Models of Set Theory II – Winter 2019/20

Peter Holy – Problem Sheet 6

Problems 1 and 2 are relevant for the proof of Lemma 8.6, which I now spelled out properly at the end of this exercise sheet (this was the lemma that I think we all got confused about in the lecture).

Problem 1: Assume that $P \in H(\chi)$ is a partial order, and that χ is a regular and uncountable cardinal. Verify that for every condition $p \in P$, and every first order formula φ in the forcing language of $H(\chi)$ – this means that only names in $H(\chi)$ are allowed to appear as parameters within φ – the following statements are equivalent:

1. $p \Vdash \left[\varphi^{H(\check{\chi})}\right]$.
2. $H(\chi) \models [p \Vdash \varphi]$.

Problem 2:

1. Let P be a notion of forcing, and let $\varphi(x)$ be a first order formula with one free variable. Show that there is a P-name $\dot{x}$ such that

$$\Vdash_P [\exists x\, \varphi(x) \to \varphi(\dot{x})]\,.$$

2. Let χ be a regular uncountable cardinal, let $P \in H(\chi)$ be a notion of forcing, and let φ be a first order sentence. Assume that $\dot{x}$ is a P-name such that

$$\Vdash_P \varphi \to \dot{x} \in H(\check{\chi}).$$

Show that we can find a P-name $\dot{y} \in H(\chi)$ such that $\Vdash_P \varphi \to \dot{x} = \dot{y}$.

Problem 3: Verify the following, under the assumptions of Problem 1:

1. If $M \prec H(\chi)$ and $P \in M$, then

$$\Vdash_P \langle \check{M}[\dot{G}], \check{M}[\dot{G}] \cap H(\chi)^V, \in\rangle \prec \langle H(\check{\chi}), H(\chi)^V, \in\rangle.$$

2. If under the above assumptions, q is (M, P)-generic, then

$$q \Vdash_P \langle \check{M}[\dot{G}], \check{M}, \in\rangle \prec \langle H(\check{\chi}), H(\chi)^V, \in\rangle.$$

Problem 4: Let $j\colon M \to N$ be an elementary embedding, where M and N are both transitive (possibly class-sized) models of ZFC^-. Verify the following:

1. $j(\alpha) \geq \alpha$ for every ordinal α.
2. If $j \neq \mathrm{id}$, then $j \restriction \mathrm{Ord}^M \neq \mathrm{id}$.
3. If $\mathrm{crit}(j) = \kappa$, then
   (a) κ is regular in M and
   (b) $j \restriction H(\mathrm{crit}(j))^M = \mathrm{id}$.

Lemma 8.6: If $M \prec H(\chi)$ for some regular and uncountable χ, and $P \in M$, then $1_P \Vdash \check{M}[\dot{G}] \prec H(\check{\chi})$.

Proof: Note first that since $|P| < \chi$, forcing with P preserves the regularity of χ, so $H(\chi)$ still is a well-defined object. Using the Tarski-Vaught criterion in the generic extensions of P, it is enough to verify that for all P-names $\dot{a} \in M$, and for any fixed first order formula φ, 1_P forces the following statement

$$(*) \quad (\exists x\, \varphi(x, \dot{a}))^{H(\check{\chi})} \to \exists x \in \check{M}[\dot{G}]\ \varphi(x, \dot{a})^{H(\check{\chi})}.$$

Using Problem 2.1, let $\dot{x}$ be a P-name such that

$$1_P \Vdash \exists x\ \left[x \in H(\check{\chi}) \wedge \varphi(x, \dot{a})^{H(\check{\chi})}\right] \to \left[\dot{x} \in H(\check{\chi}) \wedge \varphi(\dot{x}, \dot{a})^{H(\check{\chi})}\right].$$

By Problem 2.2, we find a P-name $\dot{y} \in H(\chi)$ such that

$$1_P \Vdash \left[\exists x \in H(\check{\chi})\ \varphi(x, \dot{a})^{H(\check{\chi})}\right] \to \varphi(\dot{y}, \dot{a})^{H(\check{\chi})}.$$

Using Problem 1, the existence of such a P-name $\dot{y}$ can be equivalently rewritten as a first order statement within $H(\chi)$, and thus we find such a P-name $\dot{y}$ in M by elementarity. But then, $1_P \Vdash \dot{y} \in \check{M}[\dot{G}]$, and if

$$p \Vdash \left[\dot{y} \in \check{M}[\dot{G}] \wedge \varphi(\dot{y}, \dot{a})^{H(\check{\chi})}\right],$$

then also

$$p \Vdash \exists x \in \check{M}[\dot{G}]\ \varphi(x, \dot{a})^{H(\check{\chi})},$$

for any condition $p \in P$. Taking a maximal antichain of conditions p which decide whether or not $(\exists x\, \varphi(x, \dot{a}))^{H(\check{\chi})}$ holds, the above implies that each such p forces the statement (*), and therefore that 1_P forces (*), as desired. $\square$