Models of Set Theory II – Winter 2019/20

Peter Holy – Problem Sheet 8

Problem 1: Show that there is no ω-Aronszajn tree.

Problem 2:

1. Prove Lemma 11.4, stating that in case T is a normal Suslin tree, and $\langle P, \leq\rangle = \langle T, \geq\rangle$, then $P \times P$ does not satisfy the countable chain condition.
2. Provide an example of a maximal antichain A in a normal tree T for which there exists an extension T' of T such that A is not maximal in T'.

Problem 3: Show that for every infinite set X, there exists a uniform ultrafilter on X.

Definition:

- Let κ be a regular cardinal. An ultrafilter U is *$<\kappa$-complete* in case whenever $\lambda < \kappa$ and $\langle X_i \mid i < \lambda\rangle$ is a sequence of elements of U, then $\bigcap_{i<\lambda} X_i \in U$.
- An ultrafilter U on κ is *nonprincipal* in case $\{\alpha\} \notin U$ for every $\alpha < \kappa$.
- An uncountable cardinal κ is *measurable* if there exists a $<\kappa$-complete nonprincipal ultrafilter on κ.

Problem 4: Verify the following:

1. Use the theory of normal measures developed in Peter Koepke's set theory lecture to show that if κ is measurable, then $\{\alpha < \kappa \mid \alpha$ is inaccessible$\}$ is a stationary subset of κ.

Hint: Note that κ is inaccessible, as was shown in Peter Koepke's lecture. Let U be a normal, nonprincipal, $<\kappa$-complete ultrafilter on κ, as one obtains from measurability (again, this was shown in Peter Koepke's lecture). Now assume that $N = \{\alpha < \kappa \mid \alpha \text{ is not inaccessible}\} \in U$. Define a regressive function $f\colon N \to \kappa$ by mapping each α in N to the least $\xi < \alpha$ such that either $\mathrm{cof}(\alpha) = \xi$ or $2^\xi \geq \alpha$. By the normality of U, f has to be constant on a set in U. Use this to derive a contradiction. Finally, use another result about normal ultrafilters from Peter Koepke's lecture to easily derive that every set in U is a stationary subset of κ.

2. Every supercompact cardinal is measurable and $\{\alpha < \kappa \mid \alpha \text{ is measurable}\}$ is a stationary subset of κ.

Hint: Given $j\colon H(\nu) \to H(\theta)$ with $j(\mathrm{crit}(j)) = \kappa$, for $X \subseteq \mathrm{crit}(j)$ in $H(\nu)$, define an ultrafilter U on the subsets X of $\mathrm{crit}(j)$ in $H(\nu)$ by setting $X \in U$ iff $\mathrm{crit}(j) \in j(X)$.