot{\mathfrak{M}}=(\kappa, \dot{f})$ and $\Vdash_{P} \dot{\mathfrak{M}} \models \exists \lambda \forall \xi \exists \alpha<\lambda \dot{f}(\alpha)=\xi$. By lemma $4.3, \exists \lambda<\kappa \exists f: \lambda \xrightarrow{\text { onto }} \kappa$, a contradiction.

Definition 4.5 For any poset $P$, let $P_{p}$ denote the poset below $p$. If $B$ is a boolean algebra, $p \in B$, let $\bar{p}$ denote the (boolean) complement of $p$.

Lemma 4.6 If $B$ is a boolean algebra, then

$$
\left(\mathrm{BFA}_{\kappa}(B) \wedge \llbracket \kappa \in \operatorname{Card} \rrbracket_{B}=p\right) \rightarrow \mathrm{BFA}_{\kappa}\left(B_{p}\right)
$$

Proof: By corollary 4.4, since $\llbracket \kappa$ is collapsed $\rrbracket_{P_{\bar{p}}}=\mathbf{1}$, we get $\neg \operatorname{BFA}_{\kappa}\left(P_{\bar{p}}\right)$. So there exists a collection $\mathcal{I}=\left\{I_{i}: i<\kappa\right\}$ of macs of $P_{\bar{p}}$, each of size at most $\kappa$, such that in $\mathbf{V}$, there exists no filter on $P_{\bar{p}}$ meeting every $I_{i}, i<\kappa$. We will use $\mathcal{I}$ to show that $\mathrm{BFA}_{\kappa}\left(P_{p}\right)$ holds:

Let $\mathcal{J}=\left\{J_{i}: i<\kappa\right\}$ be an arbitrary collection of macs of $P_{p}$, each of size at most $\kappa$, and let $\mathcal{L}:=\left\{I_{i} \cup J_{i}: i<\kappa\right\}$. For each $i<\kappa, I_{i} \cup J_{i}$ is a mac of $P$ of size at most $\kappa$. By $\mathrm{BFA}_{\kappa}(P)$, there exists a filter $G \in \mathbf{V}$ meeting each $I_{i} \cup J_{i}, i<\kappa$. If for every $i<\kappa, G \cap I_{i} \neq \emptyset$, then $G \cap P_{\bar{p}}$ is a filter on $P_{\bar{p}}$ meeting every set in $\mathcal{I}$, contradicting our choice of $\mathcal{I}$. Hence there is $i<\kappa$ such that $G \cap J_{i} \neq \emptyset$. Since $\bar{p} \perp J_{i}$, this implies $\forall i<\kappa G \cap J_{i} \neq \emptyset$.

Now $G \cap P_{p}$ is a filter on $P_{p}$ meeting every set in $\mathcal{J}$. Since $\mathcal{J}$ was chosen arbitrary, this implies $\mathrm{BFA}_{\kappa}\left(P_{p}\right)$.

Lemma 4.7 [14] Assume $\Vdash_{P} " \dot{\mathfrak{M}}=(\kappa, \dot{E})$ is a well-founded structure $\wedge$ $\kappa \in \operatorname{Card"}, \operatorname{cf}(\kappa)>\omega$ and $\mathrm{BFA}_{\kappa}(P)$ holds. Then, for every sufficiently generic filter $G \subseteq P, \mathfrak{M}^{*}:=\left(\kappa, \dot{E}^{G}\right)$ is well-founded.

Proof: For each $\alpha<\kappa$, let $\dot{r_{\alpha}}$ be a name for the canonical rank function for $(\alpha, \dot{E})$, i.e. $\Vdash_{P} " \operatorname{dom}\left(\dot{r_{\alpha}}\right)=\check{\alpha} \wedge \forall \beta<\check{\alpha} \dot{r_{\alpha}}(\beta)=\sup \left\{\dot{r_{\alpha}^{\prime}}(\gamma)+1: \gamma \dot{E} \beta\right\}$ ". As $\Vdash_{P} \kappa \in$ Card, $\Vdash_{P}$ range $\left(\dot{r_{\alpha}}\right) \subseteq \kappa$, so we can find $\dot{\mathfrak{N}}$ such that $\Vdash_{P} \dot{\mathfrak{N}}=$ $\left(\kappa, \dot{E},\left(\dot{r}_{\alpha}\right)_{\alpha<\kappa}\right)$. By lemma 4.3, for each $\alpha<\kappa, \dot{r_{\alpha}}{ }^{G}$ is a rank function for $\left(\alpha, \dot{E}^{G}\right)$, witnessing that $\left(\alpha, \dot{E}^{G}\right)$ is well-founded. Since $\operatorname{cf}(\kappa)>\omega$, this implies that $\left(\kappa, \dot{E}^{G}\right)$ is well-founded, since ill-foundedness had to be witnessed by ill-foundedness on $\alpha$ for some $\alpha<\kappa$.

Theorem 4.8 (Bagaria) [2] Let $P$ be a partial ordering and $\kappa$ an infinite cardinal of uncountable cofinality. Then the following are equivalent:

1. $\mathrm{BFA}_{\kappa}(P)$
2. $\operatorname{Abs}\left(\Sigma_{1}\left(H_{\kappa^{+}}\right), P\right)$

1 $\rightarrow$ 2: Let $\varphi \in \Sigma_{1}(\{A\})$ with $A \in H_{\kappa^{+}}$, let $B:=\mathcal{B}(P)$ and assume $\llbracket \varphi(A) \rrbracket_{B}=\mathbf{1}$. We show that $\mathbf{V} \models \varphi(A)$ which suffices by lemma 1.60 :

Let $p:=\llbracket \kappa \in \operatorname{Card} \rrbracket_{B}$. By corollary $4.4, p \neq \mathbf{0}$, by lemma 4.6, $B F A_{\kappa}\left(B_{p}\right)$ holds. Since $\llbracket \varphi(A) \rrbracket_{B_{p}}=\mathbf{1}$, we can without loss of generality work with $B_{p}$ instead of $B$, i.e. we can assume that $\llbracket \kappa \in \operatorname{Card} \rrbracket_{B}=\mathbf{1}$.

Let $\lambda$ be large enough s.t. $\llbracket \dot{H}_{\lambda} \models \varphi(A) \rrbracket_{B}=\mathbf{1}$. Then there exists $\dot{\mathfrak{M}}$ such that $\llbracket \dot{\mathfrak{M}} \prec\left(\dot{H}_{\lambda}, \in,(\check{y})_{y \in \operatorname{tcl}(\{\tilde{A}\})}\right) \wedge|\dot{M}|=\kappa \rrbracket_{B}=\mathbf{1}$ in $\mathbf{V}^{B}$. There exists $\dot{E}$ such that $\llbracket\left(\dot{M}, \in,(\check{y})_{y \in \operatorname{tcl}(\{\check{A}\})}\right) \cong\left(\kappa, \dot{E},\left(\check{d}_{y}\right)_{y \in \operatorname{tcl}(\{\check{A}\})}\right) \rrbracket_{B}=\mathbf{1}$. By lemma 4.3 and lemma 4.7, using a sufficiently generic filter $G \subseteq B$ given
by $\operatorname{BFA}_{\kappa}(B), \mathfrak{M}^{*}:=\left(\kappa, \dot{E}^{G},{\check{d_{A}}}^{G}\right) \models \varphi\left({\check{d_{A}}}^{G}\right)$ and $\dot{E}^{G}$ is well-founded. Let $\mathfrak{M}^{\prime}$ be the transitive collapse of $\mathfrak{M}^{*}$ and note that by induction on $\operatorname{rank}(y)$, applying lemma 4.3 , it follows that for every $y \in \operatorname{tcl}(\{A\}), d_{y}{ }^{\mathfrak{M}^{\prime}}=y$. Hence $\mathfrak{M}^{\prime} \models \varphi(A)$ and, by upward absoluteness, $\mathbf{V} \models \varphi(A)$.
$\mathcal{Z} \rightarrow \mathbf{1}: \quad$ Suppose $I=\left\{I_{\alpha}: \alpha<\kappa\right\}$ is a collection of macs of $B:=\mathcal{B}(P)$, each of size at most $\kappa$. Let $X:=\bigcup I$. Let $\bar{B}=\left(\bar{B}, \leq_{\bar{B}}, \perp_{\bar{B}}\right)$ be an elementary substructure of $\left(B, \leq_{B}, \perp_{B}\right)$ containing $X$ of size $\kappa$. Let $K=\left(\kappa, \leq_{K}, \perp_{K}\right)$ be an isomorphic copy of ( $\bar{B}, \leq_{\bar{B}}, \perp_{\bar{B}}$ ) and let $\pi$ denote the isomorphism. Notice that for every $\alpha<\kappa, \pi^{\prime \prime} I_{\alpha}$ is a mac of $K$. Let $G$ be generic for $B$ over V. Then, $\left(H_{\kappa^{+}}\right)^{\mathbf{V}[G]} \models " \exists X \subseteq \kappa$, a filter generic for $\left\{\pi^{\prime \prime} I_{\alpha}: \alpha<\kappa\right\}$ ", which is equivalent to $" \exists X \subseteq \kappa\left(\forall x, y \in X \exists z \in X z \leq_{K} x \wedge z \leq_{K} y\right) \wedge(\forall x \in$ $\left.X \forall y \in \kappa x \leq_{K} y \rightarrow y \in X\right) \wedge\left(\forall \alpha<\kappa X \cap \pi^{\prime \prime} I_{\alpha} \neq \emptyset\right) "$. As $\leq_{K} \in H_{\kappa^{+}}$and $\left\langle\pi^{\prime \prime} I_{\alpha}: \alpha<\kappa\right\rangle \in H_{\kappa^{+}}$, this is a $\Sigma_{1}\left(H_{\kappa^{+}}\right)$sentence holding in any $B$-generic extension of $\mathbf{V}$. By $\operatorname{Abs}\left(\Sigma_{1}\left(H_{\kappa^{+}}\right), B\right)$, this sentence holds in $H_{\kappa^{+}}$.
Since $\pi \in \mathbf{V}$, this implies $\mathrm{BFA}_{\kappa}(B)$.

## Observations:

- $P \kappa^{+} c c \rightarrow\left(\operatorname{BFA}_{\kappa}(P) \leftrightarrow \mathrm{FA}_{\kappa}(P)\right)$
- $\operatorname{Abs}\left(\Sigma_{1}\left(H_{\kappa^{+}}\right), P\right) \rightarrow \operatorname{BFA}_{\kappa}(P)$ also works for $\mathrm{cf} \kappa=\omega$


## Corollary 4.9

1. $\mathrm{MA}_{\omega_{1}} \leftrightarrow \operatorname{Abs}\left(\Sigma_{1}\left(\mathrm{H}_{\omega_{2}}\right), \mathrm{ccc}\right)$
2. $\operatorname{cf} \kappa>\omega \rightarrow\left(\operatorname{MA}_{\kappa} \leftrightarrow \operatorname{Abs}\left(\Sigma_{1}\left(H_{\kappa^{+}}\right), \operatorname{ccc}\right)\right)$
3. $\left.\operatorname{Abs}\left(\Sigma_{1}\left(H_{2^{\aleph_{0}}}\right), \operatorname{ccc}\right)\right) \rightarrow \mathrm{MA}$
4. $\left(\mathrm{MA} \wedge \nexists \kappa\left(2^{\aleph_{0}}=\kappa^{+} \wedge \mathrm{cf} \kappa=\omega\right)\right) \rightarrow \operatorname{Abs}\left(\Sigma_{1}\left(H_{2^{\aleph_{0}}}\right), \mathrm{ccc}\right)$
5. $\left(\mathrm{MA} \wedge \exists \kappa\left(2^{\aleph_{0}}=\kappa^{+} \wedge \mathrm{cf} \kappa=\omega\right)\right) \rightarrow \operatorname{Abs}\left(\Sigma_{1}\left(H_{\kappa}\right), \mathrm{ccc}\right)$

Proof: 1-3 are immediate. For 4, if $2^{\aleph_{0}}$ is a limit cardinal, then the regular cardinals are unbounded in $2^{\aleph_{0}}$, hence $\mathrm{MA} \rightarrow \operatorname{Abs}\left(\Sigma_{1}\left(H_{2^{\aleph_{0}}}\right)\right.$, ccc $)$. If $\exists \kappa 2^{\aleph_{0}}=\kappa^{+} \wedge \mathrm{cf} \kappa>\omega$, then $\mathrm{MA} \rightarrow \mathrm{MA}_{\kappa} \rightarrow \operatorname{Abs}\left(\Sigma_{1}\left(H_{2^{\aleph_{0}}}\right)\right.$, ccc $)$. For 5 , if $2^{\aleph_{0}}=\kappa^{+}$and $\kappa$ is a limit cardinal, $\operatorname{Abs}\left(\Sigma_{1}\left(H_{\kappa}\right), \operatorname{ccc}\right)$ again follows by unboundedness of regular cardinals in $\kappa$.

Fact 2.72 implies the following:
Corollary 4.10 Let $\nu>\aleph_{0}$ be a regular cardinal such that $\ddagger \kappa \nu=\kappa^{+} \wedge$ $\operatorname{cf} \kappa=\omega$. Then $\operatorname{Abs}\left(\Sigma_{1}\left(H_{2^{\aleph_{0}}}\right)\right.$, ccc $\left.)+2^{\aleph_{0}}=\nu\right)$ is equiconsistent with $Z F C$.

## Corollary 4.11

- BPFA $\leftrightarrow \operatorname{Abs}\left(\Sigma_{1}\left(\mathrm{H}_{\omega_{2}}\right)\right.$, proper $)$,
- BSPFA $\leftrightarrow \operatorname{Abs}\left(\Sigma_{1}\left(\mathrm{H}_{\omega_{2}}\right)\right.$, semiproper $)$,
- $\mathrm{BMM} \leftrightarrow \operatorname{Abs}\left(\Sigma_{1}\left(\mathrm{H}_{\omega_{2}}\right)\right.$, stationary-preserving $)$.


### 4.3 PE

In this section, we will show that $\operatorname{Abs}\left(\Sigma_{1}\left(H_{\lambda^{+}}\right), P\right)$ is already implied by the seemingly weaker principle $\mathrm{PE}_{\lambda}(P)$ :

Definition 4.12 We say that $\mathfrak{A}$ has a $\mathcal{P}$-potential nontrivial endomorphism if there is a forcing notion $P \in \mathcal{P}$ such that $\Vdash_{P} " \exists f: A \rightarrow A \wedge f$ is a nontrivial homomorphism".

Definition 4.13 $\mathrm{PE}_{\lambda}(\mathcal{P})$ is the statement that for any stucture $\mathfrak{A}$ of size at most $\lambda$, if $\mathfrak{A}$ has a $\mathcal{P}$-potential nontrivial endomorphism, then $\mathfrak{A}$ has a nontrivial endomorphism.

Theorem 4.14 (Shelah) [14]
For any poset $P$ and any cardinal $\lambda$ of uncountable cofinality,

$$
\operatorname{PE}_{\lambda}(P) \leftrightarrow \operatorname{BFA}_{\lambda}(P) .
$$

Proof: $\mathrm{BFA}_{\lambda}(P) \rightarrow \mathrm{PE}_{\lambda}(P)$ follows from theorem 4.8.
For the other direction, assume that $P$ is a complete boolean algebra and let $\left(A_{i}: i \in I\right)$ be a system of $\lambda$ maximal antichains of $P$ of size at most $\lambda$. We may assume that this is a directed system, i.e. for any $i, j \in I$, there is $k \in I$ such that $A_{k}$ refines both $A_{i}$ and $A_{j}$. Let us write $i \leq j$ for " $A_{i}$ refines $A_{j}{ }^{\prime \prime}$. $(I, \leq)$ now is a partially ordered directed set. Assuming $\mathrm{PE}_{\lambda}(P)$, we will find a filter meeting all the sets $A_{i}$ :

- Let $M:=\dot{U}_{i<\lambda} A_{i}$ be the disjoint union of $\left(A_{i}, i<\lambda\right)$.
- For $i \in I, z \in A_{i}$, let $R_{i, z}:=\left\{(x, y): x, y \in A_{i}, x=y \vee x=z\right\}$.
- If $i \leq j$, let $h_{i}^{j}$ be the projection function from $A_{i}$ to $A_{j}$ : for $p \in A_{i}$, $h_{i}^{j}(p)$ is the unique element of $A_{j}$ which is compatible to (and in fact weaker than) $p$.

Note that the following holds:

- The functions $h_{i}^{j}$ commute, i.e. if $i \leq j \leq k$, then $h_{i}^{k}=h_{i}^{j} \circ h_{j}^{k}$.
- If $i \leq j$ and $p \in A_{i}$, then $p \leq h_{i}^{j}(p)$.

Let $\mathfrak{M}:=\left(M,\left(A_{i}\right)_{i \in I},\left(R_{i, z}\right)_{i \in I, z \in A_{i}},\left(h_{i}^{j}\right)_{i, j \in I, i<j}\right)$, where we treat the sets $A_{i}, R_{i, z}, h_{i}^{j}$ as relations on $M$.

If $G$ is a filter which meets all the sets $A_{i}$ and for each $i \in I, G \cap A_{i}=$ $\left\{y_{i}(G)\right\}$, then we define a function $f_{G}: M \rightarrow M$ as follows: for $x \in A_{i}$, let $f_{G}(x)=y_{i}(G)$. The following shows that $f_{G}$ is an endomorphism of $\mathfrak{M}$ :

- $A_{i}(x) \rightarrow A_{i}\left(f_{G}(x)\right)$.
- $R_{i, z}(x, y) \rightarrow R_{i, z}\left(f_{G}(x), f_{G}(y)\right)$, because $R_{i, z} \rightarrow x, y \in A_{i}$, hence $f_{G}(x)=y_{i}(G)=f_{G}(y)$.
- $y=h_{i}^{j}(x) \rightarrow f_{G}(y)=h_{i}^{j}\left(f_{G}(x)\right)$, because $y_{j}(G) \geq y_{i}(G)$ for $i<j$.

So $\mathfrak{M}$ has a potential nontrivial endomorphism, as $|\mathfrak{M}|=\lambda$, by $\mathrm{PE}_{\lambda}(P)$ we know that $\mathfrak{M}$ really has a nontrivial endomorphism. Finally we will show how a nontrivial endomorphism $F$ of $\mathfrak{M}$ defines a filter $G^{*}$ meetings all the sets $A_{i}$ : Choose $y_{0}=f\left(x_{0}\right) \neq x_{0}, x_{0} \in A_{i_{0}}$. Then we prove the following:

1. $\forall j \leq i_{0} F \upharpoonright A_{j} \neq \mathrm{id} \upharpoonright A_{j}$.

