Relativization and Absoluteness, part 2

Note: α, β, γ and δ always denote ordinals, κ and λ always denote cardinals.

Exercise 1 *Show by induction on* $\operatorname{rank}(x)$*: Given a class* $\mathbf{M}$ *(which is not necessarily transitive),* $\operatorname{rank}^{\mathbf{M}}(x) \leq \operatorname{rank}(x)$ *for all* $x \in \mathbf{M}$.

Exercise 2 *Assume* $\mathbf{M}$ *is a transitive model of* ZFC*. Show that* $|\gamma|^{\mathbf{M}} \geq |\gamma|$, $\operatorname{cf}^{\mathbf{M}}(\gamma) \geq \operatorname{cf}(\gamma)$ *and* $(\gamma^+)^{\mathbf{M}} \leq \gamma^+$.

Exercise 3 *Assume* γ *is a limit ordinal. Show that* $|\alpha|$ *and* $\operatorname{cf}(\alpha)$ *are absolute for* $R(\gamma)$ *and absolute for* $H(\lambda)$.

Exercise 4 *Show that* $|\alpha|$ *and* $\operatorname{cf}(\alpha)$ *are absolute for* $H(\kappa)$.

Exercise 5 *Show that if* $\kappa^+ \in R(\gamma)$*, then* $(\kappa^+)^{R(\gamma)} = \kappa^+$ *and if* $\kappa^+ \in H(\lambda)$*, then* $(\kappa^+)^{H(\lambda)} = \kappa^+$.

Exercise 6
Assume κ *is strongly inaccessible and show that* ω_α *is absolute for* $R(\kappa)$.

Exercise 7 *If* $\omega_\alpha \in R(\gamma)$*, then* $(\omega_\alpha)^{R_\gamma} = \omega_\alpha$*. Similar for* $H(\lambda)$. [1]

Definition 1 $r\colon A \to \mathbf{ON}$ *is a ranking function for a relation* R *on* A *iff* xRy *implies* $r(x) < r(y)$.

Exercise 8 *Show that for any set* A*, the usual rank function restricted to* A *and the usual rank function relativized to* A *are both ranking functions for* $\in$ *on* A.

Exercise 9 *Show that* R *is well-founded on* A *iff there exists a ranking function for* R*. Note that this gives a* Σ_1 *definition of well-foundedness (the usual definition is* Π_1*) and use this to show that well-foundedness is absolute for transitive models of* ZFC.

Forgotten last time

Work in ZFC.

Exercise 10 *Define* $x\mathbf{R}y$ *iff* $x \in \operatorname{trcl}(y)$*. Show that* $\mathbf{R}$ *is well-founded* [2] *and set-like (on* $\mathbf{V}$*). Let* $\mathbf{G}$ *be the Mostowski collapsing function of* $(\mathbf{V}, \mathbf{R})$*. Show that* $\mathbf{G}(x) = \operatorname{rank}(x)$ *for each* x.[3]

[1] Hint: By induction on α. Strictly speaking, the ω_α are defined by transfinite recursion over all ordinals, but to define a particular ω_α, it of course suffices to do transfinite recursion over $\alpha + 1$. Maybe start by thinking about $\omega_1, \omega_2, \ldots, \omega_\omega, \ldots$. Try to observe that the claim of the exercise will hold once ω_α is defined in a "reasonable" way.

[2] Hint: Show (and use) that $x\mathbf{R}y$ implies $\operatorname{rank}(x) < \operatorname{rank}(y)$.

[3] Hint: Use induction on $\operatorname{rank}(x)$. Show first that $\mathbf{G}(x)$ is an ordinal for each x.