Predicate Calculus pt. 3

Exercise 5 from last time.

Definition 1 *Two $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$ are* elementary equivalent *iff they satisfy the same $\mathcal{L}$-sentences.*

Exercise 1 *If $\mathcal{L}$ is finite and $\mathcal{A}$ and $\mathcal{B}$ are both finite and elementary equivalent $\mathcal{L}$-structures, then they are isomorphic.*

Hint: There is an $\mathcal{L}$-sentence which completely describes $\mathcal{A}$ and $\mathcal{B}$. Use this sentence to construct an isomorphism.

Exercise 2 *The statement of the previous exercise remains valid if one drops the assumption of finiteness of $\mathcal{L}$.*

Hint: If $\mathcal{L}$ is infinite, assume for a contradiction that $\mathcal{A}$ and $\mathcal{B}$ are elementary equivalent, but not isomorphic. Then for every bijection from A to B there is a sign Z_F from L which does not commute with F. Consider now the finite sublanguage $\mathcal{L}' = \{Z_F \colon F \text{ is a bijection from } A \text{ to } B\}$ of $\mathcal{L}$.

Exercise 3 *Let R be the ordered field of reals $(\mathbb{R}, +, \cdot, 0, 1, <)$. Show that there is a structure S in the language of R which is elementary equivalent to R but not Archemedean (i.e. S has elements which are larger than every natural number n).*

Hint: Introduce a new constant c. Then every finite subset of $\mathrm{Th}(R) \cup \{n < c \colon n \in \mathbb{N}\}$ has a model.

Exercise 4 *Let R be as in the previous exercise and let $f \colon \mathbb{R} \to \mathbb{R}$ be a function with $f(0) = 0$. Consider the structure (R, f) and let (R^*, f^*) be elementary equivalent to (R, f) but not Archemedean (this is possible by the proof of the previous exercise) with underlying set $\mathbb{R}^*$. We call an element ϵ of $\mathbb{R}^*$ infinitesimal iff $-\frac{1}{n} < \epsilon < \frac{1}{n}$ for all natural numbers $n > 0$. Show that f is continuous at 0 iff f^* maps infinitesimals to infinitesimals.*

Hint: Show both directions indirectly. Note that every single natural number (and therefore it's reciprocal) is definable in R (or R^*) by repeated addition of 1, but the set of natural numbers isn't. Thus one has to be careful when formalizing this proof - instead of saying $R \models \exists n \in \mathbb{N}\ \varphi(n)$ you have to say there exists $n \in \mathbb{N}$ $R \models \varphi(n)$ for example.

Exercise 5 *The* cut rule

$$\frac{\Delta \succ \Gamma \cup \{\phi\},\ \Delta \cup \{\phi\} \succ \Gamma}{\Delta \succ \Gamma}$$

is valid as its conclusion is valid if both its premises are valid. Show validity of the cut rule without using the completeness theorem, but by induction on the lengths of the proofs of its premises.

Exercise 6 *Let L_1 and L_2 be two languages and $L = L_1 \cap L_2$. Let T_1 be a consistent L_1-theory, let T_2 be a consistent L_2-theory and assume that T_1 and T_2 prove the same L-sentences. Show that $T_1 \cup T_2$ is consistent.*

Hint: Use the interpolation theorem in its general form, allowing sentences that also use constant symbols, function symbols and equality.

Exercise 7 *Let L be a language, $L_1 = L \cup \{P\}$ and $L_2 = L \cup \{P'\}$. Let $T(P)$ be an L_1-theory and $T(P')$ the L_2-theory obtained from $T(P)$ by replacing every occurence of P in the axioms of $T(P)$ by P'. $T(P)$* implicitely *defines P iff*

$$T(P) \cup T(P') \vdash \forall x\, (P(x) \iff P'(x)).$$

Show that if $T(P)$ defines P implicitely, then it does so explicitely. The latter means that there is an L-formula $\phi(x)$ so that

$$T(P) \vdash \forall x\, (P(x) \iff \phi(x)).$$

Hint: Replace x by a new constant and use the Interpolation theorem (again in its general form).

Exercise 8 *Let $\mathcal{B}$ be a substructure of $\mathcal{A}$. Show that for all universal formulas $\phi(x_1, \ldots, x_n)$ and all $b_1, \ldots, b_n \in B$,*

$$\mathcal{A} \models \phi[b_1, \ldots, b_n] \to \mathcal{B} \models \phi[b_1, \ldots, b_n].$$

What is the corresponding statement for existential formulas?