Ideal Topologies

Peter Holy

presenting joint work with

Marlene Koelbing (Vienna),
Philipp Schlicht (Bristol),
and Wolfgang Wohofsky (Vienna).

University of Udine

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Ideals

Idea: An ideal on κ is a collection of *small* subsets of κ.

Definition 1

Let κ be a cardinal. A collection $\mathcal{I} \subseteq \mathcal{P}(\kappa)$ is an *ideal* (on κ) if:

- $\emptyset \in \mathcal{I}$, $\kappa \notin \mathcal{I}$,
- $\forall A, B \quad A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$, and
- $\forall A, B \quad A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

We will also demand our ideals to be *non-principal*, that is $\{\alpha\} \in \mathcal{I}$ for every $\alpha < \kappa$.

Examples of ideals

If $\kappa = \omega$, the most natural ideal on ω is the bounded ideal, which is in general defined as

$$\mathrm{bd}_\kappa = \{x \subseteq \kappa \mid \exists \alpha < \kappa \; x \subseteq \alpha\} = \bigcup_{\alpha < \kappa} \mathcal{P}(\alpha).$$

However, if $\kappa > \omega$ is regular, which we will usually assume from now on, there always is at least one other very natural ideal on κ with very nice properties, namely the non-stationary ideal

$$\mathsf{NS}_\kappa = \{x \subseteq \kappa \mid x \text{ is non-stationary in } \kappa\} =$$

$$\{x \subseteq \kappa \mid \exists C \subseteq \kappa \text{ club } x \cap C = \emptyset\}.$$

Other examples exist for example if κ is a large cardinal – we could take a measurable cardinal κ and use the ideal given as the complement of some measurable ultrafilter.

$<\kappa$-complete ideals

Let us assume from now on that κ is a regular and uncountable cardinal, and that $\mathcal{I}$ is an ideal on κ which is $<\kappa$-*complete*, that is if $\alpha < \kappa$ and $\langle X_\beta \mid \beta < \alpha\rangle$ is a collection of elements of $\mathcal{I}$, then $\bigcup_{\beta<\alpha} X_\beta \in \mathcal{I}$. Note that together with being non-principal, this in particular implies that $\mathcal{I} \supseteq \mathrm{bd}_\kappa$.

Note: Since the intersection of $<\kappa$-many clubs in κ is still a club, the union of $<\kappa$-many non-stationary sets is still non-stationary, i.e. NS_κ is $<\kappa$-complete.

Cantor spaces

Let ${}^{\kappa}2 = \{g \mid g\colon \kappa \to 2\}$, where of course $2 = \{0,1\}$. This collection is usually given a topology based on the ideal bd_κ: The κ-*Cantor space* is the set ${}^{\kappa}2$ with the topology given by the basic open sets (which are also easily seen to be closed)

$$[f] = \{g \in {}^{\kappa}2 \mid f \subseteq g\}$$

for $f \in {}^{<\kappa}2 = \bigcup_{\alpha<\kappa} {}^{\alpha}2$.

However, we would obtain the same topology if we took as basic open sets all sets of the form $[f]$ where f is a partial function from κ to 2 of size less than κ, i.e. a function with domain in bd_κ.

If $\kappa = \omega$, this is certainly the most natural topology on the space ${}^{\omega}2$. However, if $\kappa > \omega$, we can equally consider topologies based on ideals other than bd_κ.

Ideal Topologies

Definition 2

- The $\mathcal{I}$*-topology* is the topology with the basic open sets of the form $[f]$ where $\mathrm{dom}(f) \in \mathcal{I}$ (as before, each $[f]$ is also closed).
- We denote sets of the form $[f]$ with $\mathrm{dom}(f) \in \mathcal{I}$ as $\mathcal{I}$*-cones*.

- Open sets are (as always) arbitrary unions of basic open sets, and we call open sets in the $\mathcal{I}$-topology $\mathcal{I}$-open sets, and similarly for other notions: for example, $\mathcal{I}$-closed sets are the complements of $\mathcal{I}$-open sets, ...
- Note that the $\mathcal{I}$-topology *refines* the bounded topology: it has more open sets (and thus also more closed sets, ...).
- In case $\mathcal{I} = \mathrm{NS}_\kappa$, the basic open sets are thus *induced* by functions with non-stationary domain.

Basic cardinality observations

In the bounded topology on ${}^{\kappa}2$, one usually assumes $2^{<\kappa} = \kappa$, and then there are κ-many basic open sets, and 2^{κ}-many open sets (while there are $2^{2^{\kappa}}$-many subsets of ${}^{\kappa}2$). If $\mathcal{I}$ contains an unbounded subset of κ however, we get the maximal possible number of open sets:

Observation 3

Assume that $\mathcal{I}$ contains an unbounded subset A of κ. Then,

1. there are 2^{κ}-many disjoint $\mathcal{I}$-cones with union ${}^{\kappa}2$, and
2. there are $2^{2^{\kappa}}$-many $\mathcal{I}$-open sets.

