Models of Set Theory II – Winter 2019/20

Peter Holy – Problem Sheet 3

Problem 1:

1. Show that every axiom of ZF holds in any class forcing extension, except possibly for Powerset, Replacement and Separation (however, you are not asked to produce counterexample for these), where the Union axiom is to be considered in its weak form, stating that for every set X there exists a set containing $\bigcup X$.

2. Show that the axiom of Choice holds in any class forcing extension that satisfies the axioms of ZF.

Problem 2: Verify the following:

1. $q \leq p \;\rightarrow\; q \leq^* p$.

2. $q \leq^* p \iff q \Vdash p \in \dot{G}$.

3. $p \perp q \iff \nexists r \leq^* p \; r \leq^* q$.

4. If $p \Vdash \varphi$ and $q \leq^* p$, then $q \Vdash \varphi$.

5. If $P \approx Q$, then P and Q have the same properties w.r.t. forcing, that is, they produce the same generic extensions.

6. $\langle p_2, \dot{q}_2 \rangle \leq^* \langle p_1, \dot{q}_1 \rangle$ iff $p_2 \leq^* p_1$ and $p_2 \Vdash \dot{q}_2 \leq^* \dot{q}_1$.

Problem 3: Verify the following:

1. P is separative if and only if P is antisymmetric and whenever $p, q \in P$, then
$$p \not\leq q \rightarrow \exists r \leq p \; r \perp q.$$

2. Lemma 6.5 from the lecture: Let P be a set-size partial order. Then, there exists a separative partial order Q and a dense* embedding $i\colon P \to Q$, i.e. $P \approx Q$.

Problem 4: Verify the following:

1. If $\dot{Q}$ is a (set-size) P-name for a partial order, then $P * \dot{Q}$ is a set.

2. Let $(P * \dot{Q})^{class} = \{\langle p, \dot{q}\rangle \mid p \in P \wedge p \Vdash \dot{q} \in \dot{Q}\}$. There is a dense* embedding from $P * \dot{Q}$ into $(P * \dot{Q})^{class}$.

3. If κ is inaccessible, $|P| < \kappa$ and $\Vdash_P |\dot{Q}| < \kappa$, then $|P * \dot{Q}| < \kappa$.

4. Given a $P * \dot{Q}$-generic K, let $G = \{p \in P \mid \langle p, \dot{1}_{\dot{Q}}\rangle \in K\}$ and $H = \{\dot{q}^G \mid \langle 1_P, \dot{q}\rangle \in K\}$. Then,
   - $H \in V[K]$,
   - G is P-generic, H is $\dot{Q}^G$-generic over $V[G]$, and
   - $K = G * H$.