Set Theory pt. 2

Exercise 1 *Show that no set x is equal to it's power set $\mathcal{P}(x)$.*

Exercise 2 *Show that the comprehension axiom (Aussonderung) follows from the other axioms of* ZFC *(in particular you will need the replacement axiom (Ersetzung)) together with the assertion that the empty set exists $(\exists x \forall y \neg(y \in x))$.*

Exercise 3

- *If $f\colon A \to B$ is injective (1-1), then there exists $g\colon B \to A$ which is surjective (onto). This can be done without using the Axiom of Choice.*
- *If $f\colon \mathbb{N} \to B$ is surjective, then there exists $g\colon B \to \mathbb{N}$ which is injective. This can be done without using the Axiom of Choice.*
- *If $f\colon A \to B$ is surjective, then there exists $g\colon B \to A$ which is injective. This (in general) needs the Axiom of Choice.*

Exercise 4 (Schröder-Bernstein Theorem)
Show without using the Axiom of Choice: If $f\colon A \to B$ and $g\colon B \to A$ are injections, then there exists a bijection $h\colon B \to A$.

Hint: We may assume for simplicity of notation that $A \subseteq B$ and f is just the inclusion map. Let

$$C = \{g^n(x)\colon n \in \mathbb{N}, x \in B \setminus A\}.$$

Let $h(x) = g(x)$ for $x \in C$ and $h(x) = x$ for $x \in B \setminus C$. Show that this is a reasonable definition and that h is as desired.[1]

For the remaining exercises, let P be the set of all finite subsets of $\mathbb{N}$ and let $\beta\colon \mathbb{N} \to P$ be a bijection. Define $mE_\beta n \iff m \in \beta(n)$ and consider the structure $(\mathbb{N}, E_\beta)$ as a structure in the language of set theory.

Exercise 5 *Which axioms of* ZFC *hold in $(\mathbb{N}, E_\beta)$? Don't check the axiom of foundation, it's validity will depend on the particular choice of β; we will explore that in the following exercises.*

[1] $A \setminus B$ denotes the set-theoretic difference of A and B, i.e. A intersected with the complement of B. $g^0(x) = x$ and for any $n \in \mathbb{N}$, $g^{n+1}(x) = g(g^n(x))$. Note that (for our purposes) $0 \in \mathbb{N}$.

Exercise 6 *If we choose β to be such that*

$$\beta(2^{n_1} + \ldots + 2^{n_k}) = \{n_1, \ldots, n_k\}$$

for pairwise disjoint n_i, then $(\mathbb{N}, E_\beta)$ is well-founded, i.e. there is no infinite sequence $\langle x_i \colon i \in \mathbb{N}\rangle$ of elements of $\mathbb{N}$ such that $x_{i+1} E_\beta x_i$ for all $i \in \mathbb{N}$. Also $(\mathbb{N}, E_\beta)$ satisfies the axiom of foundation. This can either be checked directly or one may observe that in general, if $(\mathbb{N}, E_\beta)$ is well-founded, then it satisfies the axiom of foundation.

Exercise 7 *Find a bijection β from $\mathbb{N}$ to P such that $(\mathbb{N}, E_\beta)$ does not satisfy the axiom of foundation.*

Exercise 8 *Find a bijection β as above such that $(\mathbb{N}, E_\beta)$ is not well-founded, but satisfies the axiom of foundation.*

Hint: Replace $\mathbb{N}$ by $\mathbb{Z}$ and use an appropriate bijection $\beta^* \colon \mathbb{Z} \to P^*$ with $m \in \beta^*(n) \to m < n$, where P^* denotes the set of all finite subsets of $\mathbb{Z}$. Obtain β from β^* by using a bijection between $\mathbb{Z}$ and $\mathbb{N}$.