Set Theory pt. 3

Exercises 7 and 8 from last time, maybe also some more details about exercise 6.

Exercise 1 *Let M be a model of* ZFC*. An element a of M is a* nonstandard natural number *if $M \models a \in \omega$, but for all $n = 0, 1, \ldots$, $M \models \neg a = \underline{n}$. Show:*

- *If* ZFC *is consistent, it has a model with nonstandard natural numbers.*
- *There is no least nonstandard natural number in M.*

Definition 1 *$(W, <)$ is a well-ordering (Wohlordnung) iff $(W, <)$ is a linear ordering so that every nonempty subset S of W has a $<$-least element, i.e. there is $s \in S$ such that for all $t \in S$, $t \neq s$ implies $s < t$.*

Exercise 2
If $(A, <)$ is a well-order and $B \subseteq A$, then $(B, < \restriction B)$ is a well-order.

Notation: $< \restriction B$ denotes the restriction of $<$ to B, i.e.

$$< \restriction B = \{(x, y) \in <: x \in B \wedge y \in B\}.$$

Exercise 3 *Let $(W, <)$ be a well-ordering and let $f\colon W \to W$ be a strictly monotonous and increasing function (i.e. $x < y \to f(x) < f(y)$). Show that $f(x) \geq x$ for all $x \in W$, where $a \geq b$ abbreviates $b < a \vee b = a$. Find an example to illustrate that "strictly" is necessary.*

Exercise 4 *Let $(W, <)$ be a well-ordering, assume that W is infinite and that $f\colon W \to W$ is a strictly monotonous function.*[1] *Show that f is increasing. Find an example to illustrate that "infinite" is necessary.*

Exercise 5 *If $(W, <)$ is a well-order and $f\colon W \to W$ is an isomorphism respecting $<$ (i.e. f is a bijection and $w_0 < w_1 \iff f(w_0) < f(w_1)$), then f is the identity on W.*

Exercise 6 *Show that if $(W, <)$ is a well-ordering and W is infinite, then there exists a proper subset T of W such that $(W, <)$ is isomorphic to $(T, <)$.*

Exercise 7 *Show that for every set x there exists a smallest set y which is transitive and contains x as subset, where smallest means $\subseteq$-smallest, i.e. for every z which is transitive and contains x as subset, $z \supseteq y$. We call this y the* transitive closure *of x. Show also that there exists a smallest set y which contains x as element.*

[1] strictly monontonous of course means either strictly monotonous and increasing or strictly monotonous and decreasing