# Models of Set Theory II – Winter 2019/20

Peter Holy – Problem Sheet 5

**Problem 1:** Verify the following:

1. If $\varphi$ is a first order formula, $P$ is a notion of (set) forcing, $G$ is $P$-generic, $x \in V[G]$, $V[G]$ models $\varphi(x)$, and $\Vdash_P \exists y\, \varphi(y)$, then there is a name $\dot{x}$ for $x$ (i.e. $\dot{x}^G = x$) such that $\Vdash_P \varphi(\dot{x})$.

2. If $\dot{q}_0$ and $\dot{q}_1$ are $P$-names which are forced to be elements of a partial order $\dot{Q}$, and $p_0 \perp p_1$ are incompatible elements of $P$, then there is a $P$-name $\dot{q}$ for an element of $\dot{Q}$ such that $p_0 \Vdash \dot{q} = \dot{q}_0$ and $p_1 \Vdash \dot{q} = \dot{q}_1$.

3. If $P$ contains at least two incompatible conditions, and $\Vdash_P |\dot{Q}| \geq 2$, then $P * \dot{Q}$ is not antisymmetric.

**Problem 2:** Show that if $\kappa$ is a regular uncountable cardinal, $P$ is a notion of (set) forcing which preserves the regularity of $\kappa$, $\dot{x} \in \mathrm{H}(\kappa)$ is a $P$-name, and $G$ is $P$-generic, then $\dot{x}^G \in \mathrm{H}(\kappa)^{V[G]}$.

**Problem 3:**

1. Show that if $\kappa$ is a regular uncountable cardinal, and $P$ is a notion of forcing that destroys the regularity of $\kappa$ in each of its generic extensions (that is $\kappa$ is no longer regular after forcing with $P$), then $\mathrm{FA}_\kappa(P)$ fails.

2. Show that if $\kappa$ is a regular uncountable cardinal, $S$ is a stationary subset of $\kappa$, and $P$ is a notion of forcing that destroys the stationarity of $S$ in each of its generic extensions while preserving $\kappa$ as a cardinal, then $\mathrm{FA}_\kappa(P)$ fails.

*Hint:* For the second item, make use of Theorem 7.8.

**Problem 4:** Verify the following:

1. If $P$ is a $\sigma$-closed (that means $<\omega_1$-closed) partial order, then $P$ is proper.

2. Assume that $P$ is a notion of forcing such that for any condition $p \in P$, there exists an uncountable antichain of conditions $q \leq p$. Then, if $M \prec H(\theta)$ for some regular $\theta > |P|$ with $M$ countable, and $q$ is $(M, P)$-generic, then $q \notin M$.