- If $h_{j}^{i_{0}}(x)=x_{0}$, then $h_{j}^{i_{0}}(F(x))=y_{0}$, so $F(x) \neq x$.

2. $\forall j \leq i_{0} F \upharpoonright A_{j}$ is constant; let $p_{j}$ denote the value of $F$ on $A_{j}$.

- Let $x \in A_{j}, F(x) \neq x$. Then for all $y \in A_{j}$, we have $(x, y) \in R_{j, x}$, so $(F(x), F(y)) \in R_{j, x}$. Because $F(x) \neq x, F(x)=F(y)$ follows.

3. $\left\{p_{j}: j \leq i_{0}\right\}$ generates a filter $G^{*}$ meeting all sets $A_{i}$. (which proves the theorem)

- If $j \leq i \leq i_{0}$, then $h_{j}^{i}\left(p_{j}\right)=p_{i}$ and $p_{j} \leq p_{i}$. Since $\left\{j \in I: j \leq i_{0}\right\}$ is directed, $\left\{p_{j}: j \leq i_{0}\right\}$ is directed and generates a filter $G^{*}$. For any $i \in I$ there is $j \leq i$ satisfying $j \leq i_{0}$, so $A_{i} \cap G^{*}=\left\{h_{j}^{i}\left(p_{j}\right)\right\}$.

Observation: $\quad$ Since by theorem $4.8, B F A_{\lambda}(P) \leftrightarrow \operatorname{Abs}\left(\Sigma_{1}\left(H_{\lambda^{+}}\right), P\right)$, it follows that for any partial order $P$ and any cardinal $\lambda$ of uncountable cofinality,

$$
P E_{\lambda}(P) \leftrightarrow \operatorname{Abs}\left(\Sigma_{1}\left(H_{\lambda^{+}}\right), P\right) .
$$

## 5 The Consistency Strength of BPFA

In the following theorems, we will show that BPFA is equiconsistent with a reflecting cardinal. In the first theorem, we will show that a model with a reflecting cardinal allows us to construct a forcing extension of that model in which $\operatorname{Abs}\left(\Sigma_{1}\left(\mathrm{H}_{\omega_{2}}\right)\right.$, proper $)$ holds, and therefore BPFA holds by theorem 4.8. The proof of this theorem is based on the proof given in [14], where it is shown that a model with a reflecting cardinal allows us to construct a forcing extension of that model in which PE(proper) holds, yielding BPFA to hold by theorem 4.14 above:

Lemma 5.1 Let $P$ be a forcing notion, $P \in H_{\lambda}$ and $\chi>2^{\lambda}$ regular. Then $P$ is proper iff $H_{\chi} \models P$ proper.

Proof: Since $H_{\lambda} \in H_{\chi}$, the description of properness in fact 2.15, 2 is absolute between $H_{\chi}$ and $\mathbf{V}$.

Lemma 5.2 Assume that $\kappa$ is reflecting, $\lambda<\kappa$ is a regular cardinal, $\varphi(x)$ is a $\Sigma_{1}$-formula and $a \in H_{\lambda}$. If there exists a proper forcing notion $P$ such that $\Vdash_{P} \varphi(\check{a})$, then there is such a proper forcing notion in $H_{\kappa}$.

Proof: Fix $P$, let $\chi$ be sufficiently large s.t. $\chi$ regular, $H_{\chi} \models " P$ proper, $\exists \mu P \in H_{\mu}, 2^{\mu}$ exists, $\Vdash_{P} \varphi(\check{a}) "$. Now we use the fact that $\kappa$ is reflecting: we can find $\delta<\kappa, \delta>\lambda, \delta$ regular such that $H_{\delta} \vDash " \exists \nu \exists Q \in H_{\nu}$ Q proper $\wedge$ $\vdash_{Q} \varphi(\check{a}) \wedge 2^{\nu}$ exists." Fix such $Q$. By lemma $5.1, Q$ is really proper. By corollary $2.44, \Vdash_{Q} \quad H_{\delta} \models \varphi(\check{a})$. Since $\varphi$ is $\Sigma_{1}$, it is upwards absolute and hence $\Vdash_{Q} \varphi(\check{a})$.

Lemma 5.3 Let $P \in H_{\kappa}$. Then " $\kappa$ reflecting" implies " $\Vdash_{P} \kappa$ reflecting".
Proof: Let $P \in H_{\lambda}, \lambda<\kappa$. Assume that $\Vdash_{P} " H_{\chi} \models \varphi(\dot{a}) \wedge \dot{a} \in H_{\kappa}$. By lemma 2.42, we may assume that $\dot{a} \in H_{\kappa}$. By corollary 2.44, we have $H_{\chi} \vDash \Vdash_{P} \varphi(\dot{a})$, so - because $\kappa$ is reflecting - there is $\delta<\kappa, \delta>\lambda$ such that $H_{\delta} \models \Vdash_{P} \varphi(\dot{a})$, hence by corollary 2.44, $\Vdash_{P} H_{\delta} \models \varphi(\dot{a})$. Because $|P|<\lambda<\delta, \delta$ remains a cardinal in any $P$-generic extension.

Theorem 5.4 (Shelah) [14]

$$
\operatorname{Con}(\exists \kappa \text { reflecting }) \rightarrow \operatorname{Con}\left(\operatorname{Abs}\left(\Sigma_{1}\left(\mathrm{H}_{\omega_{2}}\right), \text { proper }\right)\right) \leftrightarrow \operatorname{Con}(\mathrm{BPFA}) .
$$

Proof: The equivalence on the right follows from theorem 4.8 above. For the first implication, assume that $\kappa$ is reflecting. We define a countable support iteration ( $P_{i}, \dot{Q}_{i}: i<\kappa$ ) of proper forcing notions with the following properties for all $i<\kappa$ :

1. $P_{i} \in H_{\kappa}$.
2. $\dot{Q}_{i}$ is a $P_{i}$-name, $\Vdash_{i} " \dot{Q}_{i}$ is proper, $\dot{Q}_{i} \in H_{\kappa} "$.
3. $\Vdash_{i} 2^{\aleph_{1}}<\kappa$.
4. At each stage $i$ we choose a $\Sigma_{1}$ formula $\varphi_{i}(x)$ and some $\dot{a}_{i}$ such that $\Vdash_{i} \dot{a}_{i} \in \mathrm{H}_{\omega_{2}}$, such that if for some stage $i_{0}$ there are $\varphi(x) \in \Sigma_{1}(x)$ and $\dot{a}$ such that $\Vdash_{i_{0}} \dot{a} \in \mathrm{H}_{\omega_{2}}$, then there is some $j>i_{0}$ such that $\varphi_{j} \equiv \varphi$ and $\Vdash_{j} \dot{a}_{j}=\dot{a}$.
5. $\Vdash_{i}$ "if there is a proper forcing notion in $H_{\kappa}$ forcing $\varphi_{i}\left(\dot{a}_{i}\right)$ to hold, then $\dot{Q}_{i}$ is such a forcing notion".
6. $P_{\kappa} \models \kappa$-сс.
7. Whenever $\dot{b}$ is a $P_{\kappa}$-name for an element of $\mathrm{H}_{\omega_{2}}$, then there are $i<\kappa$ and a $P_{i}$-name $\dot{a}$ such that $\Vdash_{\kappa} \dot{b}=\dot{a}$.

Such construction is possible:

- 2 follows from 5 letting $\dot{Q}_{i}$ be a name for a trivial forcing notion if $\Vdash_{i}$ "there is no proper forcing notion in $H_{\kappa}$ forcing $\varphi\left(\dot{a}_{i}\right)$ to hold".
- Let $P_{0}$ be a trivial forcing. Then 1 follows by induction on $i<\kappa$ - if $P_{i} \in H_{\kappa}$ and $\Vdash_{i} \dot{Q}_{i} \in H_{\kappa}$, then $\left|P_{i} * \dot{Q}_{i}\right|<\kappa$ and hence $P_{i+1} \in H_{\kappa}$ : First, we may, since $\kappa$ is inaccessible, assume without loss of generality, that $\Vdash_{i} \dot{Q}_{i} \subseteq \lambda_{i}$ for some $\lambda_{i}<\kappa$. To each $\dot{x}$ such that $\Vdash_{i} \dot{x} \in \dot{Q}_{i}$, we assign a function $f_{\dot{x}}: P_{i} \rightarrow \lambda_{i}+1$ such that for any $p \in P_{i}$

1. $p \Vdash \dot{x}=\gamma \rightarrow f_{\dot{x}}(p)=\gamma$
2. $\left(\nexists \gamma<\lambda_{i} p \Vdash \dot{x}=\gamma\right) \rightarrow f_{\dot{x}}(p)=\lambda_{i}$

Now if $f_{\dot{x_{0}}}=f_{x_{1}}$, then $\Vdash_{i} \dot{x_{0}}=\dot{x_{1}}$. As there are less than $\kappa$-many such functions ( $\kappa$ is inaccessible), this implies $\left|P_{i+1}\right|<\kappa$.
If $i$ is a limit ordinal, $i<\kappa$, then because $P_{i}$ is a countable support iteration, $\left|P_{i}\right| \leq\left(\sup _{\alpha<i}\left|P_{\alpha}\right|\right)^{\aleph_{0}} \cdot i^{\aleph_{0}}<\kappa$, hence $P_{i} \in H_{\kappa}$.

- 3 follows from 1 and lemma 5.3 , because they imply $\Vdash_{i} \kappa$ inaccessible.
- For 4, note that $\left|\mathrm{H}_{\omega_{2}}\right|=2^{\aleph_{1}}$, hence $\Vdash_{i}$ "there are only $2^{\aleph_{1}}$ formulas with parameter in $\mathrm{H}_{\omega_{2}} "$. By $3, \Vdash_{i} 2^{\aleph_{1}}<\kappa$.
- 6 follows from 1 using theorem 2.36.
- For 7, note that by theorem 2.35, since $P_{\kappa}$ is proper, it does not collapse $\omega_{1}$; by $7, P_{\kappa}$ does not collapse $\kappa$; it follows that $\vdash_{\kappa} \omega_{1}<\kappa$. Let $\dot{b}$ be a $P_{\kappa}$-name for an element of $\mathrm{H}_{\omega_{2}}$. We can assume that $\vdash^{\kappa} \dot{b} \subseteq \omega_{1}$, since we can code every element of $\mathrm{H}_{\omega_{2}}$ into a subset of $\omega_{1}$ (similar to definition 1.30) and the decoding is $\Delta_{1}$ (similar to lemma 1.31). Furthermore we can assume that $\dot{b}$ is a nice name of the form $\dot{b}=$ $\bigcup_{\alpha<\omega_{1}}\{\check{\alpha}\} \times A_{\alpha}$ where each $A_{\alpha}$ is an antichain of $P_{\kappa}$. Since by 6 , every antichain of $P_{\kappa}$ is of size $<\kappa$, it follows that $\dot{b} \in H_{\kappa}$. Since $P_{\kappa}$ is a countable support iteration and $\mathrm{cf} \kappa>\omega$, each condition in the $P_{\kappa}$-name $\dot{b}$ has bounded support in $\kappa$, hence there is $\gamma<\kappa$ such that $s(p) \subset \gamma$ for each condition $p$ appearing in $\dot{b}$. Let $\dot{b}^{\prime}$ be the $P_{\gamma}$-name obtained from $\dot{b}$ by replacing each condition $p$ appearing in $\dot{b}$ by $p \upharpoonright \gamma$. It follows that $\Vdash^{\kappa} \dot{b}^{\prime}=\dot{b}$.

From these properties we can now show $\Vdash_{\kappa} \operatorname{Abs}\left(\Sigma_{1}\left(\mathrm{H}_{\omega_{2}}\right)\right.$, proper $)$ : Let $\dot{a}$ be a $P_{\kappa}$-name for an element of $\mathrm{H}_{\omega_{2}}$, let $\varphi(x)$ be a $\Sigma_{1}$ formula, let $Q$ be a $P_{\kappa}$-name for a proper forcing notion and assume that

$$
\vdash_{\kappa} " \Vdash_{\dot{Q}} \varphi(\dot{a}) "
$$

By 7, we may assume that for some large enough $i<\kappa$, $\dot{a}$ is a $P_{i}$-name. By 4, we may assume $\Vdash_{\kappa} \dot{a}=\dot{a_{i}}$. Now letting $R$ be the $P_{i}$-name $\left(P_{\kappa} / G_{i}\right) * \dot{Q}$, applying fact 2.40 , we get

$$
\Vdash_{i} " \vdash_{R} \varphi_{i}\left(\dot{a_{i}}\right) "
$$

By lemma $5.3, \Vdash_{i} \kappa$ reflecting, so by the definition of $\dot{Q}_{i}$, by lemma 5.2 and since $R$ is proper (by fact 2.41), we get that $\Vdash_{i+1} \varphi_{i}\left(\dot{a}_{i}\right)$. So by lemma 2.33, since $\varphi_{i}$ is $\Sigma_{1}($ and hence upwards absolute $), \Vdash_{\kappa} \varphi_{i}\left(\dot{a_{i}}\right)$, hence $\Vdash_{\kappa} \varphi(\dot{a})$.
[14] also mentions that by a similar proof, the following can be shown:
Fact 5.5 [14]

$$
\operatorname{Con}(\exists \kappa \text { reflecting }) \rightarrow \operatorname{Con}(\mathrm{PE}(\text { semiproper })) \rightarrow \operatorname{Con}(\mathrm{BSPFA})
$$

Theorem 5.6 (Shelah) [14] If BPFA holds, then $\omega_{2}$ is reflecting in $\mathbf{L}$.
Proof: Assume BPFA. Let $\kappa:=\omega_{2}$. It is clear that $\kappa$ is regular in $\mathbf{L}$.
Claim 5.7 Without loss of generality we may assume:

1. $0^{\sharp}$ does not exist, i.e. the covering lemma holds for $\mathbf{L}$.
2. $\aleph_{2}{ }^{\aleph_{1}}=\aleph_{2}$.
3. There is $A \subseteq \omega_{2}$ such that

$$
\left(x \subset \mathbf{O r d} \wedge|x| \leq \aleph_{1}\right) \rightarrow x \in \mathbf{L}[A] .
$$

Proof:

1. If $0^{\sharp}$ exists, then $L_{\kappa} \prec \mathbf{L}$ (see 1.77), hence $\kappa$ is reflecting in $\mathbf{L}$ (see 1.69).
2. Let $Q:=\operatorname{coll}\left(\aleph_{2},\left\{\aleph_{2}^{\aleph_{1}}\right\}\right)$. Since $Q$ is $\omega_{2}$-closed, $Q$ is proper and $Q$ adds no new sets of size $\aleph_{1}$, hence $\mathbf{V}^{Q} \models \operatorname{Abs}\left(\Sigma_{1}\left(\mathrm{H}_{\omega_{2}}\right)\right.$, proper) and hence $\mathbf{V}^{Q} \models$ BPFA. Moreover, $\aleph_{2}^{\mathbf{V}}=\aleph_{2}^{\mathbf{V}^{Q}}$ and $\mathbf{V}^{Q} \models \aleph_{2}^{\aleph_{1}}=\aleph_{2}$.
3. Assuming 2 , there are only $\aleph_{2}$ functions from $\omega_{1}$ to $\omega_{2}$, so we can code every function from $\omega_{1}$ to $\omega_{2}$ into $A \subseteq \omega_{2}$; it follows that $\forall f: \omega_{1} \rightarrow$ $\omega_{2} f \in \mathbf{L}[A]$ and hence $\aleph_{2}^{\mathbf{L}[A]}=\aleph_{2}$.
Now, assuming 1 , every set $x$ of ordinals of size $\leq \aleph_{1}$ can be covered by some $y \in \mathbf{L},|y|=\aleph_{1}$. Let $j: y \rightarrow \operatorname{ot}(y)$ be order-preserving, then $j[x] \in \mathbf{L}[A]$, since by the above, every bounded subset of $\omega_{2}$ is already in $\mathbf{L}[A]$ (and $j[x]$ is such). Since $j$ (and hence $j^{-1}$ ) $\in \mathbf{L}$, it follows that $x \in \mathbf{L}[A] . \square_{\text {Claim } 5.7}$

Proof of theorem 5.6 continued:
Let $\varphi(x)$ be a formula, $a \in L_{\kappa}$ and assume that $\chi>\kappa, L_{\chi} \models \varphi(a)$ and $\chi$ is a regular cardinal in $\mathbf{L}$. By lemma 1.71 we may assume that $\chi$ is a (regular) cardinal in $\mathbf{L}[A]$ or even in $\mathbf{V}$. We have to find an $\mathbf{L}$-cardinal $\chi^{\prime}<\kappa$ such that $a \in L_{\chi^{\prime}}$ and $L_{\chi^{\prime}} \models \varphi(a)$ :

Let $Q_{0}:=\operatorname{coll}\left(\aleph_{1},\left\{L_{\chi}[A]\right\}\right)$ i.e. the set of countable partial functions from $\omega_{1}$ to $L_{\chi}[A]$ ordered by extension (note that we only defined $\operatorname{coll}(\gamma, S)$ for $S \subset \mathbf{O r d}$, but exactly the same definition works for $S \subset \mathbf{V}$ ).

In $\mathbf{V}^{Q_{0}}$, let $T$ be the following tree: Elements of $T$ are of the form $\left(\left\langle\mu_{i}: i<\alpha\right\rangle,\left\langle f_{i j}: i \leq j<\alpha\right\rangle\right)$, abbreviated as $\left\langle\mu_{i}, f_{i j}: i \leq j<\alpha\right\rangle$, where the $\mu_{i}$ are ordinals less than $\chi$, the $f_{i j}$ are a system of commuting orderpreserving embeddings ( $f_{i j}: \mu_{i} \rightarrow \mu_{j}$ ) and $\alpha<\omega_{1} . T$ is ordered by the relation "is an initial segment of".