Proof: $F = \{[f] \mid f \colon A \to 2\}$ witnesses (1). For any $X \subseteq F$, let $O_X = \bigcup_{f \in X}[f]$. If $X \neq Y$ are both subsets of F, then $O_X \neq O_Y$. Hence, $\{O_X \mid X \subseteq F\}$ witnesses (2). □

Note: In particular, (1) implies that whenever $\mathcal{I}$ contains an unbounded subset of κ, then the $\mathcal{I}$-topology is (in a very strong sense) not compact.

Tall ideals

Tallness is a very natural property of ideals:

Definition 4

An ideal $\mathcal{I}$ is *tall* if every unbounded set has an unbounded subset in $\mathcal{I}$.

Observation 5

NS_κ is tall.

Proof: Let X be an unbounded subset of κ, and enumerate X in increasing order as $X = \{x_\alpha \mid \alpha < \kappa\}$. Let $Y = \{x_{2\cdot\alpha+1} \mid \alpha < \kappa\}$ contain every ordinal in X with an odd index in its enumeration. Then, Y is an unbounded subset of κ. The closure of $X \setminus Y$ is a club subset of κ, and is disjoint from Y. Hence, Y is a nonstationary, unbounded subset of X, as desired. $\square$

The Unbounded subsets of κ

Let ub_κ denote the collection of unbounded subsets of κ. Identifying subsets of κ with their characteristic functions, we have $\mathrm{ub}_\kappa \subseteq {}^\kappa 2$.

Observation 6

$\mathcal{I}$ is tall if and only if ub_κ is $\mathcal{I}$-open.

Proof: First, assume that $\mathcal{I}$ is a tall ideal. Let c_i^A denote the constant function with domain A and value i. Then,

$$\mathrm{ub}_\kappa = \bigcup\{[c_1^A] \mid A \in \mathcal{I} \cap \mathrm{ub}_\kappa\} \text{ is } \mathcal{I}\text{-open.}$$

Assume ub_κ is $\mathcal{I}$-open. Given $A \in \mathrm{ub}_\kappa$, there is an $\mathcal{I}$-cone $[f] \subseteq \mathrm{ub}_\kappa$ with $A \in [f]$. Since $[f] \subseteq \mathrm{ub}_\kappa$, f has to take value 1 on some $B \in \mathrm{ub}_\kappa$ with $B \subseteq \mathrm{dom}(f) \in \mathcal{I}$. Hence, $B \subseteq A$ is unbounded, as desired. □

A more intricate argument shows that for no $\mathcal{I}$ is ub_κ an $\mathcal{I}$-F_σ set (a κ-union of $\mathcal{I}$-closed sets). Hence, if $\mathcal{I}$ is tall, then there is an $\mathcal{I}$-open set that is not $\mathcal{I}$-F_σ.

The Club subsets of κ

Let Club_κ denote the collection of club subsets of κ.

A similar argument as for ub_κ shows: If $\mathcal{I} = \mathrm{NS}_\kappa$, then Club_κ is not $\mathcal{I}$-F_σ.

However, as soon as $\mathcal{I}$ contains a stationary subset of κ, we have the following contrasting result:

Observation 7

$\mathcal{I}$ contains a stationary subset of κ if and only if Club_κ is $\mathcal{I}$-closed.

Proof: Assume first that Club_κ is $\mathcal{I}$-closed. Then, the complement of Club_κ contains an $\mathcal{I}$-cone $[f]$ that contains $\emptyset = c_0^\kappa$, however $[f]$ contains no club subset of κ. Thus, f has to take constant value 0, and it has to do so on a stationary subset of κ in $\mathcal{I}$.

It remains to show that if $\mathcal{I}$ contains a stationary subset of κ, then Club_κ is $\mathcal{I}$-closed.

We show that every element of the complement of Club_κ is contained in an $\mathcal{I}$-open set that is disjoint from Club_κ, showing that this complement is $\mathcal{I}$-open, and hence that Club_κ is $\mathcal{I}$-closed, as desired.

Fix $S \in \mathcal{I}$ stationary. Let $x \subseteq \kappa$ not be in Club_κ, i.e., x not closed unbounded. In case x is not closed, let $\alpha < \kappa$ be such that $x \restriction \alpha$ is not closed; then $[x \restriction \alpha] \cap \mathrm{Club}_\kappa = \emptyset$.

If x is bounded, fix $\alpha < \kappa$ such that $x \subseteq \alpha$, and let S' be the stationary set $S \setminus \alpha \in \mathcal{I}$. Then, $[x \restriction S'] = [c_0^{S'}]$, and hence $[x \restriction S'] \cap \mathrm{Club}_\kappa = \emptyset$. $\square$

Stationary tallness

Stationary tallness relates to NS_κ as does tallness to bd_κ:

Definition 8

$\mathcal{I}$ is stationary tall if every stationary set S has a stationary subset in $\mathcal{I}$.

Observation 9

If $\mathcal{I}$ contains a club subset C of κ, then $\mathcal{I}$ is stationary tall.

Proof: If S is stationary, $S \cap C \in \mathcal{I}$ is stationary. □

An ideal $\mathcal{I}$ is *maximal* if whenever A and B are disjoint subsets of κ, at least one of them is in $\mathcal{I}$.

