Set Theory pt. 4

More details on exercise 8 from one before last time.

Exercise 1 *Show that the Axiom of Choice follows from Zorn's Lemma.*

Hint: Let x be a collection of non-empty sets and consider a maximal partial choice function.

Definition 1 *If A is a set of ordinals, we let $\sup A$ denote the supremum of A, i.e. the least ordinal α so that for all γ in A, $\gamma \leq \alpha$. We define addition, multiplication and exponentiation of ordinals as follows (γ always denotes a limit ordinal):*

$$\alpha + 0 = \alpha \qquad \alpha + (\beta + 1) = (\alpha + \beta) + 1 \qquad \alpha + \gamma = \sup\{\alpha + \beta \colon \beta < \gamma\}$$

$$\alpha \cdot 0 = 0 \qquad \alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha \qquad \alpha \cdot \gamma = \sup\{\alpha \cdot \beta \colon \beta < \gamma\}$$

$$\alpha^0 = 1 \qquad \alpha^{\beta+1} = \alpha^\beta \cdot \alpha \qquad \alpha^\gamma = \sup_{\beta<\gamma} \alpha^\beta$$

Exercise 2 *Show that*[1]

- *If A is a set of ordinals, $\sup A = \bigcup A$; therefore $\sup A$ always exists,*
- $1 + \omega = \omega$,
- $\omega + \omega \cdot \omega = \omega \cdot \omega$,
- $\omega \cdot \omega^\omega = \omega^\omega$.

Note: In the following, α, β, γ, δ always denote ordinals without further mention.

Exercise 3 (transfinite induction) *Show that the principle of transfinite induction, which will be necessary for some of the subsequent exercises, is a theorem of* ZFC*:*

For any formula φ (which may also use parameters),

$$(\forall\gamma\ ((\forall\beta < \gamma\ \varphi(\beta)) \rightarrow \varphi(\gamma))) \rightarrow \forall\gamma\, \varphi(\gamma).$$

[1] like usually in mathematics, exponentiation binds stronger than multiplication, which in turn binds stronger than addition, i.e. if we write $8 + 2^5 \cdot 7$, this is supposed to mean $8 + ((2^5) \cdot 7)$

Hint: Assume for a contradiction that the left-hand side of the implication holds but γ is least such that $\neg\varphi(\gamma)$.

Exercise 4 *If $\alpha < \beta$, then $\gamma + \alpha < \gamma + \beta$ and $\alpha + \gamma \leq \beta + \gamma$. Give an example why $\leq$ cannot be replaced by $<$.*

Exercise 5 *Assume $\alpha < \beta$ and show that there is a unique $\delta \leq \beta$ such that $\alpha + \delta = \beta$.*

Exercise 6 *If β is a limit ordinal, then $\alpha + \beta$, $\alpha \cdot \beta$ and $\beta \cdot \alpha$ are limit ordinals, for any α. If β is a successor ordinal, then $\alpha + \beta$ is a successor ordinal for any α. If α and β are both successor ordinals, then $\alpha \cdot \beta$ is a successor ordinal. What about ordinal exponentiation?*

Exercise 7 *Does $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$ hold?*

Exercise 8

- *Let α be any ordinal and show that there exists a largest δ such that $\omega^\delta \leq \alpha$.*
- *Let α be any ordinal and let δ be maximal such that $\omega^\delta \leq \alpha$. Then there exists a largest $n < \omega$ such that $\omega^\delta \cdot n \leq \alpha$.*

Exercise 9 (Cantor's Normal Form Theorem) *Each ordinal $\alpha \neq 0$ can be written in the following form*

$$\alpha = \omega^{\beta_1} \cdot k_1 + \ldots + \omega^{\beta_n} \cdot k_n,$$

where $1 \leq n < \omega$, $\alpha \geq \beta_1 > \ldots > \beta_n \geq 0$ and $1 \leq k_i < \omega$ for each $i = 1, \ldots, n$.

Exercise 10 *Show that there is a least ordinal number ε_0 so that $\omega^{\varepsilon_0} = \varepsilon_0$. Show (using induction and Cantor's Normal Form Theorem) that every ordinal number below ε_0 can be written in a form which only uses the constant 0 and the functions $x + y$, $x \cdot y$ and ω^x.*