If $B$ is a branch of $T$ (in $\mathbf{V}^{Q_{0}}$ or any bigger universe) of length $\delta_{B}$, then $B$ defines a directed system $\left\langle\mu_{i}, f_{i j}: i \leq j<\delta_{B}\right\rangle$ of well-orders. Let
$\left(\gamma_{B},<_{B}\right)$ be the direct limit of this system. If $\delta_{B}=\omega_{1}$, then $\left(\gamma_{B},<_{B}\right)$ is a well-order, since any infinite descending sequence in the direct limit would already appear in some node of $T$ ( $\omega_{1}$ is regular), contradicting wellfoundedness of $\left(\mu_{i}, \in\right)$ for some $i<\omega_{1}$.

Let $Q_{1}:=P_{T}$ be the forcing "sealing the $\omega_{1}$-branches of $T$ " as defined in theorem 2.58 , which is possible since $T$ fulfills all the necessary requirements, i.e. $T$ has height $\omega_{1}$, has uncountably many $\omega_{1}$-branches and every node of $T$ is on some $\omega_{1}$-branch. We let $P:=Q_{0} * Q_{1}$. $P$ is proper.

In $\mathbf{V}^{P}$, we define a model $\mathfrak{M}$ as follows: Let $\Omega$ be a sufficiently large regular cardinal of $\mathbf{V}^{P}$ and let $(M, \in)$ be an elementary submodel of $\left(H_{\Omega} \mathbf{V}^{P}, \in\right)$ of size $\aleph_{1}$ containing all necessary information, in particular $M \supseteq L_{\chi}[A]$, which is possible since in $\mathbf{V}^{P},\left|L_{\chi}[A]\right|=\aleph_{1}$.

We now expand $(M, \in)$ to $\mathfrak{M}$ by adding the following functions, relations and constants:

- a constant for each element of $L_{\xi}$, where $\xi$ is chosen such that $a \in L_{\xi}$;
- relations $M_{0}, M_{1}$ which are interpreted as $M \cap H_{\Omega}{ }^{\mathbf{V}}, M \cap H_{\Omega} \mathbf{V}^{Q_{0}}$;
- constants $\chi, A, \kappa, T, g, b$, where $b$ is the function enumerating the branches of $T$ and $g$ is the specializing function $g: T \rightarrow \omega$, both defined in theorem 2.58;
- a function $c: \chi \times \omega_{1} \rightarrow \chi$ such that, for all $\delta<\chi$, if $\operatorname{cf}(\delta)=\aleph_{1}$, then $c(\delta, \cdot): \omega_{1} \rightarrow \delta$ is increasing and cofinal in $\delta$.

Since $M$, the underlying set of $\mathfrak{M}$, has cardinality $\aleph_{1}$, we can find an isomorphic model $\overline{\mathfrak{M}}=\left(\omega_{1}, \bar{E}, \bar{\chi}, \ldots\right)$. In $\mathbf{V}$, we have names for all those elements of $\mathbf{V}^{P}: \dot{\mathfrak{M}}, \dot{\bar{E}}, \dot{\bar{\chi}}, \ldots$; because BPFA holds, by lemma 4.3 , there exists a sufficiently generic filter $G \in \mathbf{V}$ for $\overline{\mathfrak{M}}$, such that, applying lemma 4.7:

1. $\left(\omega_{1}, \dot{\bar{E}}^{G}\right)$ is well-founded.
2. If $\psi$ is a closed formula such that $\Vdash_{P} \mathfrak{M} \mid=\psi$, then $\dot{\mathfrak{M}}^{G} \models \psi$.

Now let $\mathfrak{M}^{\prime}:=\left(M^{\prime}, \in, \chi^{\prime}, \ldots\right)$ be the Mostowski collapse of $\dot{\mathfrak{M}}^{G}$ and let $\mathfrak{M}_{0}^{\prime}:=\left(M_{0}^{\prime}, \in\right)$ and $\mathfrak{M}_{1}^{\prime}:=\left(M_{1}^{\prime}, \in\right)$ be inner models of $\mathfrak{M}^{\prime}$.

## Claim 5.8

1. $\vdash_{P} \mathfrak{M} \vDash " \mathbf{L}[A] \models \kappa=\aleph_{2} "$
2. $\aleph_{1} \mathfrak{M}^{\prime}=\aleph_{1} \mathbf{V}$

Proof: Since $\mathfrak{M}$ is elementary in some sufficiently large $H_{\Omega}$ and $\mathbf{L}[A]$ is absolute between $\mathbf{V}^{P}$ and $\mathbf{V}, 1$ is equivalent to $\mathbf{L}[A] \models \kappa=\aleph_{2}$, which was one of our assumptions in claim 5.7. 2 now follows from 1 :

- $\Vdash_{P} \mathfrak{M} \models " \mathbf{L}[A] \models \kappa=\aleph_{2} " \rightarrow \Vdash_{P} \mathfrak{M} \models \kappa \leq \aleph_{2}$
- $\Vdash_{P} \aleph_{1} \mathbf{V} \in$ Card, so $\Vdash_{P} \mathfrak{M} \models \aleph_{1} \mathbf{V} \in \mathbf{C a r d}$

Since $\aleph_{1} \mathbf{V}<\kappa, \Vdash_{P} \mathfrak{M} \models \aleph_{1}=\aleph_{1} \mathbf{V}$, hence $\aleph_{1}{ }^{\mathfrak{M}}=\aleph_{1}{ }^{\mathbf{V}}$. $\square_{\text {Claim 5.8 }}$
With the following two claims, we will finish the proof of theorem 5.6:
Claim 5.9 $L_{\chi^{\prime}} \models \varphi(a)$.
Claim 5.10 $\mathbf{L} \vDash \chi^{\prime} \in$ Card.
Proof of Claim 5.9: $\chi^{\prime} \in M^{\prime}$, since $\mathfrak{M}^{\prime}$ satisfies a large fragment of $Z F C$, this implies $L_{\chi^{\prime}} \subseteq M^{\prime}$. For each $y \in L_{\xi}$, let $c_{y}$ be the associated constant symbol. By induction on $\operatorname{rank}(y), \forall y \in L_{\xi} y=c_{y}{ }^{\mathfrak{M}}$. Since $\Vdash_{P} \mathfrak{M} \models " L_{\chi} \models \varphi(a)$ ", we thus have $\mathfrak{M}^{\prime} \models " L_{\chi^{\prime}} \models \varphi(a)$ ". But $L_{\chi^{\prime}} \subseteq M^{\prime}$, so $L_{\chi^{\prime}} \models \varphi(a)$. $\square_{\text {Claim } 5.9}$

Proof of Claim 5.10: In $\mathbf{L}\left[A^{\prime}\right]$, let $\mu$ be the cardinality of $\chi^{\prime}$ and - again in $\mathbf{L}\left[A^{\prime}\right]$ - let $\nu:=\mu^{+}$. We will use the following to finish our proof:

Claim 5.11 $\nu \subset M^{\prime}$.
We will show that $\chi^{\prime}$ is a cardinal in $\mathbf{L}\left[A^{\prime}\right]$, which is clearly sufficient: Assume $\chi^{\prime} \notin \operatorname{Card}^{\mathbf{L}\left[A^{\prime}\right]}$, then $\mu<\chi^{\prime}$ and since $\nu$ is a cardinal in $\mathbf{L}\left[A^{\prime}\right]$, we can find $\gamma<\nu$ such that $L_{\gamma}\left[A^{\prime}\right] \vDash \exists f: \mu \xrightarrow{\text { onto }} \chi^{\prime}$. By claim 5.11, $\gamma \in M^{\prime}$, so by the absoluteness properties of relative constructibility and because $\mathfrak{M}^{\prime}$ satisfies a large fragment of $Z F C$, we have $L_{\gamma}\left[A^{\prime}\right] \subseteq M^{\prime}$, so $\mathfrak{M}^{\prime} \models \mathbf{L}\left[A^{\prime}\right] \models \chi^{\prime} \notin$ Card. But we also have $\Vdash_{P} \mathfrak{M} \models \mathbf{L}[A] \models \chi \in \mathbf{C a r d}$ by our choice of $\chi$, a contradiction. $\square_{\text {Claim } 5.10}$

Claim 5.12 Assume $\mu<\chi^{\prime}$. Then the following hold:

1. $\forall \delta\left(\operatorname{cf}^{\mathbf{L}[A]}(\delta)>\aleph_{0} \rightarrow \operatorname{cf}(\delta)>\aleph_{0}\right)$
2. $\Vdash_{P} " \forall \delta<\chi\left(\operatorname{cf}^{\mathbf{L}[A]}(\delta)>\aleph_{0} \rightarrow \operatorname{cf}(\delta)=\aleph_{1}\right) "$
3. $\left(\mathfrak{M}^{\prime} \models \operatorname{cf}^{\mathbf{L}\left[A^{\prime}\right]}(\mu)>\aleph_{0}\right) \rightarrow\left(\mathfrak{M}^{\prime} \models \operatorname{cf}(\mu)=\aleph_{1}\right)$
4. $\left(\mathfrak{M}^{\prime} \models \operatorname{cf}(\mu)=\aleph_{1}\right) \rightarrow \operatorname{cf}(\mu)=\aleph_{1}$

Proof:

1. Follows directly from our choice of $A$.
2. Use 1 and the fact that, since $P$ is proper, $\operatorname{cf}(\delta)>\aleph_{0} \rightarrow \vdash_{P} \operatorname{cf}(\delta)>\aleph_{0}$ : For a contradiction, assume that $\operatorname{cf}(\delta)>\aleph_{0}$ and in some $P$-generic extension $\mathbf{V}[H], \operatorname{cf}(\delta)=\aleph_{0}$, so there exists a sequence $s$ of length $\omega$ which is cofinal in $\delta$. In $\mathbf{V}$, let $S$ be the set of all countable subsets of $\delta$, i.e. $S:=[\delta]^{\aleph_{0}}$. Clearly, $S$ is stationary in $[\delta]^{\aleph_{0}}$. In $\mathbf{V}[H]$, let $C:=\{c: c \supseteq s\} \cap[\delta]^{\aleph_{0}}$. Clearly, $C$ is club in $[\delta]^{\aleph_{0}}$ and $C \cap S=\emptyset$, since not set in $C$ could be an element of $\mathbf{V}$ for this would imply $\operatorname{cf}(\delta)=\aleph_{0}$. Now since $C \cap S=\emptyset, P$ cannot be proper, a contradiction.
Since $\delta<\chi$ and $\Vdash_{P} \chi<\aleph_{2}$, it follows that $\Vdash_{P} \operatorname{cf}(\delta)=\aleph_{1}$.
3. Follows from 2 , the fact that $\mathfrak{M} \prec H_{\Omega}$ for some sufficiently large $\Omega$, and the fact that $\mu<\chi^{\prime}$.
4. If $\mathfrak{M}^{\prime} \models \operatorname{cf}(\mu)=\aleph_{1}$, then the function $c^{\prime}(\mu, \cdot)$ is increasing and cofinal in $\mu$ in length $\omega_{1} \mathfrak{M}^{\prime}=\omega_{1}{ }^{\mathbf{V}}$, see claim 5.8, 2. $\square_{\text {Claim 5.12 }}$

Proof of Claim 5.11: We will distinguish two cases, according to $\operatorname{cf}(\mu)$ :

1. $\operatorname{cf}(\mu)=\aleph_{0}$ :

From claim 5.12, 3 and 4, we get $\mathfrak{M}^{\prime} \models \mathbf{L}\left[A^{\prime}\right] \models \operatorname{cf}(\mu)=\aleph_{0}$. Let $\mathfrak{M}^{\prime} \models " \nu_{1}$ is the $\mathbf{L}\left[A^{\prime}\right]$-successor of $\mu "$. We will show that $\nu_{1}=\nu$, which suffices since $M^{\prime}$ is transitive.

Assume for a contradiction that $\nu_{1}<\nu$. Working in $\mathbf{L}\left[A^{\prime}\right]$, we have $\left|[\mu]^{\aleph_{0}}\right|=\nu$, since $\left|[\mu]^{\aleph_{0}}\right| \leq 2^{\mu}=\nu$ and $\left|[\mu]^{\aleph_{0}}\right|=\mu^{\aleph_{0}} \geq \nu$ since cf $(\mu)=$ $\aleph_{0}$. Furthermore $\left|L_{\nu_{1}}\left[A^{\prime}\right]\right|=\nu_{1}<\nu$. So we can find $y \in[\mu]^{\aleph_{0}}$ such that $y \in L_{\gamma}\left[A^{\prime}\right] \backslash L_{\nu_{1}}\left[A^{\prime}\right]$ for some $\gamma<\nu$.
Working in $\mathbf{V},|\gamma|=\aleph_{1}$, hence $\left|L_{\gamma}\left[A^{\prime}\right]\right|=\aleph_{1}$, so let $L_{\gamma}\left[A^{\prime}\right]=\bigcup_{i<\omega_{1}} X_{i}$, where $\left\langle X_{i}: i<\omega_{1}\right\rangle$ is a continuous increasing chain of countable elementary submodels of $L_{\gamma}\left[A^{\prime}\right]$ with $y, A^{\prime} \in X_{0}$. In $\mathfrak{M}_{1}^{\prime}$, we can find a continuous increasing sequence $\left\langle Y_{i}: i<\omega_{1}\right\rangle$ of countable elementary submodels of $L_{\mu}\left[A^{\prime}\right]$ with $\bigcup_{i<\omega_{1}} Y_{i}=L_{\mu}\left[A^{\prime}\right]$ and $A^{\prime} \in Y_{0}$ (note that $\mathfrak{M}_{1}^{\prime} \models \mu<\chi^{\prime}$, hence $\left.\mathfrak{M}_{1}^{\prime} \models|\mu|=\aleph_{1}\right)$.
Because both sequences $X_{i}$ and $Y_{i}$ are club in $\left[L_{\mu}\left[A^{\prime}\right]\right]^{\aleph_{0}}$, we can find $i<\omega_{1}$ such that $X_{i} \cap L_{\mu}\left[A^{\prime}\right]=Y_{i}$. Fix such $i$ :
Let $j:\left(X_{i}, \in, A^{\prime}, Y_{i}\right) \rightarrow\left(L_{\hat{\gamma}}[\hat{A}], \in, \hat{A}, L_{\hat{\mu}}[\hat{A}]\right)$ be the collapsing isomorphism ( $A^{\prime}$ and $Y_{i}$ viewed as unary predicates). Note that $Y_{i}=X_{i} \cap$ $L_{\mu}\left[A^{\prime}\right]$ is a transitive subset of $X_{i}$ (i.e. $x \in Y_{i} \wedge z \in x \cap X_{i} \rightarrow z \in Y_{i}$ ), so $j\left\lceil Y_{i}\right.$ is the Mostowski collapse of $\left(Y_{i}, \in\right)$; since $Y_{i} \in \mathfrak{M}_{1}^{\prime}$ and $\mathfrak{M}_{1}^{\prime}$ satisfies a large fragment of $Z F C$, it follows that $j \upharpoonright Y_{i} \in M_{1}^{\prime}$ and $\hat{A} \in M_{1}^{\prime}$. Also, $j(y) \in L_{\hat{\gamma}}[\hat{A}] \subseteq M_{1}^{\prime}$ (since $\hat{\gamma}<\aleph_{1}$ ), so we can compute $y=\left\{\alpha:\left(j \upharpoonright Y_{i}\right)(\alpha) \in j(y)\right\}$ in $\mathfrak{M}_{1}^{\prime}$ (since $y \subseteq \mu<\chi^{\prime} \in M_{1}^{\prime}$ ). But

$$
\mathfrak{M}^{\prime} \models "[\mu]^{\aleph_{0}} \cap M_{1}^{\prime}=[\mu]^{\aleph_{0}} \cap M_{0}^{\prime}=[\mu]^{\aleph_{0}} \cap \mathbf{L}\left[A^{\prime}\right] " .
$$

The first equality holds because $Q_{0}$ is a $\sigma$-closed forcing notion and hence does not add any new countable sets, the second because of our assumption in claim 5.7, 3:

$$
\Vdash_{P} \mathfrak{M} \models "\left[|\chi|^{\mathbf{L}[A]}\right]^{\aleph_{0}} \cap M_{0}=\left[|\chi|^{\mathbf{L}[A]}\right]^{\aleph_{0}} \cap \mathbf{L}[A] "
$$

holds because $\mathfrak{M} \prec H_{\Omega}$ for some sufficiently large $\Omega$ and because, by the above-mentioned assumption, $\left[|\chi|^{\mathbf{L}[A]}\right]^{\aleph_{0}}=\left[|\chi|^{\mathbf{L}[A]}\right]^{\aleph_{0}} \cap \mathbf{L}[A]$.
Hence $\mathfrak{M}^{\prime} \models y \in \mathbf{L}\left[A^{\prime}\right]$, so, since $y \subseteq \mu$ and $\mathbf{L}\left[A^{\prime}\right] \vDash \mu^{+}=\nu_{1}$, $\mathfrak{M}^{\prime} \equiv y \in L_{\nu_{1}}\left[A^{\prime}\right]$, a contradiction to our choice of $y$.
2. $\operatorname{cf}(\mu)=\aleph_{1}$ :

