The Edinburgh Topology I

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The Edinburgh topology

Let κ be a regular and uncountable cardinal. Let NS_κ denote the non-stationary ideal on κ. In a nutshell, the Edinburgh topology is obtained from the bounded topology by working with non-stationary rather than bounded subsets of κ. One could also generalize this further by using arbitrary $<\kappa$-closed ideals on κ.

Definition

Let $\heartsuit = \{f : A \to 2 \mid A \in \mathrm{NS}_\kappa\}$. The Edinburgh topology on 2^κ is provided by the basis $\{[f] \mid f \in \heartsuit\}$ of (Edinburgh) clopen subsets of 2^κ, where $[f] = \{g \in 2^\kappa \mid f \subseteq g\}$.

Many, but not all of our results also apply to the generalized Baire space ${}^\kappa\kappa$ rather than ${}^\kappa 2$.

Some basic observations

- The Edinburgh topology refines the bounded topology.
- The basis of the Edinburgh topology has size 2^κ.
- Thus, there are 2^{2^κ}-many Edinburgh open sets.

In the following, we will abbreviate Edinburgh by the letter E, for example, we will E-open, E-closed, etc.

I want to provide some arguments showing that the Edinburgh topology leads to an interesting structure theory of Edinburgh Borel sets.

A normal form for E-closed sets

For $x \in 2^\kappa$, let $I_x = \{x \restriction A \mid A \in \mathrm{NS}_\kappa\}$ be the (principal) ideal on $\heartsuit$ generated by x.

Proposition

If $P \subseteq \heartsuit$, then $[P] = \{x \in 2^\kappa \mid I_x \subseteq P\}$ is an E-closed subset of 2^κ. Conversely, every E-closed subset of 2^κ is of the form $[P]$ for some $P \subseteq \heartsuit$ that is closed under restrictions.

Proof: If $X \subseteq 2^\kappa$ is E-closed, let $P = \{x \restriction A \mid x \in X \wedge A \in \mathrm{NS}_\kappa\}$. Now if $x \in X$, then clearly $I_x \subseteq P$. If $x \notin X$, then since X is E-closed, there is $A \in \mathrm{NS}_\kappa$ with $X \cap [x \restriction A] = \emptyset$. But then, $x \restriction A \notin P$, hence also $I_x \not\subseteq P$. The first statement of the proposition is verified similarly. □

We will see in the following that this normal form is actually useful.

An Edinburgh open set that is not Edinburgh F_σ

Let $U = \{x \in 2^\kappa \mid x \subseteq \kappa \text{ is unbounded}\}$. Note: U is Edinburgh open!

Proposition

U is not Edinburgh F_σ, i.e. no κ-union of E-closed sets.

Proof: Assume for a contradiction that it is, i.e. $U = \bigcup_{\alpha<\kappa}[P_\alpha]$, with each $P_\alpha \subseteq \heartsuit$. We inductively construct an unbounded subset of κ which is not in U. We say that $f \in \heartsuit$ is *bounded* in κ if $\{\gamma < \kappa \mid f(\gamma) = 1\}$ is. Starting with $f_0 = \emptyset$, we construct a continuous and increasing κ-sequence of bounded f_α's in $\heartsuit$ so that $f_{\alpha+1}(\gamma) = 1$ for some $\gamma \geq \alpha$, and so that $f_{\alpha+1} \notin P_\alpha$ for all $\alpha < \kappa$: If some P_α contained all bounded extensions of f_α, then $[P_\alpha]$ would have to contain a bounded set. In the end, $f = \bigcup_{\alpha<\kappa} f_\alpha$ is an unbounded subset of κ which is not in U, yielding our desired contradiction. $\square$

An Edinburgh G_δ set that is neither E-open nor E-F_σ

Let $\mathcal{C} = \{x \in 2^\kappa \mid x \subseteq \kappa \text{ is club}\}$. Since $\mathcal{C}$ is G_δ, it is also E-G_δ. Since every E-open set contains a nonstationary set, $\mathcal{C}$ is not E-open.

Proposition

$\mathcal{C}$ is not Edinburgh F_σ.

Proof Sketch: Assume for a contradiction that $\mathcal{C} = \bigcup_{\alpha<\kappa}[P_\alpha]$ with each $P_\alpha \subseteq \heartsuit$. A slightly more careful diagonalization argument as for the set U above, this time using closed and bounded subsets of κ, adding only a single function value 1 in each step, and making sure this happens exactly on a club subset C of κ, yields that $C \notin \bigcup_{\alpha<\kappa}[P_\alpha]$, our desired contradiction. □

The club filter is not Edinburgh Borel

Note first that the club filter is both Edinburgh dense and co-dense. Moreover, as usual, one can verify the Baire category theorem for the Edinburgh topology. That is, every κ-intersection of Edinburgh open dense sets is Edinburgh dense. The same proof shows also that every κ-intersection of Edinburgh open dense sets contains both an element of the club filter, and of the nonstationary ideal. It follows, as usual, that the club filter cannot have the Edinburgh Baire property. However, again as usual, every Edinburgh Borel set does have the Edinburgh Baire property.

Corollary

The club filter is not Edinburgh Borel.
In particular, not every subset of 2^κ is Edinburgh Borel.