Cardinals

Exercise 1 (Schroeder-Bernstein Theorem) *$A \precsim B$ and $B \precsim A$ implies $A \approx B$, i.e. if there is a 1-1 function from A to B and a 1-1 function from B to A then in fact there is a bijection from A to B.*

Hint: Let f be an injection from A to B, let g be an injection from B to A. Let $S_0 = B \setminus \operatorname{range} f$. If $S_0 = \emptyset$, we are of course finished. Assume otherwise and define (inductively) $S_{n+1} = f''(g''S_n)$ for all natural numbers $n \geq 0$. Define h as follows:

$$h(x) = \begin{cases} g(x) & \text{if } x \in \bigcup_{n\in\omega} S_n \\ f^{-1}(x) & \text{if } x \in B \setminus (\bigcup_{n\in\omega} S_n) \end{cases}$$

Try to illustrate how h operates by drawing an appropriate picture and show that h is a bijection from B to A. Note that the proof does not use the Axiom of Choice.

Exercise 2 *$A \precsim B$ iff $|A| \leq |B|$.*

Exercise 3 *X is infinite iff $\exists Y \subsetneq X$ and $\exists f : Y \to X$ which is a bijection.*

Exercise 4 *If $\kappa < |X|$ then there exists $Y \subseteq X$ with $|Y| = \kappa$.* [1]

Exercise 5 *If $A \subseteq \alpha$, then $\operatorname{type}(A, \in) \leq \alpha$.*

Exercise 6 *If $A \subseteq \kappa$, then $|A| = \kappa$ iff $\operatorname{type}(A, \in) = \kappa$.*

Exercise 7

- *For all ordinals α, $\aleph_\alpha \geq \alpha$.*
- *For every γ, there exists $\kappa \geq \gamma$ with $\kappa = \aleph_\kappa$.*

Definition 1 *We say that X is T-finite (T stands for Tarski here) iff every nonempty $S \subseteq \mathcal{P}(X)$ has a $\subset$-maximal element, i.e. an $u \in S$ such that there is no $v \supsetneq u$ in S. We say that X is T-infinite iff X is not T-finite.*

Exercise 8 *Every finite set is T-finite.*

Exercise 9 *Every infinite set is T-infinite.*

[1] α, β, γ, δ usually denote ordinals, while κ and λ usually denote cardinals.