We let $\gamma<\nu$ and show that $\gamma \in M^{\prime}$ :
Since $\mathbf{L}\left[A^{\prime}\right] \vDash|\gamma|=\mu$, in $\mathbf{L}\left[A^{\prime}\right]$ we can find an increasing sequence $\left\langle A_{\xi}: \xi<\mu\right\rangle$ such that $\gamma=\bigcup_{\xi<\mu} A_{\xi}$ where each $A_{\xi}$ has cardinality $<\mu$ in $\mathbf{L}\left[A^{\prime}\right]$. Let $\alpha_{\xi}:=\operatorname{ot}\left(A_{\xi}\right)$ for each $\xi<\mu$, then the inclusion maps from $A_{\xi}$ into $A_{\zeta}$ naturally induce order-preserving functions $f_{\xi \zeta}: \alpha_{\xi} \rightarrow \alpha_{\zeta}$.
Let $B:=\left\langle\alpha_{\xi}, f_{\xi \zeta}: \xi \leq \zeta<\mu\right\rangle$ and write $B \upharpoonright \beta$ for $\left\langle\alpha_{\xi}, f_{\xi \zeta}: \xi \leq \zeta<\beta\right\rangle$, $\beta<\mu$. Like above, it follows that the direct limit of the system $B$ is a well-ordered set and has order type $\gamma$, since each element of $\gamma$ is canonically connected to some element of the direct limit and vice versa. $B \in \mathbf{L}\left[A^{\prime}\right]$, but each initial segment $B \upharpoonright \beta$ is already in $L_{\mu}\left[A^{\prime}\right]$, since it can be coded by a bounded subset of $\mu$. Since $L_{\mu}\left[A^{\prime}\right] \subseteq$ $L_{\chi^{\prime}}\left[A^{\prime}\right] \subseteq M_{1}^{\prime}$, we know that $\forall \beta<\mu B \upharpoonright \beta \in M_{1}^{\prime}$.
In $\mathfrak{M}_{1}^{\prime}, \operatorname{cf}(\mu)=\aleph_{1}$ since $\mu \leq \chi^{\prime}<\aleph_{2} \mathfrak{M}_{1}^{\prime}$, so we let $\left\langle\xi_{i}: i<\omega_{1}\right\rangle$ be an increasing cofinal subsequence of $\mu$. Let $\beta_{i}:=\alpha_{\xi_{i}}$ and $h_{i j}:=$ $f_{\xi_{i} \xi_{j}}$. Note that the direct limit of the system $\left\langle\beta_{i}, h_{i j}: i \leq j<\omega_{1}\right\rangle$ is still a well-ordered set of ot $\gamma$, since every element of the direct limit of $B$ can be identified with an element of this thinned out direct limit canonically. For each $\delta<\omega_{1}$, we know that the sequence $b_{\delta}:=$ $\left\langle\beta_{i}, h_{i j}: i \leq j<\delta\right\rangle$ is in $M_{1}^{\prime}$ and $\mathfrak{M}_{1}^{\prime} \models b_{\delta} \in T^{\prime}$.
Now we can (in V) find an uncountable set $C \subseteq \omega_{1}$ and $n \in \omega$ such that $\forall \delta \in C g^{\prime}\left(b_{\delta}\right)=n$. By the properties of $g$ and hence $g^{\prime}$, we have that for each $\delta_{1}<\delta_{2} \in C$, there is a unique branch $B_{\alpha}{ }^{\prime}=\left\{b^{\prime}(\alpha, \beta): \beta<\omega_{1}\right\}$ with $\left\{b_{\delta_{1}}, b_{\delta_{2}}\right\} \subseteq B_{\alpha}{ }^{\prime}$. $\alpha$ might depend on $\left\{\delta_{1}, \delta_{2}\right\}$, but since for $\alpha \neq \beta$, $B_{\alpha}{ }^{\prime} \cap B_{\beta}{ }^{\prime}=\emptyset$, we must have the same $\alpha$ for all $\delta \in C$.
So $\left\langle b_{\delta}: \delta \in C\right\rangle$ is cofinal on some branch $B_{\alpha}{ }^{\prime} \in M^{\prime}$. So $\gamma$, the order-type of the directed system defined by $B_{\alpha}{ }^{\prime}$ has to be in $M^{\prime}$.
$\square_{\text {Claim 5.11 }} \square_{\text {Theorem }} 5.6$
Corollary 5.13 By fact 5.5, we also get the following:

- $\operatorname{Con}(\mathrm{BPFA}) \leftrightarrow \operatorname{Con}(\exists \kappa$ reflecting $) \leftrightarrow \operatorname{Con}(\mathrm{BSPFA})$


## 6 More on $\Sigma_{3}^{1}$ Absoluteness

## Theorem 6.1 [2]

For any poset $P, \operatorname{Abs}\left(\Sigma_{1}\left(\mathrm{H}_{\omega_{2}}\right), P\right) \rightarrow \boldsymbol{\Sigma}_{\mathbf{3}}^{\mathbf{1}}$ Absoluteness for $P$.
Proof: Let $a \subseteq \omega$, let $\varphi(a) \equiv \exists x \subseteq \omega \forall y \subseteq \omega \exists z \subseteq \omega \psi(x, y, z, a)$ be a $\Sigma_{3^{-}}^{1}$ formula, let $B:=\mathcal{B}(P)$, assume $\llbracket \varphi(\check{a}) \rrbracket_{B}=\mathbf{1}$ and $\operatorname{Abs}\left(\Sigma_{1}\left(\mathrm{H}_{\omega_{2}}\right), P\right)$ holds. Note that this implies $\llbracket \omega_{1}^{\mathbf{V}} \in \mathbf{C a r d} \rrbracket_{B}=\mathbf{1}$. By corollary 1.58,

$$
\llbracket \exists x \subseteq \omega\left(L_{\omega_{1}}[\check{a}, x] \models \forall y \subseteq \omega \exists z \subseteq \omega \psi(x, y, z, \check{a})\right) \rrbracket_{B}=\mathbf{1} .
$$

Since the map $\alpha \mapsto L_{\alpha}[a, x]$ is $\Sigma_{1}(a, x)$-definable, the sentence

$$
\begin{equation*}
\exists x \subseteq \omega\left(L_{\omega_{1}}[a, x] \models \forall y \subseteq \omega \exists z \subseteq \omega \psi(x, y, z, a)\right) \tag{7}
\end{equation*}
$$

is $\Sigma_{1}\left(a, \omega_{1}\right)$ - it is equivalent to: $\exists x \subseteq \omega \forall \alpha \in \omega_{1} \forall y \in L_{\alpha}[a, x] y \subseteq \omega \rightarrow$ $\exists \beta \in \omega_{1} \exists z \in L_{\beta}[a, x](z \subseteq \omega \wedge \psi(x, y, z, a))$.
$\operatorname{By} \operatorname{Abs}\left(\Sigma_{1}\left(\mathrm{H}_{\omega_{2}}\right), P\right),(7)$ holds in $\mathbf{V}$. So, for some $x_{0} \subseteq \omega$,

$$
\mathbf{V} \models " L_{\omega_{1}}\left[a, x_{0}\right] \models \forall y \subseteq \omega \exists z \subseteq \omega \psi\left(x_{0}, y, z, a\right) "
$$

By upward absoluteness, $\mathbf{V} \models \forall y \subseteq \omega \exists z \subseteq \omega \psi\left(x_{0}, y, z, a\right)$, hence $\mathbf{V} \models \varphi(a)$.

## Corollary 6.2

1. $\mathrm{MA}_{\omega_{1}} \rightarrow \operatorname{Abs}\left(\Sigma_{2}\left(\mathrm{H}_{\omega_{1}}\right), \mathrm{ccc}\right)$
2. $\mathrm{BPFA} \rightarrow \operatorname{Abs}\left(\Sigma_{2}\left(\mathrm{H}_{\omega_{1}}\right)\right.$, proper $)$
3. $\operatorname{BSPFA} \rightarrow \operatorname{Abs}\left(\Sigma_{2}\left(\mathrm{H}_{\omega_{1}}\right)\right.$, semiproper $)$
4. $\mathrm{BMM} \rightarrow \operatorname{Abs}\left(\Sigma_{2}\left(\mathrm{H}_{\omega_{1}}\right)\right.$, stationary-preserving $)$

## Observation:

It is easy to see that the implications 2-4 above cannot be reversed, since $\mathrm{BMM} \rightarrow \mathrm{BSPFA} \rightarrow \mathrm{BPFA} \rightarrow \mathrm{MA}_{\omega_{1}} \rightarrow \neg \mathrm{CH}\left(\mathrm{MA}_{2^{\aleph}}\right.$ is inconsistent with ZFC by lemma 2.73), but each of the absoluteness principles in 2-4 remains true after collapsing the continuum to $\omega_{1}$ by $\operatorname{coll}\left(\omega_{1},\left\{2^{\aleph_{0}}\right\}\right)$, which is $\sigma$-closed (and therefore proper), hence they are all consistent with $C H$. After the next theorem, we will be able to show that also the implication in 1 above cannot be reversed.

## 6.1 $\quad \Sigma_{3}^{1}$-absoluteness for proper forcing

Theorem 6.3 (Friedman) [12][11]
$\mathbf{\Sigma}_{\mathbf{3}}^{\mathbf{1}}$-absoluteness for proper forcing is equiconsistent with $Z F C$.
Proof: By an $\omega_{1}$-iteration $P_{0}$ of proper forcings with countable support iteration, we produce a generic $G_{0}$ such that $\mathbf{L}\left[G_{0}\right]$ satisfies $\Sigma_{3}^{1}$-absoluteness for proper forcing for formulas with parameters in $\mathbf{L}$ which is possible since by GCH , there are only $\aleph_{1}$ reals in $\mathbf{L}$, properness is preserved by countable support iteration and $\boldsymbol{\Sigma}_{\mathbf{3}}^{\mathbf{1}}$-formulas are upwards absolute for set-forcing extensions by corollary 1.59 .

We can assume that $\left|P_{0}\right|=\aleph_{1}$ in $\mathbf{L}\left[G_{0}\right]$, as if necessary we can follow $P_{0}$ by a Lévy Collapse with countable conditions to $\omega_{1}: P_{0} \in H_{\theta}^{\mathrm{L}}$ for some regular L-cardinal $\theta$. Let $C_{0}:=\operatorname{coll}\left(\omega_{1},\left\{H_{\theta}^{\mathbf{L}}\right\}\right)^{\mathbf{L}\left[G_{0}\right]}$ and let $H_{0}$ be generic for $C_{0}$ over $\mathbf{L}\left[G_{0}\right] . \quad C_{0} \subseteq H_{\theta}^{\mathbf{L}\left[G_{0}\right]}$, since each $f \in C_{0}$ is a function with countable domain from $\omega_{1}$ into $H_{\theta}^{\mathbf{L}}$, i.e. $f \in H_{\theta}^{\mathbf{L}\left[G_{0}\right]}$, hence $H_{0} \subseteq H_{\theta}^{\mathbf{L}\left[G_{0}\right]}$ and in $\mathbf{L}\left[G_{0}, H_{0}\right]$, both $G_{0}$ and $H_{0}$ have cardinality $\aleph_{1}$, so we can find $X_{0} \subseteq \omega_{1}$ such that $\mathbf{L}\left[X_{0}\right]=\mathbf{L}\left[G_{0}, H_{0}\right]$.

Now repeat the above over the model $\mathbf{L}\left[X_{0}\right]$, guaranteeing with a proper countable support iteration of length $\omega_{1}$ that proper absoluteness holds in $\mathbf{L}\left[\left(X_{0}, X_{1}\right)\right]$ for $\Sigma_{3}^{1}$-formulas with real parameters from $\mathbf{L}\left[X_{0}\right]$ and $X_{1} \subseteq \omega_{1}$. Repeat this for $\omega_{1}$ stages, producing $\mathbf{L}\left[\left\langle X_{i}: i<\omega_{1}\right\rangle\right]$, a model where proper absoluteness holds for $\Sigma_{3}^{1}$-formulas with parameters in $\bigcup_{i<\omega_{1}} \mathbf{L}\left[\left\langle X_{j}: j<i\right\rangle\right]$ : If $\lambda$ is a limit ordinal and for each $\gamma<\lambda$, we have a model $\mathbf{L}\left[\left\langle X_{i}: i<\right.\right.$ $\gamma\rangle$ ] satisfying proper absoluteness for $\Sigma_{3}^{1}$-formulas with real parameters in $\bigcup_{j<\gamma} \mathbf{L}\left[\left\langle X_{i}: i<j\right\rangle\right]$, then $\mathbf{L}\left[\left\langle X_{i}: i<\lambda\right\rangle\right]$ is our desired model at stage $\lambda$. If $\lambda=\gamma+1 \geq 2$ and we have a suitable model $\mathfrak{M}=\mathbf{L}\left[\left\langle X_{i}: i<\gamma\right\rangle\right]$, then, by lemma $1.53, \mathfrak{M} \models$ GCH and we can proceed over $\mathfrak{M}$ as we did over $\mathbf{L}$ above to obtain a suitable model at stage $\lambda$.

Every real in $\mathbf{L}\left[\left\langle X_{i}: i<\omega_{1}\right\rangle\right]$ belongs to $\mathbf{L}\left[\left\langle X_{j}: j<i\right\rangle\right]$ for some $i<\omega_{1}$ : For every $\beta<\omega_{1}$, let $Y_{\beta}:=\left\langle X_{i} \cap \beta: i<\beta\right\rangle$. Since each $Y_{\beta}$ is countable, it can be coded into a subset of $\omega$, hence we can find $X \subseteq \omega_{1}$ such that for each limit ordinal $\beta<\omega_{1}, X \cap \beta$ codes $Y_{\beta}$ and $X$ codes $\left\langle X_{i}: i<\omega_{1}\right\rangle$.

Assume $r$ is a real in $\mathbf{L}\left[\left\langle X_{i}: i<\omega_{1}\right\rangle\right]=\mathbf{L}[X]$. Then $r$ belongs to some countable, elementary submodel $\mathfrak{M}$ of $\mathbf{L}_{\omega_{1}}[X]$, which is transitive by corollary 1.51. By lemma $1.52, M=L_{\beta}[X \cap \beta]$ for some $\beta<\omega_{1}$, hence $r \in M \subseteq \mathbf{L}[X \cap \beta]=\mathbf{L}\left[\left\langle X_{i} \cap \beta: i<\beta\right\rangle\right] \subseteq \mathbf{L}\left[\left\langle X_{i}: i<\beta\right\rangle\right]$.

Thus $\mathbf{L}\left[\left\langle X_{i}: i<\omega_{1}\right\rangle\right]$ is a model of $\boldsymbol{\Sigma}_{\mathbf{3}}^{\mathbf{1}}$-absoluteness for proper forcing.

The same proof (using a revised countable support iteration of semiproper forcings) yields the following:

Theorem 6.4 (Friedman) [12]
$\mathbf{\Sigma}_{\mathbf{3}}^{\mathbf{1}}$-absoluteness for semiproper forcing is equiconsistent with $Z F C$.
As announced before, we can now conclude that the implication in corollary $6.2,1$ cannot be reversed:

Corollary 6.5 $\operatorname{Abs}\left(\Sigma_{2}\left(\mathrm{H}_{\omega_{1}}\right), \mathrm{ccc}\right) \nrightarrow \mathrm{MA}_{\omega_{1}}$
Proof: By lemma 2.73, $\mathrm{MA}_{\omega_{1}} \rightarrow \neg \mathrm{CH}$.
We show that $\operatorname{Abs}\left(\Sigma_{2}\left(H_{\omega_{1}}\right)\right.$, ccc $)$ is consistent with CH :
Applying theorem 6.3, let $\mathfrak{M}$ be a model for $\operatorname{Abs}\left(\Sigma_{2}\left(H_{\omega_{1}}\right)\right.$, proper $)$ and let $C:=\operatorname{coll}\left(\omega_{1},\left\{2^{\aleph_{0}}\right\}\right)^{\mathfrak{M}}$. Then, since $C$ is proper, $\operatorname{Abs}\left(\Sigma_{2}\left(H_{\omega_{1}}\right)\right.$, proper $)$ still holds in $\mathfrak{M}^{C}$, implying $\mathfrak{M}^{C} \models \operatorname{Abs}\left(\Sigma_{2}\left(H_{\omega_{1}}\right)\right.$, ccc $) \wedge \mathrm{CH}$.

## 6.2 $\quad \Sigma_{3}^{1}$-absoluteness for $\omega_{1}$-preserving forcing

Theorem 6.6 (Friedman, Bagaria) [9] Suppose $\omega_{1}=\omega_{1}{ }^{\mathbf{L}}$. Then $\Sigma_{3}^{1}$ absoluteness fails for some $\omega_{1}$-preserving forcing.

Proof: For each countably infinite ordinal $\alpha$, let $g_{\alpha}$ be the $<_{\mathbf{L}}$-least function mapping $\omega$ onto $\alpha$. For each $n \in \omega$, fix $\lambda_{n}$ such that $S_{n}:=\left\{\alpha: g_{\alpha}(n)=\lambda_{n}\right\}$ is stationary, which is possible by fact 1.26 , since for each fixed $n \in \omega$, $g_{\alpha}(n)<\alpha$, hence $\alpha \mapsto g_{\alpha}(n)$ is regressive. By theorem 2.49, we may add, preserving $\omega_{1}$, a club subset $C_{n}$ of $S_{n}$. Let $G_{n}$ be generic for this forcing. In $\mathbf{V}\left[G_{n}\right]$,

$$
\exists C_{n} \subseteq \omega_{1} \text { club }: \forall \xi, \chi \in C_{n} g_{\xi}(n)=g_{\chi}(n)
$$

By using the method of almost disjoint coding of a function (section 2.2.6), we may add a real $b_{n}$ such that whenever $a \in \mathcal{P}(\omega) \cap \mathbf{L}$ codes a countable ordinal (in the sense of definition 1.30), then $b_{n} \odot a$ codes the least member of $C_{n}$ that is strictly larger than that ordinal. Before we can continue with the main proof, we need the following:

Lemma 6.7 If $L_{\eta}\left[b_{n}\right]$ is admissible, then $L_{\eta}$ is amenable.
Proof: Pairing, Union, Infinity, Cartesian Product and $\Sigma_{0}$-Comprehension all hold in $L_{\eta}$ because they hold both in $L_{\eta}\left[b_{n}\right]$ and $\mathbf{L}$, hence the (unique) required sets can be found in their intersection, $L_{\eta}$.

Lemma 6.8 If $L_{\eta}$ is amenable, $L_{\eta} \models \theta \cong \omega$, then $L_{\eta} \vDash \exists c \subseteq \omega$ c codes $\theta$.
Proof: $L_{\eta} \models \exists f: \theta \cong \omega$, by $\Sigma_{0}$-Comprehension, $L_{\eta} \vDash \exists f:(\theta+1) \cong \omega$. Choose such $f$ and note that $(\theta+1)=\operatorname{tcl}(\{\theta\})$. $L_{\eta} \vDash \exists \omega \times \omega$, so we can define $E:=\left\{(x, z) \in \omega \times \omega: f^{-1}(x) \in f^{-1}(z)\right\}$ using $\Sigma_{0}$-Comprehension. It follows that $f:(\operatorname{tcl}(\{\theta\}), \in) \cong(\omega, E)$. Let $r:=\{i \in \omega: \exists x \in E \Gamma(a)=i\}$. Then $r \in L_{\eta}$ and $r$ codes $\theta$.