Observation 10

Every maximal ideal is both tall and stationary tall.

Proof: Assume that S is a stationary subset of κ. Write S as disjoint union of two stationary sets $S_0 \cup S_1$, using Solovay's theorem. One of them has to be in $\mathcal{I}$ by maximality. Tallness follows by an analogous argument. □

The Club Filter can be $\mathcal{I}$-closed

$\mathcal{C}_\kappa$ denotes the collection of subsets of κ that contain a club.

Observation 11

$\mathcal{I}$ is stationary tall if and only if $\mathcal{C}_\kappa$ is $\mathcal{I}$-closed.

Proof: Let $x \subseteq \kappa$ not be in $\mathcal{C}_\kappa$, so that $\kappa \setminus x$ is stationary. Using that $\mathcal{I}$ is stationary tall, we can fix a stationary $S \subseteq \kappa \setminus x$ in $\mathcal{I}$. Since S is disjoint from x, all elements of $[x \restriction S] = [c_0^S]$ are disjoint from S. Then, since each set in $\mathcal{C}_\kappa$ intersects S, $\mathcal{C}_\kappa$ is disjoint from $[x \restriction S]$, as desired.

Assume now that $\mathcal{C}_\kappa$ is $\mathcal{I}$-closed. Then its complement, the collection of co-stationary subsets of κ, is $\mathcal{I}$-open. Given an arbitrary stationary set A, let B be its (co-stationary) complement. There has to be some $\mathcal{I}$-cone $[f]$ with $[f] \cap \mathcal{C}_\kappa = \emptyset$ and with $B \in [f]$. Hence, $f(\alpha) \neq 0$ for every $\alpha \in B$. For every element of $[f]$ to be co-stationary, f has to take value 0 on a stationary set. But this means that $\mathrm{dom}(f) \in \mathcal{I}$ has to contain a stationary subset of A, showing that $\mathcal{I}$ is stationary tall. □

The Club Filter can be $\mathcal{I}$-open

Observation 12

$\mathcal{I}$ contains a club subset of κ if and only if $\mathcal{C}_\kappa$ is $\mathcal{I}$-open.

Proof: Assume that $C \in \mathcal{I}$ is club. Then, $\mathcal{I}$ is in fact *club tall*: every club D contains the club $D \cap C \in \mathcal{I}$. So we can write

$$\mathcal{C}_\kappa = \bigcup\{[c_1^E] \mid E \text{ is a club in } \mathcal{I}\}.$$

The reverse direction is verified similar as for our earlier observations. □

Often however, it yields non-$\mathcal{I}$-Borel sets

Usually, the club filter is the standard example of a complicated set – in the bounded topology, it is not *Borel* (Halko-Shelah). This result can be generalized. $\mathcal{I}$-Borel sets are (iteratively) generated from the $\mathcal{I}$-open sets by taking κ-unions and complements. For a stationary $S \subseteq \kappa$, let

$$\mathcal{C}_\kappa^S = \{A \subseteq \kappa \mid \exists C \subseteq \kappa \text{ club } A \supseteq S \cap C\}.$$

Proposition 13 (without proof)

$\mathcal{I}$ is stationary tall if and only if $\mathcal{C}_\kappa^S$ is *$\mathcal{I}$-Borel* for every stationary set S.

That is, if $\mathcal{I}$ is not stationary tall, there are non-$\mathcal{I}$-Borel sets.

The Club subsets of κ can also be $\mathcal{I}$-open

Observation 14 (without proof)

Club_κ is $\mathcal{I}$-open if and only if the following property (*) holds: $\mathcal{I}$ contains a club subset of κ, and for every nonstationary set N of limit ordinals, there is a regressive function $f\colon N \to \kappa$ such that

$$\bigcup_{\alpha \in N} [f(\alpha), \alpha] \in \mathcal{I}.$$

Ideals satisfying (*) do exist:

Observation 15

There is an ideal $\mathcal{I}$ satisfying Property (*).

Proof: Let $\mathcal{I}$ be the ideal generated by the club of limit ordinals below κ together with the sets $A^\oplus = \{\alpha + 1 \mid \alpha \in A\}$ for nonstationary A. By the $<\kappa$-completeness of NS_κ, $\kappa \notin \mathcal{I}$. It is not hard to verify that $\mathcal{I}$ satisfies property (*)... (let's skip the details) □

Some Questions

The first question is asking whether all sets could possibly be $\mathcal{I}$-Borel:

Question 16

If $\mathcal{I}$ is stationary tall, is there still a set that is not $\mathcal{I}$-Borel?

We have noted that ub_κ is never $\mathcal{I}$-F_σ. However, it is always $\mathcal{I}$-G_δ (the complement of an $\mathcal{I}$-F_σ set), for it is already G_δ in the bounded topology. We don't really know how to obtain more complicated $\mathcal{I}$-Borel sets though – none of the known methods seems to answer the following:

Question 17

Is there a proper $\mathcal{I}$-Borel hierarchy? If so, what is its length and structure?