Models of Set Theory II – Winter 2019/20

Peter Holy – Problem Sheet 11

Problem 1: Show that the converse of Lemma 15.2 holds true for separative partial orders as well – that is, if κ is an infinite cardinal such that forcing with a separative partial order P does not add any new functions from κ to V, then P is $<\kappa^+$-distributive.

Problem 2: Show that the forcing which adds a club subset of a given stationary subset S of ω_1 is proper if and only if S contains a club subset of ω_1.

Problem 3: Suppose that T is a tree of height ω_1 such that $[T]$ is non-stationary. As in the proof of Theorem 17.6, let $i\colon [T] \to T$ witness that $[T]$ is non-stationary, and for every $b \in [T]$, pick $t_b \in b$ with $i(b) \leq_T t_b$ and $t_b \notin a$ for all $a \in [T]$ with $i(a) <_T i(b)$. Moreover, also let

$$S = \{t \in T \mid \forall b \in [T]\ (t \in b \to t \leq_T t_b)\}.$$

If S has countable height, let P_S denote the trivial forcing, and let P_S be the forcing to specialize S, as introduced in the Proof of Theorem 11.9 otherwise. Show that by forcing with P_S, T becomes almost special.

Problem 4:

1. Let $\mathcal{T}$ be the collection of all trees T of height ω_1 for which every level T_α of T is of the form $T_\alpha \subseteq \{\alpha\} \times \omega_1$, and which have no ω_1-branches. Let $\mathcal{T}^*$ be the tree which is the disjoint union of all the trees in $\mathcal{T}$ and their orderings, together with a new element $\emptyset$ that is below all other elements of $\mathcal{T}^*$. Show that $\mathcal{T}^*$ is of height ω_1 and has no ω_1-branches.
2. Show that there is a ccc partial order which specializes all trees of height and size ω_1 witout cofinal branches, and which almost specializes all trees of height and size ω_1 with a non-stationary set of cofinal branches.
3. Argue why we might not be able to simply use the partial order that forces MA_{ω_1}, as defined in the proof of Theorem 7.3, as a witness in the above.