Lemma 6.9 If $M=L_{\eta}\left[b_{n}\right]$ is a countable admissible set such that

$$
M \models \text { every ordinal is constructibly countable, }
$$

then $\eta \in C_{n}$.
Proof: Let $\theta<\eta$ and let $c \in L_{\eta}$ be a subset of $\omega$ coding $\theta$. By admissibility of $M, b_{n} \odot c \in M . b_{n} \odot c$ codes a member $\zeta$ of $C_{n}$ greater than $\theta . E_{b_{n} \odot c}=$ : $E \in M$ by $\Sigma_{0}$-Comprehension and since $\Gamma \in M$. We can define the graph of $\pi_{E}$, the collapsing map of $E$, as a binary $\Sigma_{1}$-predicate on $M: \pi_{E}(x, z) \leftrightarrow$

$$
\exists F: \operatorname{tcl}_{E}(x) \rightarrow \operatorname{tcl}(\{z\}) \forall y \in \operatorname{tcl}_{E}(x) F(y)=\{F(t): t E y\} \wedge F(x)=z .
$$

Hence by $\Sigma_{1}$-Collection, range $\left(\pi_{E}\lceil\right.$ field $(E))=\operatorname{tcl}(\{\zeta\}) \in M$. Since $M$ is transitive, $\zeta \in M$, henceforth $\zeta<\eta$. Hence $C_{n}$ is unbounded in $\eta$. As $C_{n}$ is closed, $\eta \in C_{n}$.

Proof of theorem 6.6 continued: Let $\varphi\left(\eta, b_{n}\right)$ be a formula expressing

$$
L_{\eta}\left[b_{n}\right] \models K P \wedge \text { every ordinal is constructibly countable. }
$$

Our construction has added a real $b_{n}$ such that
$\forall \theta_{1}<\theta_{2}<\theta_{3}<\omega_{1}\left(\varphi\left(\theta_{1}, b_{n}\right) \wedge \varphi\left(\theta_{2}, b_{n}\right) \wedge \varphi\left(\theta_{3}, b_{n}\right)\right) \rightarrow L_{\theta_{3}}\left[b_{n}\right] \models g_{\theta_{1}}(n)=g_{\theta_{2}}(n):$
Let $i \in\{1,2\}$. If $\varphi\left(\theta_{3}, b_{n}\right)$ holds, then $L_{\theta_{3}} \models \theta_{i} \cong \omega$, since $g_{\theta_{i}}$ was defined to be the $<_{\mathbf{L}}$-least function mapping $\omega$ onto $\theta_{i}$, it follows that $g_{\theta_{i}} \in L_{\theta_{3}} \subseteq$ $L_{\theta_{3}}\left[b_{n}\right]$. Since $\varphi\left(\theta_{i}, b_{n}\right)$ implies $\theta_{i} \in C_{n}$ (lemma 6.9), the above follows.

The above statement (call it $\nu\left(b_{n}, n\right)$ ) is $\Pi_{1}\left(b_{n}, n\right)^{\mathrm{H}_{\omega_{1}}}$, hence $\exists b \subseteq \omega \nu(b, n)$ is $\Sigma_{2}(n)^{\mathrm{H}_{\omega_{1}}}$. Assuming that $\Sigma_{3}^{1}$-absoluteness holds for $\omega_{1}$-preserving forcing, it will be true in the ground model, witnessed by some $B_{n} \subseteq \omega$.

Consider $L_{\omega_{1}}\left[B_{n}\right]$ : It is a model of $K P \subseteq Z F^{-}$and believes that every ordinal is constructibly countable. Therefore there is a club $D_{n}:=\{\eta<$ $\left.\omega_{1}: L_{\eta}\left[B_{n}\right] \prec L_{\omega_{1}}\left[B_{n}\right]\right\}$ in the ground model such that each $\eta \in D_{n}$ satisfies $\varphi\left(\eta, B_{n}\right)$. Let $D:=\bigcap_{n<\omega} D_{n} . D$ is a club. If $\theta_{1}<\theta_{2}<\theta_{3} \in D$, then $\forall n \in \omega L_{\theta_{3}} \models g_{\theta_{1}}(n)=g_{\theta_{2}}(n)$, a contradiction. Hence $\Sigma_{3}^{1}$-absoluteness fails for some $\omega_{1}$-preserving forcing.

Corollary $6.10 \Sigma_{3}^{1}$-absoluteness for $\omega_{1}$-preserving forcing implies that $\omega_{1}$ is inaccessible to reals.

Proof: In the above theorem, start with $\mathbf{L}[x], x \subseteq \omega$ instead of $\mathbf{L}$. Like above, show that $\Sigma_{3}^{1}(x)$-absoluteness for $\omega_{1}$-preserving forcing implies that $\omega_{1}>\omega_{1}^{\mathbf{L}[x]}$ (using fact 1.46).

Corollary $6.11 \Sigma_{3}^{1}$-absoluteness for $\omega_{1}$-preserving forcing is equiconsistent with the existence of a reflecting cardinal.

Proof: Follows directly from corollary 6.10 and theorem 6.15 below.

## 6.3 $\quad \Sigma_{3}^{1}$-absoluteness for class forcing

We are going to show that $\boldsymbol{\Sigma}_{\mathbf{3}}^{\mathbf{1}}$-absoluteness for $\omega_{1}$-preserving class forcing is false. Our proof will make use of the following two theorems, which we will not give proofs of in this paper:

Theorem 6.12 (Jensen's Coding Theorem) ([10], section 4.3)
Suppose $(M, A)$ is a model of $Z F C$. Then there is an $(M, A)$-definable class forcing $P$ such that if $G \subseteq P$ is $P$-generic over $(M, A)$ :

- $(M[G], A, G) \vDash Z F C$.
- $(M[G], A, G) \models \exists r \subseteq \omega \mathbf{V}=\mathbf{L}[r] \wedge A, G$ are definable from $r$.

Moreover, $P$ preserves cardinals and cofinalities.

Theorem 6.13 ([5], Theorem 1) Let $M$ be a transitive model of ZFC s.t. $M \models \mathbf{V}=\mathbf{L}[b]$ for some $b \subseteq \omega$. Then there is an $M$-definable class $P$ of conditions such that if $N$ is a $P$-generic extension of $M$, then:

- $N$ is a model of $Z F C, N \models \exists a \subseteq \omega \mathbf{V}=\mathbf{L}[a]$.
- $\forall \alpha \in \mathbf{O r d}^{N} L_{\alpha}[a] \not \vDash Z F C$.

Moreover, $P$ preserves $\omega_{1}$.

Theorem 6.14 [9] Suppose $M$ is a model of $Z F C$. Then there is an $\omega_{1-}$ preserving class-generic extension $N$ of $M$ and a $\mathbf{\Sigma}_{\mathbf{3}}^{\mathbf{1}}$-sentence with real parameters from $M$ which is true in $N$ and false in $M$.

Proof: By theorem 6.12, $M$ can be extended to a model of the form $\mathbf{L}[r]$, $r \subseteq \omega$. By theorem 6.13, $\mathbf{L}[r]$ can be extended to a model of the form $\mathbf{L}[s]$, $s \subseteq \omega$, such that in $\mathbf{L}[s], \forall \alpha \in$ Ord $L_{\alpha} \not \vDash Z F C$.

Hence the following holds in $\mathbf{L}[s]$ (let $\sharp Z F C \subseteq \omega:=\{\sharp \varphi: \varphi \in Z F\}$ ):

$$
\mathrm{H}_{\omega_{1}} \models \exists s \subseteq \omega \forall \alpha \in \operatorname{Ord} \exists g \in \omega g \in \sharp Z F C \wedge \operatorname{Sat}\left(L_{\alpha}[s], F_{\neg}(g)\right) .
$$

As the above statement is $\Sigma_{2}^{\mathrm{H}_{\omega_{1}}}$, it is equivalent to some sentence $\varphi \in \boldsymbol{\Sigma}_{\mathbf{3}}^{\mathbf{1}}$.

Work in $M$ and assume for a contradiction that $\varphi$ holds. Let $s \subseteq \omega$ be some witness for $\varphi$. Note that $\varphi$ implies that $\omega_{1}$ is not inaccessible to reals, since otherwise $L_{\omega_{1}}[s] \models Z F C$, hence by a Löwenheim-Skolem argument, $\exists \alpha<\omega_{1} L_{\alpha}[s] \models Z F C$, contradicting $\varphi$. By corollary 6.10 , it follows that already $\boldsymbol{\Sigma}_{\mathbf{3}}^{\mathbf{1}}$-absoluteness for $\omega_{1}$-preserving set forcing fails in $M$.

## 6.4 $\quad \Sigma_{3}^{1}$-absoluteness for proper forcing $\& \omega_{1}$ inaccessible to reals

Theorem 6.15 (Friedman) [12] " $\boldsymbol{\Sigma}_{\mathbf{3}}^{\mathbf{1}}$-absoluteness for proper forcing and $\omega_{1}$ is inaccessible to reals" is equiconsistent with the existence of a reflecting cardinal.

Proof: Assume first that there exists a reflecting cardinal $\kappa$. Like in theorem 3.1, let $G$ be generic for $\operatorname{coll}(\omega, \kappa)$ to obtain a model $\mathbf{V}[G]$ where $\boldsymbol{\Sigma}_{\mathbf{3}}^{\mathbf{1}}$-absoluteness holds for arbitrary set-forcing and (by corollary 6.10) $\omega_{1}$ is inaccessible to reals.

For the other direction, assume that $\boldsymbol{\Sigma}_{\mathbf{3}}^{\mathbf{1}}$-absoluteness holds for proper forcings and that $\omega_{1}$ is inaccessible to reals. We may assume that $\omega_{1}$ is not Mahlo in $\mathbf{L}$, since otherwise $L_{\omega_{1}} \models \exists \kappa$ reflecting (see corollary 1.75). We will show that $\omega_{1}$ is reflecting in $\mathbf{L}$ :

Let $\kappa$ denote $\omega_{1}^{\mathbf{V}}$ and assume that for some $\mathbf{L}$-cardinal $\lambda \geq \kappa$, some closed formula $\varphi(x)$ and some $x \in L_{\kappa}, L_{\lambda} \models \varphi(x)$. We may assume that $0^{\sharp}$ does not exist, as otherwise $\kappa$ is reflecting in $\mathbf{L}$ by corollary 1.77.

In a countably closed set-forcing extension, there is $A \subseteq \omega_{1}$ such that:

- $\lambda<\omega_{2}$, in fact, $\lambda$ is less than the height of the least transitive model of $Z F^{-}$containing $A$ and $\kappa$,
- $\mathcal{P}\left(\omega_{1}\right) \subseteq \mathbf{L}[A]$, in particular, $\omega_{2}=\omega_{2}^{\mathbf{L}[A]}$.
$A$ is obtained as follows: Let $\delta>\lambda$ be a singular strong limit cardinal of uncountable cofinality. Since $0^{\sharp}$ does not exist, we have $\delta^{+}=\left(\delta^{+}\right)^{\mathbf{L}}$ and $2^{\delta}=\delta^{+}$(see lemma 1.81 and lemma 1.28). Now we force with Coll $:=$ $\operatorname{coll}\left(\omega_{1},\{\delta\}\right)$, since $\delta^{<\omega}=\delta$, Coll is $\delta^{+}$-cc (see lemma 2.47, 2) and hence $\delta^{+}=\omega_{2}^{\mathbf{V}^{\text {Coll }}}=\left(\delta^{+}\right)^{\mathbf{L}}$. As Coll is $\sigma$-closed, $\mathcal{P}(\omega)^{\mathbf{V}^{\text {Coll }}}=\mathcal{P}(\omega)^{\mathbf{V}}$, since $\delta$ is a strong limit cardinal in $\mathbf{V}, 2^{\aleph_{0}}<\delta$, hence in $\mathbf{V}^{\text {Coll }},|\mathcal{P}(\omega)|=2^{\aleph_{0}}=\omega_{1}$. In $\mathbf{V}^{\text {Coll }}$, any $S \subseteq \omega_{1}$ has a name $\dot{S}$ in $\mathbf{V}$, of the form $\bigcup_{\gamma<\omega_{1}}\{\check{\gamma}\} \times A_{\gamma}$ with $A_{\gamma} \subseteq$ Coll. Since in $\mathbf{V},|C o l l|=\delta$, there are only $\delta^{+}$such names, implying $\mathbf{V}^{\text {Coll }} \models 2^{\omega_{1}}=\aleph_{2}$. So in $\mathbf{V}^{\text {Coll }}$, we can find $B \subseteq \omega_{2}$ coding every subset of $\omega_{1}$. Now we code $B$ by $A \subseteq \omega_{1}$ with a $\sigma$-closed almost disjoint forcing to obtain a $\sigma$-closed (and therefore proper) extension in which every subset of $\omega_{1}$ belongs to $\mathbf{L}[A]$ : In $\mathbf{V}^{\text {Coll }}$, let $G \subseteq \omega_{1}$ code a surjection from $\omega_{1}$ onto $\delta$. Since $\delta^{+}=\left(\delta^{+}\right)^{\mathbf{L}}, \mathbf{L}[G] \models \omega_{2}=\delta^{+}$. So we can choose an almost disjoint family $\mathcal{A}$ on $\omega_{1}$ of size $\delta^{+}, \mathcal{A} \in \mathbf{L}[G]$. Now we force with the $\sigma$-closed almost disjoint coding $P_{\mathcal{A}, B}$ and work in $\mathbf{W}:=\mathbf{V}^{\operatorname{Coll} * P_{\mathcal{A}, B}}:$ Choose $A \subseteq \omega_{1}$ such that $A$ codes both $G$ and the set coding $B$ obtained by forcing with $P_{\mathcal{A}, B}$. Since every subset of $\omega_{1}$ in $\mathbf{V}^{\text {Coll }}$ is an element of $\mathbf{L}[A]$, it follows that

W. For $x \in H_{\omega_{2}}$, let $E_{x} \subseteq \omega_{1}$ code $x$. $E_{x}$ has a nice name in $\mathbf{V}^{\text {Coll }}$, of the form $\dot{E}_{x}=\bigcup_{\gamma \in \omega_{1}}\{\check{\gamma}\} \times A_{\gamma}$, where each $A_{\gamma}$ is an antichain of $P_{\mathcal{A}, B}$. Since $\omega_{1}{ }^{\aleph_{0}}=2^{\aleph_{0}}=\omega_{1}$ in $\mathbf{V}^{\text {Coll }}$, by lemma 2.64, $P_{\mathcal{A}, B}$ is $\omega_{1}$-centered and therefore $\omega_{2}$-cc. Hence $\forall \gamma<\omega_{1}\left|A_{\gamma}\right| \leq \omega_{1}$, i.e. $\dot{E}_{x} \in\left(H_{\omega_{2}}\right)^{\mathrm{V}^{\text {Coll }} \subseteq L_{\omega_{2}}[A] \text {. Since we }}$ can decode the generic $g$ for $P_{\mathcal{A}, B}$ from $A, g \in \mathbf{L}[A]$, hence $E_{x}=\dot{E}_{x}{ }^{g} \in \mathbf{L}[A]$ - in fact, $E_{x} \in L_{\omega_{2}}[A]$, hence $x \in L_{\omega_{2}}[A]$. Also note that, since $P_{\mathcal{A}, B}$ is $\sigma$ closed, $\mathbf{W} \vDash \mathrm{CH}$. Furthermore we may demand that $A$ also codes $\lambda$ in $\mathbf{W}$, which gives us the property that $\lambda$ is less than the height of the least transitive model of $Z F^{-}$containing $A$, i.e. $\lambda \in M$ for every transitive $\mathfrak{M} \models Z F^{-}$ s.t. $A \in M$, in particular this gives us $\lambda<\omega_{2}$.

In $\mathbf{L}[A]$, the following holds:
$(*)$ If $L_{\alpha}[A] \models Z F^{-} \wedge \alpha>\kappa$, then $L_{\alpha}[A] \models \exists \lambda \in \operatorname{Card}^{\mathrm{L}} L_{\lambda} \models \varphi(x)$.
Now we add $A^{*} \subseteq \kappa$ with the following improved version of (*):

$$
\begin{aligned}
& \text { (**) } \quad \text { If } L_{\alpha}\left[A^{*} \cap \gamma\right] \models " Z F^{-} \wedge \alpha>\gamma=\omega_{1} ", \text { then } \\
& \\
& L_{\alpha}\left[A^{*} \cap \gamma\right] \models \exists \lambda \in \operatorname{Card}^{\mathbf{L}} L_{\lambda} \models \varphi(x) .
\end{aligned}
$$

$A^{*}$ is obtained as follows:

$$
P:=\left\{p \stackrel{\text { bnd }}{\subseteq} \omega_{1}: \forall \gamma \leq \sup p \forall \alpha>\gamma(* * *)(p, \gamma, \alpha)\right\}
$$

where $p \stackrel{b \text { nd }}{\subseteq} \omega_{1}$ iff $p$ is a bounded subset of $\omega_{1}$ and $(* * *)(p, \gamma, \alpha)$ is the condition that if $L_{\alpha}[A \cap \gamma, p \cap \gamma] \models " Z F^{-} \wedge \gamma=\omega_{1}$ ", then

$$
L_{\alpha}[A \cap \gamma, p \cap \gamma] \models \exists \lambda \in \operatorname{Card}^{\mathbf{L}} L_{\lambda} \models \varphi(x)
$$

and $P$ is ordered by end-extension.
Lemma 6.16 The generic for $P$ is unbounded in $\omega_{1}$.
Proof: Given $p_{0}$ and $\xi<\omega_{1}, D_{\xi}:=\{p \in P: \xi \leq \sup p\}$ will be hit by the following condition $p \leq p_{0}$ : Let $\delta:=\sup p_{0}$, assume $\delta<\xi<\omega_{1}$ and let $E \subseteq[\delta+1, \delta+\omega)$ code the $\epsilon$-relation on $\xi$. If $p_{0} \cup[\delta+1, \xi]$ is a condition in $P$, we can choose $p$ like that. Otherwise, consider $p:=p_{0} \cup E \cup[\delta+\omega, \xi]$; if this a condition, we can choose $p$ like that, and indeed this is a condition in $P$, since if $\gamma$ is a limit ordinal, $\gamma \leq \xi, \alpha>\gamma, \gamma=\omega_{1}^{L_{\alpha}[A \cap \gamma, p \cap \gamma]}>\delta$ and $L_{\alpha}[A \cap \gamma, p \cap \gamma] \models Z F^{-}$, then $E \in L_{\alpha}[A \cap \gamma, p \cap \gamma]$, hence $L_{\alpha}[A \cap \gamma, p \cap \gamma] \models \xi \cong$ $\omega$, a contradiction, implying that for no such $\gamma$ and $\alpha, L_{\alpha}[A \cap \gamma, p \cap \gamma] \vDash Z F^{-}$, hence $(* * *)(p, \gamma, \alpha)$ holds for all $\gamma$ and $\alpha$ to be considered.

