Set Theory pt. 6

Exercises 2 and 3 from last time.

Exercise 1 *Show that if an ordinal α can be written as a finite expression using only* 0*,* ω^x*,* $x+y$ *and* $x \cdot y$ *as in Exercise 10 from the one before last sheet, then* $\omega^\alpha > \alpha$*. Hence* ε_0 *cannot be written as such a finite expression.*

Hint: Induction on the structure of the finite expression. To be a little more exact this time, z is said to be a finite expression of the above form iff $z = 0$ or x and y are known to be finite expressions of the above form and z is of the form ω^x, $x+y$ or $x \cdot y$. So for example in the proof of Exercise 1, one has to show that if x and y are finite expressions, α_x and α_y are the ordinals described by x and y and inductively, $\omega^{\alpha_x} > \alpha_x$ and $\omega^{\alpha_y} > \alpha_y$, then $\omega^{\alpha_x+\alpha_y} > \alpha_x + \alpha_y$.

Exercise 2 *Show that if for some ordinal* δ*,* $\alpha < \omega^\delta$ *and* $\beta \geq \omega^\delta$*, then* $\alpha + \beta = \beta$ *(*Hint: *Cantor's Normal Form Theorem will be helpful).*

Exercise 3 *Show that ordinal addition is not commutative, but a (very) weak form of commutativity holds, namely for all ordinals* α *and* β *we have that*

$$\alpha + \beta + \alpha + \beta = \beta + \alpha + \alpha + \beta.$$

Hint: Use Exercise 2 and Cantor's Normal Form Theorem.

Exercise 4 *Show that every cardinal (*Kardinalzahl*) is a limit ordinal.*

Exercise 5 *If* κ *and* λ *are cardinals and* A *and* B *are disjoint sets with* $|A| = \kappa$ *and* $|B| = \lambda$*, we write* $\kappa + \lambda$ *for* $|A \cup B|$*. Show that* $\kappa + \lambda$ *is thus well-defined and commutative. Show that for any infinte cardinals* κ *and* λ*,* $\kappa + \kappa = \kappa$ *and therefore* $\kappa + \lambda = \max\{\kappa, \lambda\}$*.*

Note: Although the symbol + is the same as before, the operation defined in the previous exercise as + (cardinal addition) is very different to that we used in the previous exercises (ordinal addition).

Model Theory

Exercise 6 *Show that if a theory has arbitrary large finite models, then it has an infinite model.*

Hint: Introduce infinitely many new constant symbols and use the Compactness Theorem.

Exercise 7 *Let $\mathcal{L}$ be the language consisting of countably many constant symbols c_n, $n \in \mathbb{N}$. Let T be the theory which says that for all $n_0 \neq n_1$ in $\mathbb{N}$, $c_{n_0} \neq c_{n_1}$. Determine how many non-isomorphic countable models T has. How many non-isomorphic models does T have of cardinality κ if κ is an uncountable cardinal? (*Hint: *Use Exercise 5)*

Exercise 8 *Let $\mathcal{L}$ be the language consisting of countably many relation symbols R_n, $n \in \mathbb{N}$. Let T be the theory which says that the R_n are (pairwise) disjoint and each R_n holds for infinitely many x. Try to write down the axioms of T and show that the number of non-isomorphic models of cardinality κ of T are uncountable if κ is ω_1, where we let ω_1 denote ω^+, the successor cardinal (*Nachfolgerkardinalzahl*) of ω. (*Hint: *Use that $|\omega \times \omega_1| = \omega_1$)*