Models of Set Theory II – Winter 2019/20

Peter Holy – Problem Sheet 4

Problem 1: Let $\langle P_\alpha, \dot{Q}_\alpha \mid \alpha < \epsilon \rangle$ be an I-supported iteration, where ϵ is a limit ordinal, and let $\gamma < \epsilon$. Verify the following:

1. $P_{\gamma+1}$ is isomorphic to $P_\gamma * \dot{Q}_\gamma$.
2. $p \in P_\epsilon$ iff $[\operatorname{dom} p \in I \cap \mathcal{P}(\epsilon)$ and $\forall \alpha < \epsilon\ p \upharpoonright \alpha \in P_\alpha]$.
3. For $p, q \in P_\epsilon$, $q \leq_\epsilon p$ iff $\forall \alpha < \epsilon\ q \upharpoonright \alpha \leq_\alpha p \upharpoonright \alpha$.
4. For $p, q \in P_\epsilon$, $q \leq^*_\epsilon p$ iff $\forall \alpha < \epsilon\ q \upharpoonright \alpha \leq^*_\alpha p \upharpoonright \alpha$.

Problem 2: Let $\langle P_\alpha, \dot{Q}_\alpha \mid \alpha < \epsilon \rangle$ be an I-supported iteration, and fix an ordinal $\alpha < \epsilon$. Let $p \in P_\epsilon$ and $q \in P_\alpha$ be such that $q \leq_\alpha p \upharpoonright \alpha$. Verify the following:

1. $p \wedge q \in P_\epsilon$.
2. $p \wedge q \leq p, q$.
3. If $p_2 \leq p_1$, and $q \leq_\alpha p_2 \upharpoonright \alpha$, then $p_2 \wedge q \leq p_1 \wedge q$.
4. If $q_2 \leq q_1$, and $q_1 \leq_\alpha p \upharpoonright \alpha$, then $q_2 \wedge p \leq q_1 \wedge p$.
5. $p \perp_\epsilon q$ iff $p \upharpoonright \alpha \perp_\alpha q$.
6. P_α is a complete subforcing of P_ϵ.

Problem 3: Verify the following: If κ is a regular uncountable cardinal, P is a partial order with the κ-cc, and σ is a P-name for a subset of V of size less than κ (that is, $\sigma^G \subseteq V$ and $|\sigma^G| < \kappa$ whenever G is P-generic), then there is a set x of size less than κ in V such that $\Vdash_P \sigma \subseteq \check{x}$.

Hint: A slightly weaker result was shown in Philipp's lecture in the proof that κ-cc forcings preserve cardinals $\geq \kappa$.

Problem 4:

1. State and verify a result analogous to Corollary 6.12 for the κ-cc when κ is a regular cardinal greater than ω_1.
2. Provide a counterexample to the following statement: If P is ω_2-cc and $\Vdash_P \dot{Q}$ is ω_2-cc, then $P * \dot{Q}$ is ω_2-cc.

   Hint: Let P be the standard forcing that collapses ω_1 to become countable.