Lemma 6.17 $P$ is proper.

Proof: We show that for club-many countable $\mathfrak{N} \prec L_{\omega_{2}}[A]$, each condition $p \in N$ can be extended to a condition $q$ such that $q$ forces the generic to intersect $D \cap N$ whenever $D$ is a dense set in $N$. We take all countable $\mathfrak{N} \prec L_{\omega_{2}}[A]$ which have $A$ as element and $\operatorname{tcl}(\{x\})$ as subset. Suppose that $p \in N$ and let $\mathfrak{N} \cong \overline{\mathfrak{N}}=L_{\gamma}[A \cap \beta]$ with $\beta=\omega_{1}{ }^{N}$.

Since we assumed that $\kappa$ is not Mahlo in $\mathbf{L}$, the set of regular cardinals below $\kappa$ is not stationary in $\mathbf{L}$, i.e. there exists a club $C \subseteq \kappa$ in $\mathbf{L}$ consisting of singular L-cardinals only. By elementarity of $\mathfrak{N}$, there exists such $C$ in $N$. Consider the structure $(N, \in, A, C)$ with $A, C$ as unary predicates interpreted as $A \cap N, C \cap N$. By lemma 1.50, $\kappa \cap N=\beta<\kappa$, hence $C \cap N \subseteq \beta$ and $C \cap \beta$ is unbounded in $\beta$, implying $\beta \in C$ and hence $\beta$ is a singular L-cardinal. As $\beta=\omega_{1}{ }^{\bar{N}}, \overline{\mathfrak{N}} \models \beta$ is a regular cardinal. So we have

- $L_{\gamma} \vDash \beta$ is a regular cardinal,
- $\mathbf{L} \models \beta$ is a singular cardinal.

If $\gamma$ were some ordinal $\geq\left(\beta^{+}\right)^{\mathbf{L}}$, then $L_{\gamma}$ would correctly compute singularity of $\beta$; it follows that $\beta<\gamma<\left(\beta^{+}\right)^{\mathbf{L}}$, implying that $\gamma$ is not an $\mathbf{L}$-cardinal. Let $\mu$ be the least ordinal such that $\gamma$ is collapsed in $L_{\mu}\left(\mu<\left(\beta^{+}\right)^{\mathbf{L}}\right)$.

We will build $q$ to be an extension of $p$ of length $\beta$, as the union of conditions of length less than $\beta$. Then $(* * *)(q,<\beta, \alpha)$ always holds due to the fact that $q$ is the union of conditions of length less than $\beta$.
$(* * *)(q, \beta,<\gamma)$ holds by elementarity of $\mathfrak{N}$ in $L_{\omega_{2}}[A]:$

$$
L_{\omega_{2}}[A] \models \forall \alpha>\omega_{1} L_{\alpha}[A] \vDash Z F^{-} \rightarrow L_{\alpha}[A] \models \exists \lambda \in \operatorname{Card}^{\mathbf{L}} L_{\lambda} \models \varphi(x)
$$

since $A \in N$ and $\operatorname{tcl}(\{x\}) \subseteq N$, the same holds in $\overline{\mathfrak{N}}$, i.e.: $\overline{\mathfrak{N}}=L_{\gamma}[A \cap \beta] \models$ $\forall \alpha>\omega_{1} L_{\alpha}[A \cap \beta] \models Z F^{-} \rightarrow L_{\alpha}[A \cap \beta] \vDash \exists \lambda \in \operatorname{Card}^{\mathbf{L}} L_{\lambda} \mid=\varphi(x)$, therefore $\forall \alpha\left(\beta<\alpha<\gamma \wedge L_{\alpha}[A \cap \beta] \vDash Z F^{-}\right) \rightarrow L_{\alpha}[A \cap \beta] \vDash \exists \lambda \in \operatorname{Card}^{\mathbf{L}} L_{\lambda} \mid=\varphi(x)$. Now if $\beta<\alpha<\gamma$ and $L_{\alpha}[A \cap \beta, q] \mid=Z F^{-}$, then also $L_{\alpha}[A \cap \beta] \mid=Z F^{-}$ (since $L_{\alpha}[A \cap \beta]=(\mathbf{L}[A \cap \beta])^{L_{\alpha}[A \cap \beta, q]}$ ) and so by the above, $L_{\alpha}[A \cap \beta] \models$ $\exists \lambda \in \operatorname{Card}^{\mathbf{L}} L_{\lambda} \models \varphi(x)$ and the same holds in $L_{\alpha}[A \cap \beta, q]$.
$(* * *)(q, \beta, \gamma)$ also holds by elementarity of $\mathfrak{N}$ in $L_{\omega_{2}}[A]$, since $L_{\omega_{2}}[A] \mid=$ $\exists \lambda \in \operatorname{Card}^{\mathbf{L}} L_{\lambda} \models \varphi(x)$, so $L_{\gamma}[A \cap \beta, q] \vDash \exists \lambda \in \operatorname{Card}^{\mathbf{L}} L_{\lambda} \models \varphi(x)$.
$(* * *)(q, \beta, \alpha)$ also holds for $\gamma<\alpha<\mu$, as in this case any cardinal of $L_{\gamma}$ is also a cardinal of $L_{\alpha}$ : Since $\mu$ is least such that $\gamma$ is collapsed in $L_{\mu}, \gamma$ is a cardinal in $L_{\alpha}$ for $\gamma<\alpha<\mu$. Assume $L_{\gamma} \models \lambda \in \mathbf{C a r d}$ and $L_{\alpha} \models \lambda \notin \mathbf{C a r d}$ for some $\lambda$, i.e. $\lambda<\gamma$ and $L_{\alpha} \models \exists \nu<\lambda \exists f: \nu \xrightarrow{\text { onto }} \lambda$. But such $f$ would be inside $L_{\gamma}$, since $\gamma$ is a cardinal in $L_{\alpha}$, a contradiction.

As we have observed above, there is $\lambda$ such that $L_{\gamma}[A \cap \beta] \vDash \lambda \in \mathbf{C a r d}^{\mathbf{L}} \wedge$ $L_{\lambda} \models \varphi(x)$. Since also $L_{\alpha} \models \lambda \in \mathbf{C a r d}^{\mathbf{L}}, L_{\alpha}[A \cap \beta, q] \models \lambda \in \mathbf{C a r d}^{\mathbf{L}}$, i.e. $(* * *)(q, \beta, \alpha)$ holds.

Thus it suffices to build $q$ extending $p$ of length $\beta$ as the union of conditions of length less than $\beta$ in a way that $\beta$ is collapsed in $L_{\mu^{\prime}}[A \cap \beta, q]$, where $\mu^{\prime} \geq \mu$ is least such that $L_{\mu^{\prime}}[A \cap \beta, q] \models Z F^{-}$(and henceforth $L_{\mu^{\prime}}[A \cap \beta] \models Z F^{-}$), for then $(* * *)(q, \beta, \alpha)$ is vacuous for $\alpha \geq \mu$, since the left hand side of the implication in $(* * *)(q, \beta, \alpha)$ cannot hold.
$\gamma$ is collapsed in $L_{\mu^{\prime}}$. If $\beta$ is collapsed in $L_{\mu^{\prime}}[A \cap \beta]$, then any union of conditions of length less than $\beta$ will again be a condition, since $\beta$ is also collapsed in $L_{\mu^{\prime}}[A \cap \beta, q]$. So assume that $\beta$ is not collapsed in $L_{\mu^{\prime}}[A \cap \beta]$, i.e. $\beta=\omega_{1}{ }^{L_{\mu^{\prime}}}[A \cap \beta]$. Since $L_{\gamma}[A \cap \beta] \models \nexists \omega_{2}$ and $\gamma<\mu^{\prime}, L_{\mu^{\prime}}[A \cap \beta] \models \gamma \cong \beta$. So we can write $L_{\gamma}[A \cap \beta]$ as the union of a continuous increasing chain of elementary submodels of $L_{\gamma}[A \cap \beta]$ of the form $\left\langle\mathfrak{M}_{i}: i<\beta\right\rangle$ where each $M_{i}$ is countable in $L_{\mu^{\prime}}[A \cap \beta]$, the chain itself belongs to $L_{\mu^{\prime}}[A \cap \beta]$ and for all $i<\beta, M_{i} \cap \beta \in M_{i+1}$. Let $B$ be the set of intersections of the models of this chain with $\beta$, a club in $\beta$. We will choose an $\omega$-sequence $p=p_{0} \geq p_{1} \geq \ldots$ of conditions below $p$ such that each $p_{n}$ belongs to $N$, each dense set in $N$ is forced by some $p_{n}$ to intersect the generic and if $q=\bigcup\left\{p_{n}: n \in \omega\right\}$, then $\{\eta \in B: \eta \in q\}$ is a cofinal subset of $B$ of order-type $\omega$. Then $\beta$ is collapsed in $L_{\mu^{\prime}}[A \cap \beta, q]$, as desired.

To define $\left\langle p_{n}: n \in \omega\right\rangle$, enumerate the dense $D \in N$ in an $\omega$-sequence $\left\langle D_{n}: n \in \omega\right\rangle$ and choose a cofinal subset $B_{0}$ of $B$ of order-type $\omega$. Inductively, choose $p_{n}$ as follows: If $p_{n}$ is defined then first extend $p_{n}$ to $p_{n}^{\prime}$ of length $\sup \left(p_{n}\right)+\omega$, such that at the $\omega$ new ordinals, $p_{n}^{\prime}$ codes some $M_{i} \cap \beta=: x_{n} \in$ $B_{0}$, where both $D_{n}$ and $p_{n}^{\prime}$ belong to $M_{i+1}$ : assume $p_{n} \in N$; since $p_{n}{ }^{\text {bnd }} \subseteq \omega_{1}$, letting $\pi$ be the collapsing map of $\mathfrak{N}$, it follows that $\pi\left(p_{n}\right)=p_{n}$, and for every bounded subset of $\omega_{1}$ in $N$, the same holds, hence every bounded subset of $\omega_{1}$ in $N$ is an element of some $M_{j}$. Since every ordinal below $\beta$ is seen to be countable in $\mathfrak{N}$, it follows that $\mathfrak{N}=x \cong \omega$ for every $x \in B$, hence also $L_{\gamma}[A \cap \beta] \models x \cong \omega$, by elementarity it follows that for every $\mathfrak{M}_{j}$ such that $x \in M_{j}$, it holds that $\mathfrak{M}_{j} \models x \cong \omega$. Since $\mathfrak{M}_{j} \vDash Z F^{-}$, it follows that if $x \in M_{j}$ then there is a real in $M_{j}$ coding $x$. So let $i$ be least such that $D_{n} \in M_{i+1}$ and $x_{n} \in B_{0}$. By our assumption about the $\mathfrak{M}_{i}$-chain, it holds that $x_{n} \in M_{i+1}$, hence $M_{i+1}$ contains $p_{n}^{\prime}$.

Since $x_{n} \in M_{i+1}$, we can further extend $p_{n}^{\prime}$ to $p_{n}^{\prime \prime} \in M_{i+1}$ by setting $p_{n}^{\prime \prime}:=p_{n}^{\prime} \cup\left\{x_{n}\right\}$. Note that if $\alpha>\delta>\sup \left(p_{n}\right), \delta \leq \sup \left(p_{n}^{\prime \prime}\right)$ and $L_{\alpha}[A \cap$ $\left.\delta, p_{n}^{\prime \prime} \cap \delta\right] \models Z F^{-}$, then it collapses $\delta$, hence $p_{n}^{\prime \prime}$ is actually a condition.

Finally, we choose $p_{n+1} \leq p_{n}^{\prime \prime}$ to belong to $D_{n} \cap M_{i+1}$ : Since $D_{n} \in N$, $\mathfrak{N}=" D_{n}$ is dense" and $D_{n}$ consists of bounded subsets of $\omega_{1}$ in $N$, which remain unchanged under $\pi$, it follows that $\pi\left(D_{n}\right)=D_{n} \cap N, \pi\left(D_{n}\right)$ is dense in $L_{\gamma}[A \cap \beta]$, hence by elementarity, $D_{n} \cap M_{i+1}$ is dense in $M_{i+1}$, so we can obtain $D_{n} \cap M_{i+1} \ni p_{n+1} \leq p_{n}$ with the additional property that no element of the form $M_{j} \cap \beta, j>i$ is an element of $p_{n+1}$ (since none of them is an element of $M_{i+1}$ ). This implies that $\{\nu \in B: \nu \in q\}$ is a cofinal subset of $B$ of order-type $\omega$, as desired.

Proof of theorem 6.15 continued: Let $A^{*}$ code $A$ and the generic $H$ for $P$ such that $A^{*}:=\{2 \beta: \beta \in A\} \cup\{2 \beta+1: \beta \in \bigcup H\}$. Thus $A^{*}$ satisfies the condition in $(* *)$, since this is guaranteed for countable $\gamma$ by the definition of $P$ and for $\gamma=\omega_{1}$ by construction of $A$.

As $\kappa$ is not Mahlo in $\mathbf{L}$, there is a club $C \subseteq \kappa, C \in \mathbf{L}$, consisting of $\mathbf{L}$ singular cardinals. Let $\left\langle\chi_{i}: i<\kappa\right\rangle$ be the increasing enumeration of $C \cup\{0\}$ and for each $i$ let $S_{i}$ be a real coding the countable ordinal $\chi_{i+1}$. Let $D:=$ $\left\{\chi_{i}+n: i<\kappa \wedge n \in S_{i}\right\}$. Then $\chi \cong \omega$ implies $\mathbf{L}[D \cap \chi] \vDash \chi \cong \omega$, since if for some $i, \chi=\chi_{i+1}$, then $\mathbf{L}\left[D \cap\left(\chi_{i}+\omega\right)\right] \models \chi \cong \omega$, if $\chi$ is a limit point of $C$, then $\mathbf{L}[D \cap \chi] \vDash \beta \cong \omega$ for all $\beta<\chi$, assuming $\neg \mathbf{L}[D \cap \chi] \vDash \chi \cong \omega$ thus implies $\mathbf{L}[D \cap \chi] \models \chi=\omega_{1}$, a contradiction since $\chi$ is a singular $\mathbf{L}$ cardinal by assumption; if for some $i<\kappa, \chi_{i}<\chi<\chi_{i+1}$, it also holds that $\mathbf{L}[D \cap \chi] \models \chi \cong \omega$ by construction of $D$. Let $A^{* *}$ code $D$ and $A^{*}$ such that $A^{* *}=\left\{2 \beta: \beta \in A^{*}\right\} \cup\{2 \beta+1: \beta \in D\}$. Now $A^{* *}$ satisfies the condition in $(* *)$ and $\chi \cong \omega \rightarrow \mathbf{L}\left[A^{* *} \cap \chi\right] \vDash \chi \cong \omega$.

Now we code $A^{* *}$ by a real $R$ such that for all $\alpha$, if $L_{\alpha}[R] \models " Z F^{-}, \exists \gamma=\omega_{1}$ and $\forall \chi\left(\chi \cong \omega \rightarrow \mathbf{L}\left[A^{* *} \cap \chi\right] \models \chi \cong \omega\right)$ ", then

$$
(* * * *) L_{\alpha}[R] \models \exists \lambda \in \operatorname{Card}^{\mathbf{L}} L_{\lambda} \models \varphi(x):
$$

We choose distinct reals $R_{\chi}, \chi<\omega_{1}$, such that $R_{\chi}$ can be defined uniformely in $\mathbf{L}\left[A^{* *} \cap \chi\right]$ : Let $R_{0}$ be the $<_{\mathbf{L}}$-least real. Having constructed $\left(R_{\xi}\right)_{\xi<\chi}$, we let $R_{\chi}:=$ the $<_{\mathbf{L}\left[A^{* *} \cap \chi\right]}$-least real distinct from $\left(R_{\xi}\right)_{\xi<\chi}$, which exists as $\mathbf{L}\left[A^{* *} \cap \chi\right] \models \chi \cong \omega$. Let $\mathcal{B}:=\left(R_{\xi}\right)_{\xi<\omega_{1}}$. By lemma 1.23, we may assume that $\left(R_{\chi}\right)_{\chi<\omega_{1}}$ in fact is an almost disjoint family on $\omega$.

Now we force with the almost disjoint coding $P_{\mathcal{B}, A^{* *}}$ to code $A^{* *}$ by a real $R$. It follows that $\forall \beta<\omega_{1} \beta \in A^{* *} \leftrightarrow\left|R \cap R_{\beta}\right|<\omega$.

If $L_{\alpha}[R] \models " Z F^{-} \wedge \gamma=\omega_{1} \wedge \forall \chi\left(\chi \cong \omega \rightarrow \mathbf{L}\left[A^{* *} \cap \chi\right] \models \chi \cong \omega\right)$ ", then $\left(R_{\xi}\right)_{\xi<\gamma}$ is definable in $L_{\alpha}[R]$ : By induction, if $\left(R_{\xi}\right)_{\xi<\chi}$ is definable for some $\chi<\gamma=\omega_{1}{ }^{L_{\alpha}[R]}$ in $L_{\alpha}[R]$, this allows us to define $A^{* *} \cap \chi$. Since $\chi<\gamma$, $L_{\alpha}[R] \models \chi \cong \omega$ and hence by our above assumption $L_{\alpha}\left[A^{* *} \cap \chi\right] \models \chi \cong \omega$, so there is a real distinct from $\left(R_{\xi}\right)_{\xi<\chi}$ in $L_{\alpha}\left[A^{* *} \cap \chi\right]$, so we let $R_{\chi}$ be the least real distinct from $\left(R_{\xi}\right)_{\xi<\chi}$ in $L_{\alpha}\left[A^{* *} \cap \chi\right]$.

We finally get that if $L_{\alpha}[R]$ is as above, then $A^{* *} \cap \omega_{1}^{L_{\alpha}[R]}$ is definable in $L_{\alpha}[R]$, so the following holds in our final forcing extension:

$$
\begin{gathered}
\forall \alpha<\omega_{1} L_{\alpha}[R] \models " Z F^{-} \wedge \exists \gamma=\omega_{1} \wedge \forall \chi\left(\chi \cong \omega \rightarrow \mathbf{L}\left[A^{* *} \cap \chi\right] \models \chi \cong \omega\right) " \rightarrow \\
L_{\alpha}[R] \models \exists \lambda \in \mathbf{C a r d}^{\mathbf{L}} L_{\lambda} \models \varphi(x) .
\end{gathered}
$$

The above is a $\Pi_{1}\left(\mathrm{H}_{\omega_{1}}\right)$-condition on $R$, so by $\operatorname{Abs}\left(\Sigma_{2}\left(\mathrm{H}_{\omega_{1}}\right)\right.$, proper $)$, it follows that the condition holds for some real $R$ in $\mathbf{V}$.

