The Mostowski collapse

Work in ZFC.

Exercise 1 *Show that given any set X and any relation R on X, R is set-like on X.*

Exercise 2 *Show that the $\in$ relation is well-founded and set-like on $\mathbf{V}$.*

Exercise 3 *Show that given any class $\mathbf{A}$, the $\in$ relation is well-founded and set-like on $\mathbf{A}$.*

Assume in the following three exercises that a class $\mathbf{A}$ is given and that $\mathbf{M}$ and $\mathbf{G}$ are (uniquely) such that $\mathbf{M}$ is transitive and $\mathbf{G}$ is an isomorphism from $(\mathbf{A}, \in \restriction \mathbf{A})$ to $(\mathbf{M}, \in)$.[1]

Exercise 4

- *If $\emptyset \in \mathbf{A}$, then $\mathbf{G}(\emptyset) = \emptyset$.*
- *If $\alpha + 1 \subseteq \mathbf{A}$, then $\mathbf{G}(\alpha) = \alpha$.*
- *If $\alpha \subseteq \mathbf{A}$, $x \subseteq \alpha$ and $x \in \mathbf{A}$, then $\mathbf{G}(x) = x$.*
- *If $Y \subseteq \mathbf{A}$ is transitive, $x \subseteq Y$ and $x \in \mathbf{A}$, then $\mathbf{G}(x) = x$.*

Exercise 5 *Under what (sufficient and neccessary) conditions (on $\mathbf{A}$) is $\mathbf{G}$ the identity function on $\mathbf{A}$?*

Exercise 6

- *Assume $\beta \in \mathbf{A}$ and $\alpha = \mathbf{A} \cap \beta$. Then $\mathbf{G}(\beta) = \alpha$.*
- *Assume $Y \in \mathbf{A}$ and $X = \mathbf{A} \cap Y$ is transitive. Then $\mathbf{G}(Y) = X$.*

Exercise 7 *Define $x\mathbf{R}y$ iff $x \in \mathrm{trcl}(y)$. Show that $\mathbf{R}$ is well-founded [2] and set-like (on $\mathbf{V}$). Let $\mathbf{G}$ be the Mostowski collapsing function of $(\mathbf{V}, \mathbf{R})$. Show that $\mathbf{G}(x) = \mathrm{rank}(x)$ for each x.*[3]

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[1] this is, by exercise 3, a special case of the situation of III, definition 5.9 of Kunen's book, where $\mathbf{R}$ is the $\in$ relation. $\in \restriction \mathbf{A}$ is the $\in$ relation on $\mathbf{A}$.

[2] Hint: Show (and use) that $x\mathbf{R}y$ implies $\mathrm{rank}(x) < \mathrm{rank}(y)$.

[3] Hint: Use induction on $\mathrm{rank}(x)$. Show first that $\mathbf{G}(x)$ is an ordinal for each x.

Leftovers

Exercise 8 *Assume $\lambda \leq \kappa$ and show that there are κ^λ-many injective functions from λ to κ (i.e. $|\{f \in {}^\lambda\kappa \colon f \text{ injective }\}| = \kappa^\lambda$).*

Hint: Given $f \in {}^\lambda\kappa$, find a way to construct an injective $g_f \in {}^\lambda\kappa$ in an injective way, i.e. it should be the case that whenever $f_0 \neq f_1 \in {}^\lambda\kappa$, $g_{f_0} \neq g_{f_1}$.

Additional hint:
Use a fixed partition of κ into λ-many disjoint pieces, each of size κ. Define g_f so that $g_f(i)$ is the $f(i)^{\text{th}}$ element of the i^{th} disjoint piece for each $i < \lambda$. Check that this works!

Exercise 9 *Show (without assuming* GCH*) using the definition*

$$\kappa^{<\lambda} := \sup\{\kappa^\delta \colon \delta < \lambda, \delta \in \text{Card}\} :$$

- *If κ is strongly inaccessible, then $\kappa^{<\kappa} = 2^{<\kappa} = \kappa$.*
- *If κ is weakly inaccessible, then $\kappa^{<\kappa} = 2^{<\kappa}$.*

Hint for the 2nd item: If κ happens to be strongly inaccessible, we are done using the first item. Thus we may assume (and use) that there is $\delta < \kappa$ such that $2^\delta \geq \kappa$.