Leftovers from the last Session

Exercise 1 *Assume $(A,<)$ is a well-order and B is a proper initial segment of A (with respect to $<$), i.e. there exists $a \in A$ such that B is the set of all predecessors of a in $(A,<)$ - in Kunen's notation: $B = \mathrm{pred}(A, a, <)$.*

Then $\mathrm{type}(B,<) < \mathrm{type}(A,<)$.

Use this to give a more formal proof of part 1 of exercise 4 from last time:

$$\text{If } \alpha < \beta, \text{ then } \gamma + \alpha < \gamma + \beta.$$

Exercise 2 *= Exercise 8 from last time.*

Ordinal Exponentiation, Cantor's Normal Form Theorem

Exercise 3 *Does $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$ hold?*

Exercise 4 *Let α be any ordinal and show that there exists a largest δ such that $\omega^\delta \leq \alpha$.*

Exercise 5 *Let α be any ordinal and let δ be maximal such that $\omega^\delta \leq \alpha$. Then there exists a largest $n < \omega$ such that $\omega^\delta \cdot n \leq \alpha$.*

Exercise 6 (Cantor's Normal Form Theorem) *Each ordinal $\alpha \neq \emptyset$ can be displayed in the following form*

$$\alpha = \omega^{\beta_1} \cdot k_1 + \ldots + \omega^{\beta_n} \cdot k_n,$$

where $1 \leq n < \omega$, $\alpha \geq \beta_1 > \ldots > \beta_n \geq 0$ and $1 \leq k_i < \omega$ for each $i = 1, \ldots, n$.

Exercise 7 *If $\alpha = \omega^\delta$ and $\beta < \alpha$, then there is $\beta_0 < \delta$ and $k \in \omega$ such that $\beta < \omega^{\beta_0} \cdot k$.*

Exercise 8 *Let α be a limit ordinal and show that the following are equivalent:*

1. $\forall \beta, \gamma < \alpha \ (\beta + \gamma < \alpha)$,
2. $\forall \beta < \alpha \ (\beta + \alpha = \alpha)$,
3. $\exists \delta \ (\alpha = \omega^\delta)$.

Hint: Show $1 \iff 3$ and $1 \iff 2$.

Note: Ordinals of the form ω^δ are called indecomposible ordinals.

Exercise 9 (A non-recursive definition of Ordinal Exponentiation) *Define a function F by*

$$F(\alpha, \beta) = \{f\colon \beta \to \alpha\colon |\{\xi\colon f(\xi) \neq 0\}| < \omega\}.$$

Thus $F(\alpha, \beta)$ is the set of functions from β to α which map all but finitely many ordinals to 0.

For $f \neq g$ both in $F(\alpha, \beta)$, we define $f \lhd g$ iff $f(\xi) < g(\xi)$ where ξ is the largest ordinal such that $f(\xi) \neq g(\xi)$.

Show that $(F(\alpha, \beta), \lhd)$ is a well-order and (by induction on β; for $\beta = 0$, don't forget about the empty function) that

$$\alpha^\beta = \mathrm{type}(F(\alpha, \beta), \lhd).$$