Assume $L_{\alpha}[R] \models " Z F^{-} \wedge \exists \gamma=\omega_{1} \wedge \forall \chi\left(\chi \cong \omega \rightarrow \mathbf{L}\left[A^{* *} \cap \chi\right] \models \chi \cong \omega\right) "$ and $\alpha \geq \omega_{1}$. Then the same holds in some countable elementary submodel of $L_{\alpha}[R]$ of the form $L_{\alpha^{\prime}}[R]$ for some $\alpha^{\prime}<\omega_{1}$. Applying our above result, it follows that, again by elementarity, $L_{\alpha}[R] \models \exists \lambda \in \operatorname{Card}^{\mathrm{L}} L_{\lambda} \models \varphi(x)$.

By our assumption that $\omega_{1}$ is inaccessible to reals, it follows that $L_{\omega_{1}}[R] \models$ $\exists \omega_{1}$. Furthermore $L_{\omega_{1}}[R] \models Z F^{-} \wedge \forall \chi\left(\chi \cong \omega \rightarrow \mathbf{L}\left[A^{* *} \cap \chi\right] \models \chi \cong \omega\right)$ ". So $L_{\omega_{1}}[R] \vDash \exists \lambda \in \operatorname{Card}^{\mathbf{L}} L_{\lambda} \models \varphi(x)$ and hence we obtain that

$$
\exists \lambda<\omega_{1} \lambda \in \operatorname{Card}^{\mathbf{L}} \wedge L_{\lambda} \models \varphi(x)
$$

i.e. $\kappa=\omega_{1}^{\mathbf{V}}$ is reflecting in $\mathbf{L}$.

## 6.5 $\quad \Sigma_{3}^{1}$-absoluteness for stationary-preserving forcing

Theorem 6.18 (Friedman) [12] $\boldsymbol{\Sigma}_{3}^{1}$-absoluteness for stationary-preserving forcing implies that $\omega_{1}$ is inaccessible to reals.

Corollary $6.19 \Sigma_{3}^{1}$-absoluteness for stationary-preserving forcing is equiconsistent with the existence of a reflecting cardinal.

Proof: Follows immediately from theorem 3.1 and theorem 6.15.
Before we start the proof of theorem 6.18, we need the following:
Lemma 6.20 [12] If $R^{\sharp}$ does not exist for some $R \subseteq \omega$ and $\lambda \subseteq$ Ord, then $\lambda$ is constructible from a real $r$ in a stationary-preserving forcing extension, moreover, in this extension, $\mathrm{H}_{\omega_{2}}=L_{\omega_{2}}[r]$.

Proof: First we produce $A \subseteq \omega_{1}$ by a countably-closed forcing such that in the extension $\mathrm{H}_{\omega_{2}}=L_{\omega_{2}}[A]$ and $\lambda \in \mathrm{H}_{\omega_{2}}$, which works very much like in theorem 6.15, we will give a sketch here (note our remark in section 1.7.5 about the relativization of results about $0^{\sharp}$, which we will use in the following): Let $\delta>\sup \lambda$ be a singular strong limit cardinal of uncountable cofinality. Since $R^{\sharp}$ does not exist, $\delta^{+}=\left(\delta^{+}\right)^{\mathbf{L}[R]}$ and $2^{\delta}=\delta^{+}$. Now we force with Coll $:=\operatorname{coll}\left(\omega_{1},\{\delta\}\right)$. It follows that $\delta^{+}=\omega_{2}^{\mathbf{V}^{\text {Coll }}}=\left(\delta^{+}\right)^{\mathbf{L}[R]}$, $\mathcal{P}(\omega)^{\mathbf{V}^{\text {coll }}}=\mathcal{P}(\omega)^{\mathbf{V}}, \mathbf{V}^{\text {Coll }}\left|=|\mathcal{P}(\omega)|=2^{\aleph_{0}}=\omega_{1}\right.$ and $\mathbf{V}^{\text {Coll }} \models 2^{\aleph_{1}}=\aleph_{2}$. So in $\mathbf{V}^{\text {Coll }}$, we can find $B \subseteq \omega_{2}$ coding every subset of $\omega_{1}$. Now we code $B$ by $A \subseteq \omega_{1}$ with a $\sigma$-closed almost disjoint forcing to obtain an extension in which $\mathrm{H}_{\omega_{2}}=L_{\omega_{2}}[A]$ and (obviously) $\lambda \in \mathrm{H}_{\omega_{2}}$ : In $\mathbf{V}^{\text {Coll }}$, let $G \subseteq \omega_{1}$ code a surjection from $\omega_{1}$ onto $\delta$. Since $\delta^{+}=\left(\delta^{+}\right)^{\mathbf{L}[R]}, \mathbf{L}[R, G] \models \delta^{+}=\omega_{2}$, so we can choose an almost disjoint family $\mathcal{A}$ on $\omega_{1}$ of size $\delta^{+}, \mathcal{A} \in \mathbf{L}[R, G]$. Now we force with the $\sigma$-closed almost disjoint coding $\mathcal{P}_{\mathcal{A}, B}$ and work in $\mathbf{W}:=\mathbf{V}^{\operatorname{Coll}^{*} * \mathcal{P}_{\mathcal{A}, B}}$ : Choose $A \subseteq \omega_{1}$ such that $A$ codes $R, G$ and the set coding $B$ obtained by forcing with $\mathcal{P}_{\mathcal{A}, B}$. Now we conclude, exactly as we did in theorem 6.15, that $\mathrm{H}_{\omega_{2}} \mathbf{W}=L_{\omega_{2}}[A]$ and $\mathbf{W} \models \mathrm{CH}$.

Let $P$ be the reshaping forcing for $A$, introduced in section 2.2.7. We will show that $P$ is stationary-preserving (in $\mathbf{W}$ ). Assuming this, let $G$ be $P$-generic, $F:=\bigcup G$ and $A^{*}:=\{2 \beta: \beta \in A\} \cup\{2 \beta+1: \beta \in F\}$. As towards the end of the proof of theorem 6.15, we can choose a sequence $\left(R_{\alpha}\right)_{\alpha<\omega_{1}}$ of almost disjoint reals such that $R_{\alpha}$ is uniformely definable in $\mathbf{L}\left[A^{*} \cap \alpha\right]$ and let $\mathcal{B}:=\left(R_{\alpha}, \alpha<\omega_{1}\right)$. We then force with the almost disjoint coding $\mathcal{P}_{\mathcal{B}, A^{*}}$ to code $A^{*}$ by a real $r$, resulting in a stationary-preserving extension in which $\lambda \in L_{\omega_{2}}[r]$ and $\mathrm{H}_{\omega_{2}}=L_{\omega_{2}}[r]$ : Obviously, $\mathrm{H}_{\omega_{2}}{ }^{\mathbf{W}} \subseteq L_{\omega_{2}}[r]$. Since $|P|=\omega_{1}$ and $\mathcal{P}_{\mathcal{B}, A^{*}}$ is $\omega_{1}$-cc, the claim follows as in the proof of theorem 6.15, using nice names for subsets of $\omega_{1}$.

Now we are going to complete the proof of lemma 6.20 by showing that $P$ is stationary-preserving in $\mathbf{W}$ :

Lemma 6.21 $P$, the reshaping for $A$, is stationary-preserving in $\mathbf{W}$.
Proof: First we repeat the definition of reshaping given in section 2.2.7:

$$
P:=\left\{p \subseteq \omega_{1}: \forall \alpha \leq \sup p \mathbf{L}[A \cap \alpha, p \cap \alpha] \models \alpha \cong \omega\right\}
$$

Working in $\mathbf{W}$, given $p \in P$, a stationary $X \subseteq \omega_{1}$ and a name $\sigma$ for a club in $\omega_{1}$ (in the extension), let $C \subseteq \omega_{1}$ be club such that:

1. $(\alpha \in C \wedge \beta<\alpha) \rightarrow\left(p \in L_{\alpha}[A] \wedge \forall q \leq p q \in L_{\alpha}[A] \rightarrow(\exists r \leq q r \in\right.$ $L_{\alpha}[A] \wedge r \Vdash \beta^{*} \in \sigma$ for some $\beta^{*}$ such that $\left.\beta<\beta^{*}<\alpha\right)$ ),
2. $\alpha \in C \rightarrow C \cap \alpha \in \mathbf{L}[A \cap \alpha]$,
3. $\alpha \in C \rightarrow L_{\alpha}[A] \models \forall \beta \exists r \subseteq \omega r$ codes $\beta$.

We are going to show that such a club can be constructed. Assuming this, we will now prove our present lemma:

Choose $\alpha$ as a limit point of $C$ such that $\alpha \in X$, which is possible since the limit points of $C$ are also club. Let $\left\langle\gamma_{n}: n \in \omega\right\rangle$ be any increasing $\omega$-sequence contained in $C$ with supremum $\alpha$. We inductively define conditions $q_{n}$ with supremum $\gamma_{n}$ as follows: Let $q_{0}$ be the $<_{\mathbf{L}[A]}$-least extension of $p \in L_{\gamma_{0}}[A]$ with supremum $\gamma_{0}$; by property 3 above, $q_{0} \in L_{\gamma_{1}}[A]$. If $q_{n} \in L_{\gamma_{n+1}}[A]$ is defined, let $q_{n}^{\prime}$ be the $<_{\mathbf{L}[A]}$-least extension of $q_{n}$ such that $\gamma_{n} \in q_{n}^{\prime}$ and $\exists \beta_{n}>\gamma_{n} q_{n}^{\prime} \Vdash \beta_{n} \in \sigma$; by property 1 above, it follows that this is possible and moreover $q_{n}^{\prime} \in L_{\gamma_{n+1}}[A]$. Also, letting $\gamma_{n}^{\prime}:=\sup q_{n}^{\prime}$, property 1 implies that $\gamma_{n}^{\prime}$ is less than the least element of $C$ above $\gamma_{n}$. Let $R_{n}$ be a real coding the ordinal $\gamma_{n+1}$ and extend $q_{n}^{\prime}$ to $q_{n}^{\prime \prime}$ of length $\gamma_{n}^{\prime}+\omega$ by setting $\gamma_{n}^{\prime}+k \in q_{n}^{\prime \prime} \leftrightarrow k \in R_{n}$. Then $q_{n+1}$ is obtained extending $q_{n}^{\prime \prime}$ to have supremum $\gamma_{n+1}$, adding all successor ordinals above $\gamma_{n}^{\prime}+\omega$. Since $q_{n}^{\prime}$ is a condition and
by definition of $q_{n}^{\prime \prime}$, it follows that $q_{n+1}$ is a condition. By property 3 above, $q_{n+1} \in L_{\gamma_{n+2}}[A]$.

Let $q:=\bigcup_{n \in \omega} q_{n}$. Then $\left\{\gamma \in C \cap\left[\gamma_{0}, \alpha\right): \gamma \in q\right\}=\left\{\gamma_{n}: n \in \omega\right\}$ (this holds since all elements of $C$ have to be limit ordinals by definition of $C$ ). By property 2 above, $\left\{\gamma_{n}: n \in \omega\right\}$ belongs to $\mathbf{L}[A \cap \alpha, q]$, therefore $\mathbf{L}[A \cap \alpha, q] \vDash \alpha \cong \omega$, i.e. $q$ is a condition.

As $q$ forces that $\sigma \cap \alpha$ is unbounded in $\alpha, q \Vdash \alpha \in \sigma$. Since $\alpha$ was chosen to belong to $X$, we have $q \Vdash X \cap \sigma \neq \emptyset$.

Now we are going to finish the proof of lemma 6.21 (and therefore also lemma 6.20 ) by showing that a club $C \subseteq \omega_{1}$ with the above-claimed properties can be constructed. We can assume that $\sigma$ is a nice name of the form $\sigma=\bigcup_{j<\omega_{1}}\{\check{j}\} \times A_{j}$ where $A_{j} \subseteq P$ is an antichain. Since $|P|=\aleph_{1}$, we have both $P$ and $\sigma$ as elements of $\mathrm{H}_{\omega_{2}}=L_{\omega_{2}}[A]$ :

Lemma 6.22 Using the notation of lemma 6.21 above, there exists a club $C \subseteq \omega_{1}$ such that:

1. $(\alpha \in C \wedge \beta<\alpha) \rightarrow\left(p \in L_{\alpha}[A] \wedge \forall q \leq p q \in L_{\alpha}[A] \rightarrow(\exists r \leq q r \in\right.$ $L_{\alpha}[A] \wedge r \Vdash \beta^{*} \in \sigma$ for some $\beta^{*}$ such that $\left.\beta<\beta^{*}<\alpha\right)$ ),
2. $\alpha \in C \rightarrow C \cap \alpha \in \mathbf{L}[A \cap \alpha]$.
3. $\alpha \in C \rightarrow L_{\alpha}[A] \mid=\forall \exists r \subseteq \omega r$ codes $\beta$.

Proof: We let $\gamma:=\omega_{2}^{\mathbf{L}[A]}$ and

$$
C:=\left\{\alpha<\omega_{1}: \alpha=\omega_{1} \cap h^{L_{\gamma}[A]}(\alpha \cup\{p, \sigma, A\})\right\},
$$

where $h^{L_{\gamma}[A]}(X)$ denotes the Skolem hull of $X$ in $L_{\gamma}[A]$ and we require our Skolem functions to always pick the least possible elements.
Let $M_{\alpha}$ denote $h^{L_{\gamma}[A]}(\alpha \cup\{p, \sigma, A\})$.
Claim 1: Property 1 (from above) holds.
Proof: Assume $\alpha \in C$. Then $p \in M_{\alpha}, L_{\gamma}[A] \models \exists \delta<\omega_{1} p \in L_{\delta}[A]$, hence by elementarity, $M_{\alpha} \models \exists \delta<\omega_{1} p \in L_{\delta}[A]$, since $\alpha=\omega_{1} \cap M_{\alpha}$, we obtain $\exists \delta<\alpha p \in L_{\delta}[A]^{M_{\alpha}}$, since $\delta \subseteq M_{\alpha}, L_{\delta}[A]^{M_{\alpha}}=L_{\delta}[A] \subseteq L_{\alpha}[A]$.

Furthermore,

$$
L_{\gamma}[A] \models \forall q \in P \forall \beta<\omega_{1} \exists r \leq q \exists \beta^{*} \beta<\beta^{*}<\omega_{1} \wedge r \Vdash_{P} \beta^{*} \in \sigma .
$$

Note that we can define $P$ in $M_{\alpha}$ as above, which gives us $P^{M_{\alpha}}=P \cap L_{\alpha}[A]$. By elementarity (of $M_{\alpha}$ in $L_{\gamma}[A]$ ), we get
$\forall q \in P \cap L_{\alpha}[A] \forall \beta<\alpha \exists r \leq q r \in P \cap L_{\alpha}[A] \exists \beta^{*} \beta<\beta^{*}<\alpha \wedge M_{\alpha} \models r \Vdash \beta^{*} \in \sigma$.

It remains to show that $\left(M_{\alpha} \models r \Vdash \beta^{*} \in \sigma\right) \rightarrow\left(r \Vdash \beta^{*} \in \sigma\right)$ :
Note that

$$
\left(r \Vdash \beta^{*} \in \sigma\right) \leftrightarrow\left(\exists a \in A_{\beta^{*}} r \leq a\right)
$$

and

$$
\left(M_{\alpha} \models r \Vdash \beta^{*} \in \sigma\right) \leftrightarrow\left(\exists a \in A_{\beta^{*}} \cap L_{\alpha}[A] r \leq a\right)
$$

But if $r \in L_{\alpha}[A]$, then $a \geq r \rightarrow a \in L_{\alpha}[A]$, since then $a=r \cap \delta$ for some $\delta<\alpha$, which gives us

$$
\left(\exists a \in A_{\beta^{*}} r \leq a\right) \leftrightarrow\left(\exists a \in A_{\beta^{*}} \cap L_{\alpha}[A] r \leq a\right)
$$

$\square_{\text {Claim } 1}$
Claim 2: $C$ is club.
Proof: Assume $\alpha_{1}<\alpha_{2}<\ldots \in C$. We show that $\alpha:=\bigcup_{i<\omega} \alpha_{i} \in C$ : Obviously, $M_{\alpha}=\bigcup_{i<\omega} M_{\alpha_{i}}$, hence $\omega_{1} \cap M_{\alpha}=\bigcup_{i<\omega}\left(\omega_{1} \cap M_{\alpha_{i}}\right)=\bigcup_{i<\omega} \alpha_{i}=$ $\alpha$, i.e. $\alpha \in C$, hence $C$ is closed.

Now assume $\alpha \in C$. We have to find $\beta>\alpha, \beta \in C$ : Let $\beta:=\omega_{1} \cap M_{\alpha+1}$, then $\beta \subseteq M_{\alpha+1}$, hence $M_{\alpha+1}=M_{\beta}$, hence $\beta=\omega_{1} \cap M_{\beta}$, i.e. $\beta \in C$. $\square_{\text {Claim 2 }}$

Claim 3: Property 2 (from above) holds.
Proof: Assume $\alpha \in C . M_{\alpha}$ is a countable elementary submodel of $L_{\gamma}[A]$, hence its transitive collapse is of the form $L_{\xi}[A \cap \alpha]$, which is an element of $\mathbf{L}[A \cap \alpha]$. Let $\pi$ denote the collapsing map.

$$
C \cap \alpha=\left\{\beta<\alpha: \beta=\omega_{1} \cap h^{L_{\gamma}[A]}(\beta \cup\{p, \sigma, A\})\right\}
$$

Now $M_{\beta}=h^{L_{\gamma}[A]}(\beta \cup\{p, \sigma, A\})=h^{M_{\alpha}}(\beta \cup\{p, \sigma, A\})$ by elementarity. Collapsing $M_{\alpha}$, it follows that $\pi\left(M_{\beta}\right)=h^{L_{\xi}[A \cap \alpha]}(\beta \cup\{p, \pi(\sigma), \pi(A)\})$. Finally,

$$
C \cap \alpha=\left\{\beta<\alpha: \beta=\alpha \cap h^{L_{\xi}[A \cap \alpha]}(\beta \cup\{p, \pi(\sigma), \pi(A)\})\right\}
$$

so $C \cap \alpha \in \mathbf{L}[A \cap \alpha] . \square_{\text {Claim } 3}$

Claim 4: Property 3 (from above) holds.
Proof: Obvious, since $\alpha \in C \rightarrow\left(L_{\alpha}[A] \vDash Z F^{-} \wedge L_{\alpha}[A] \vDash \forall \beta \beta \cong \omega\right)$. $\square_{\text {Claim 4 }} \square_{\text {Lemma 6.22 }} \square_{\text {Lemma } 6.21} \square_{\text {Lemma } 6.20}$

Proof of Theorem 6.18: Suppose that $\boldsymbol{\Sigma}_{\mathbf{3}}^{\mathbf{1}}$-absoluteness for stationarypreserving forcing holds and $\omega_{1}$ is not inaccessible to reals. Thus for some real $s, \omega_{1}=\omega_{1}^{\mathbf{L}[s]}$, hence $s^{\sharp}$ does not exist. So by lemma 6.20 , in a stationarypreserving forcing extension, $\mathrm{H}_{\omega_{2}}=L_{\omega_{2}}[r]$ for some real $r$. We argue in this extension:

For any $A \subseteq \omega_{1}$, consider the function $f_{A}: \omega_{1} \rightarrow \omega_{1}$ defined by

$$
f_{A}(\alpha)=\text { the least } \beta \text { such that } L_{\beta+1}[r, A \cap \alpha] \models \alpha \cong \omega .
$$

Note that since $\omega_{1}=\omega_{1}^{\mathbf{L}[r]}, f_{A}$ is well-defined.
We say that $B \subseteq \omega_{1}$ is faster than $A$ iff $f_{A}<f_{B}$ on a club.
Lemma 6.23 For any $A$, there is a faster $B$ in a further stationary-preserving forcing extension.

Given this lemma, we prove theorem 6.18: Set $A_{0}=R_{0}=\emptyset$. By lemma 6.23, there is $A_{1}$ which is faster than $A_{0}$ in a stationary-preserving forcing extension. $A_{1}$, together with a club $C_{1}$ witnessing that $A_{1}$ is faster than $A_{0}$ can be coded by a real $R_{1}$ by a ccc almost disjoint coding; moreover we can use an almost disjoint family $\mathcal{A}$ of size $\omega_{1}$ on $\omega$ which is an element of $L_{\omega_{2}}[r]$ to code $A_{1}, C_{1}$. In particular, we can choose $\mathcal{A}$ such that $\mathcal{A}=\left\langle a_{i}: i \in\right.$ $\left.\omega_{1}\right\rangle$ and for each $i<\omega_{1}, a_{i}=\left\{\pi\left(b_{i} \cap n\right): n \in \omega\right\}$ where $\pi$ is some fixed arithmetical bijection $2^{<\omega} \rightarrow \omega$ and $\left\langle b_{i}: i<\omega_{1}\right\rangle$ is such that for all $i<\omega_{1}$, $b_{i}:=$ the $<_{\mathbf{L}[r]}$-least real distinct from $b_{j}, j<i$. Thus in this model,

$$
\exists r \exists R_{1} \forall \alpha<\omega_{1}\left(\alpha \in C\left(R_{1}\right)\right) \rightarrow\left(f_{A\left(R_{1}\right)}(\alpha)<f_{\emptyset}(\alpha)\right)
$$

where $C\left(R_{1}\right)$ denotes the club $C_{1}$ and $A\left(R_{1}\right)$ denotes set $A_{1}$ coded by $R_{1}$ s.t. $\alpha \in C\left(R_{1}\right) \leftrightarrow\left|R_{1} \cap a_{2 \alpha}\right|<\omega$ and $\alpha \in A\left(R_{1}\right) \leftrightarrow\left|R_{1} \cap a_{2 \alpha+1}\right|<\omega$.

Claim: The above statement is equivalent to a $\Sigma_{2}\left(\mathrm{H}_{\omega_{1}}\right)$-statement.
Proof: We give a sketch. First note that for a fixed $\beta<\omega_{1}, F=\left\langle a_{i}: i<\beta\right\rangle$ is $\Delta_{1}\left(\mathrm{H}_{\omega_{1}}\right)$, since $G=\left\langle b_{i}: i<\beta\right\rangle$ is $\Delta_{1}\left(\mathrm{H}_{\omega_{1}}\right)$ since

$$
x=b_{i} \leftrightarrow x=\min _{<\mathbf{L}[r]}\left\{y \in \mathbf{L}[r]: y \subseteq \omega \wedge \forall j<i y \neq b_{j}\right\}
$$

is $\Delta_{1}\left(\mathrm{H}_{\omega_{1}}\right)$ :

- $\Pi_{1}: \forall j<i x \neq b_{j} \wedge \forall y \subseteq \omega\left(\forall j<i y \neq b_{j}\right) \rightarrow y \geq_{\mathbf{L}[r]} x$
- $\Sigma_{1}: \exists \gamma x \in \mathbf{L}_{\gamma}[r] \wedge \forall j<i x \neq b_{j} \wedge$

$$
\forall y \in \mathbf{L}_{\gamma}[r]\left(y \subseteq \omega \wedge\left(\forall j<i y \neq b_{j}\right) \rightarrow y \geq_{\mathbf{L}[r]} x\right.
$$

Furthermore,

1. $\alpha \in C\left(R_{1}\right) \leftrightarrow\left|R_{1} \cap a_{2 \alpha}\right|<\omega$,
2. $A\left(R_{1}\right) \cap \alpha=\left\{\beta<\alpha:\left|R_{1} \cap F(2 \beta+1)\right|<\omega\right\}$,
3. $f_{A\left(R_{1}\right)}(\alpha)<f_{\emptyset}(\alpha) \leftrightarrow$
$\min \left\{\beta: L_{\beta+1}\left[A\left(R_{1}\right) \cap \alpha\right] \models \alpha \cong \omega\right\}<\min \left\{\beta: L_{\beta+1} \models \alpha \cong \omega\right\}$.
So " $\alpha \in C\left(R_{1}\right)$ " is $\Delta_{0}\left(\mathrm{H}_{\omega_{1}}\right)$ and $" f_{A\left(R_{1}\right)}(\alpha)<f_{\emptyset}(\alpha)$ " is $\Pi_{1}\left(\mathrm{H}_{\omega_{1}}\right)$, since $x \subseteq \omega$ is finite iff $\exists n \in \omega \forall m \in x m<n$, which proves our claim. $\square_{\text {Claim }}$

Proof of Theorem 6.18 continued: By $\boldsymbol{\Sigma}_{3}^{1}$-absoluteness for stationarypreserving forcing, the above statement holds in the ground model V. But we can repeat this, obtaining $A_{n+1}{ }^{\prime}$ which is faster than $A_{n}{ }^{\prime}$ for each $n$. Thus on a club, $\forall n \in \omega f_{A_{n+1}{ }^{\prime}}<f_{A_{n}{ }^{\prime}}$, a contradiction. $\square_{\text {Theorem 6.18 }}$

Proof of Lemma 6.23: Consider the forcing $P$ whose conditions are pairs $(b, c)$ where

$$
\begin{gathered}
c \stackrel{b n d}{\subseteq} \omega_{1}, c \text { closed }, b \subseteq \max c \\
\forall \alpha \in c L_{f_{A}(\alpha)}[r, b \cap \alpha] \vDash \alpha \cong \omega
\end{gathered}
$$

and conditions are ordered by

$$
\left(b_{0}, c_{0}\right) \leq\left(b_{1}, c_{1}\right) \text { iff } c_{0} \text { end-extends } c_{1} \text { and } b_{0} \cap \max c_{1}=b_{1}
$$

Claim: Any condition $(b, c)$ can be extended to increase max $c$ above any countable ordinal $\gamma$.

Proof: $\mathrm{H}_{\omega_{2}}=L_{\omega_{2}}[r, A] \models \neg \omega_{1} \cong \omega$. As in claim 2 of lemma 6.22 , we obtain that $\left\{\alpha<\omega_{1}: \alpha=\omega_{1} \cap h^{L_{\omega_{2}}[r, A]}(\alpha \cup\{r, A\})\right\}$ is club, so we can choose $\alpha>\gamma$ inside. As $h^{L_{\omega_{2}}[r, A]}(\alpha \cup\{r, A\}) \vDash \neg \omega_{1} \cong \omega$, observing the transitive collapse, $L_{\xi}[r, A \cap \alpha] \vDash \neg \alpha \cong \omega$, hence $f_{A}(\alpha) \geq \xi>\alpha$. Hence we obtain a condition by adding $\alpha$ to $c$ and extending $b$ to any $b^{\prime} \subseteq \alpha$ such that $L_{\xi}\left[r, b^{\prime}\right] \models \alpha \cong \omega$ : As $L_{\xi}[r] \vDash Z F^{-}$, there is a bijection $\pi: \alpha \rightarrow$ $\alpha \times \alpha, \pi \in L_{\xi}[r]$. Choose $\pi$ as the $<_{\mathbf{L}[r]}$-least such bijection. Let $f$ be a surjection $f: \omega \rightarrow \alpha$. Note that $\operatorname{ot}(\alpha \backslash \max c)=\alpha$, so we can use $f$ to set $(\max c+\gamma) \in b^{\prime}: \leftrightarrow \pi(\gamma) \in f$ for all $\gamma<\alpha$. As $\pi \in L_{\xi}[r], b^{\prime} \in L_{\xi}\left[r, b^{\prime}\right]$ and $\alpha \in L_{\xi}[r]$, it follows that $f \in L_{\xi}\left[r, b^{\prime}\right]$, implying $L_{\xi}\left[r, b^{\prime}\right] \models \alpha \cong \omega$ and finally $L_{f_{A}(\alpha)}\left[r, b^{\prime}\right] \models \alpha \cong \omega . \square_{\text {Claim }}$

Thus if $G$ is $P$-generic, then $B:=\bigcup\{b: \exists c(b, c) \in G\}$ is faster than $A$ which is witnessed by the club $C:=\bigcup\{c: \exists b(b, c) \in G\}$.

It remains to show that $P$ is stationary-preserving: Suppose $b=(b, c) \in P$, $X$ is stationary and $\sigma$ is a $P$-name for a club. Similar to lemma 6.22, we let $\gamma:=\omega_{2}^{\mathbf{L}[r]}$ and define $C_{0} \supseteq C_{1}$ as

$$
\begin{gathered}
C_{0}:=\left\{\alpha<\omega_{1}: \alpha=\omega_{1} \cap h^{L_{\gamma}[r]}(\alpha \cup\{p, \sigma, r\})\right\}, \\
C_{1}:=\left\{\alpha<\omega_{1}: \alpha=\omega_{1} \cap h^{L_{\gamma+\omega}[r]}(\alpha \cup\{p, \sigma, r, \gamma\})\right\} .
\end{gathered}
$$

We obtain the following properties:

1. $\left(\alpha \in C_{0} \wedge \beta<\alpha\right) \rightarrow\left(p \in L_{\alpha}[r] \wedge \forall q \leq p q \in L_{\alpha}[r] \rightarrow(\exists r \leq q r \in\right.$ $L_{\alpha}[r] \wedge r \Vdash \beta^{*} \in \sigma$ for some $\beta^{*}$ such that $\left.\beta<\beta^{*}<\alpha\right)$ ),
2. $\alpha \in C_{1} \rightarrow\left(f_{A}(\alpha)>\alpha \wedge C_{0} \cap \alpha \in L_{f_{A}(\alpha)}[r]\right)$.

Let $M_{\alpha}:=h^{L_{\gamma}[r]}(\alpha \cup\{p, \sigma, r\}), N_{\alpha}:=h^{L_{\gamma+\omega}[r]}(\alpha \cup\{p, \sigma, r, \gamma\})$. Similar to lemma 6.22 it follows that $C_{0}$ and $C_{1}$ are club and satisfy property 1. We have to show that $C_{1} \subseteq C_{0}$ and that property 2 holds: If $\alpha \in C_{1}$, i.e. $\alpha=\omega_{1} \cap N_{\alpha}$, then $\alpha=\omega_{1} \cap\left(N_{\alpha} \cap L_{\gamma}[r]\right)$ and since $N_{\alpha} \prec L_{\gamma+\omega}[r]$, it follows that $\left(N_{\alpha} \cap L_{\gamma}[r]\right) \prec L_{\gamma}[r]$ : Assume $\left(N_{\alpha} \cap L_{\gamma}[r]\right) \models \varphi\left(n_{1}, \ldots, n_{k}\right)$. Since $N_{\alpha} \cap L_{\gamma}[r]=\left(L_{\gamma}[r]\right)^{N_{\alpha}}$ and $N_{\alpha} \prec L_{\gamma+\omega}[r]$, this is equivalent to $L_{\gamma}[r] \models$ $\varphi\left(n_{1}, \ldots, n_{k}\right)$. Altogether this implies that $\alpha=\omega_{1} \cap M_{\alpha}$, as $\left(\omega_{1} \cap M_{\alpha}\right) \subseteq$ $\left(\omega_{1} \cap\left(N_{\alpha} \cap L_{\gamma}[r]\right)\right)$ and $\omega_{1} \cap M_{\alpha} \supseteq \alpha$; hence, $\alpha \in C_{0}$.

To show that property 2 holds, assume $\alpha \in C_{1}$, i.e. $\alpha=\omega_{1} \cap N_{\alpha}$. Since $N_{\alpha} \prec L_{\gamma+\omega}[r]=L_{\gamma+\omega}[r, A]\left(A \in L_{\gamma}[r]\right)$ and $L_{\gamma+\omega}[r, A] \models \neg \omega_{1} \cong \omega$, observing the transitive collapse, we have $L_{\xi}[r, A \cap \alpha] \vDash \neg \alpha \cong \omega$, hence $f_{A}(\alpha) \geq \xi>\alpha$.

Now we show that, still assuming $\alpha \in C_{1}, C_{0} \cap \alpha \in L_{f_{A}(\alpha)}[r]$ :
$N_{\alpha} \prec L_{\gamma+\omega}[r]$, hence its transitive collapse is of the form $L_{\xi}[r], \xi<\omega_{1}$. Let $\pi$ denote the collapsing map. Now $\left.M_{\beta}=h^{L_{\gamma}[r]}(\beta \cup\{p, \sigma, r\})\right\}=h^{N_{\alpha} \cap L_{\gamma}[r]}(\beta \cup$ $\{p, \sigma, A\})\}$ by elementarity $\left(N_{\alpha} \cap L_{\gamma}[r] \prec L_{\gamma}[r]\right)$. Collapsing $N_{\alpha}$, it follows that $\left.\pi\left(M_{\beta}\right)=h^{L_{\pi(\gamma)}[r]}(\beta \cup\{p, \pi(\sigma), r\})\right\}$. Finally,

$$
C_{0} \cap \alpha=\left\{\beta<\alpha: \beta=\alpha \cap h^{L_{\pi(\gamma)}[r]}(\beta \cup\{p, \pi(\sigma), r\})\right\}
$$

and since $f_{A}(\alpha) \geq \xi>\pi(\gamma)>\alpha$, we have $C_{0} \cap \alpha \in L_{f_{A}(\alpha)}[r]$.

Thus we have shown that clubs $C_{0} \supseteq C_{1}$ with the above-claimed properties exist. Now, using those clubs, we finish the proof of lemma 6.23:

Choose $\alpha \in \lim C_{1} \cap X$ and let $\left\langle\gamma_{n}: n \in \omega\right\rangle$ be an increasing $\omega$-sequence contained in $C_{1}$ with supremum $\alpha$. We inductively define conditions $q_{n}=$ $\left(b_{n}, c_{n}\right)$ with $\max c_{n}=\gamma_{n}$ as follows: Set $q_{0}$ to be the $\mathbf{L}[r]$-least extension of $p \in L_{\gamma_{0}}[r]$ such that max $c_{0}=\gamma_{0}$; similar to lemma 6.21 , it follows that $q_{0} \in L_{\gamma_{1}}[r]$. If $q_{n} \in L_{\gamma_{n+1}}[r]$ is defined, let $q_{n}^{\prime}=\left(b_{n}^{\prime}, c_{n}^{\prime}\right)$ be the $<_{\mathbf{L}[r]}$-least extension of $q_{n}$ such that $\gamma_{n} \in b_{n}^{\prime}$ and $\exists \beta_{n}>\gamma_{n} q_{n}^{\prime} \Vdash \beta_{n} \in \sigma$; by property 1 above, it follows that this is possible and moreover $q_{n}^{\prime} \in L_{\gamma_{n+1}}[A]$. Also, letting $\gamma_{n}^{\prime}:=\max c_{n}^{\prime}$, property 1 implies that $\gamma_{n}^{\prime}$ is less than the least element of $C_{0}$ above $\gamma_{n}$. Again similar to lemma 6.21, set $c_{n+1}:=c_{n}^{\prime} \cup\left\{\gamma_{n+1}\right\}$ and extend $b_{n}^{\prime}$ to $b_{n+1} \subseteq \gamma_{n+1}$ such that $\left(b_{n+1}, c_{n+1}\right)$ is a condition in $L_{\gamma_{n+2}}[r]$.

Let $b:=\bigcup_{n \in \omega} b_{n}$ and $c:=\bigcup_{n \in \omega} c_{n} \cup\{\alpha\}$. Then $\left\{\gamma \in C_{0} \cap\left[\gamma_{0}, \alpha\right): \gamma \in\right.$ $b\}=\left\{\gamma_{n}: n \in \omega\right\}$ (this holds since all elements of $C_{0}$ have to be limit ordinals by definition of $C_{0}$ ). By property 2 above, $\left\{\gamma_{n}: n \in \omega\right\}$ belongs to $\mathbf{L}_{f_{A}(\alpha)}[r, b]$, therefore $\mathbf{L}_{f_{A}(\alpha)}[r, b] \models \alpha \cong \omega$, i.e. $q:=(b, c)$ is a condition.

As $q$ forces that $\sigma \cap \alpha$ is unbounded in $\alpha, q \Vdash \alpha \in \sigma$. Since $\alpha$ was chosen to belong to $X$, we have $q \Vdash X \cap \sigma \neq \emptyset$. $\square_{\text {Lemma } 6.23}$

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