Condensation does not imply Square

Peter Holy

University of Bonn

presenting joint work with Philip Welch and Liuzhen Wu

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Condensation

Lemma (Gödel)

If $M \prec (L_\alpha, \in)$, then for some $\bar{\alpha} \leq \alpha$, $M \cong (L_{\bar{\alpha}}, \in)$.

We want to consider generalizations of this principle that apply to models other than L:

Condensation in models of the form L[A]

Assume $A \subseteq \mathrm{Ord}$. If M is a substructure of $(L_\alpha[A], \in, A)$, we say that M *condenses* if for some $\bar{\alpha} \leq \alpha$, $M \cong (L_{\bar{\alpha}}[A], \in, A)$.

Local Club Condensation at κ

If $\kappa = \lambda^+$, LCC at κ is the statement that there is $A \subseteq \kappa$ s.t. $H_\kappa = L_\kappa[A]$ and if $\alpha \in [\lambda, \kappa)$ and $\mathcal{A} = (L_\alpha[A], \in, A, \ldots)$ is a structure for a countable language, then there exists a continuous chain $\langle \mathcal{B}_\gamma \mid \gamma < \lambda \rangle$ of condensing substructures of $\mathcal{A}$ whose domains have union $L_\alpha[A]$, where each $B_\gamma = \mathrm{dom}(\mathcal{B}_\gamma)$ is s.t. $|B_\gamma| < \lambda$ and $\gamma \subseteq B_\gamma$.

Strong Condensation

Lemma (Friedman, Holy, Wu)

If $\kappa = \lambda^+$, λ is regular and LCC *at κ holds, then there is a structure $\mathcal{M}$ for a countable language with domain H_κ such that X condenses whenever X is a substructure of $\mathcal{M}$ and is transitive below λ.*

If $\kappa = \omega_2$, every substructure of such $\mathcal{M}$ will be transitive below ω_1, hence we obtain the following.

Corollary

If LCC *at ω_2 holds, then there is a structure $\mathcal{M}$ for a countable language with domain H_{ω_2} such that every substructure X of $\mathcal{M}$ condenses.*

This is what Hugh Woodin introduced as Strong Condensation for ω_2.

Theorem (Wu)

Assuming the consistency of a stationary limit of measurable cardinals, Strong Condensation for ω_2 is consistent with the failure of $\square_{\omega_1}$.

Square

Definition

If $\lambda \geq \omega_1$, $\square_\lambda$ is the statement that there exists a sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ such that

1. Whenever α is a limit ordinal, C_α is a closed unbounded subset of α.
2. Whenever β is a limit point of C_α then $C_\beta = C_\alpha \cap \beta$.
3. For every α, $\mathrm{ot}(C_\alpha) \leq \lambda$.

$\square_\lambda$ holds in L for every uncountable cardinal λ. (Jensen) All known proofs of this fact use some sort of fine structural machinery. It is generally believed that this is in fact necessary; we support this belief (as did Liuzhen Wu for $\square_{\omega_1}$) by showing that Local Club Condensation does not imply $\square_\lambda$, under sufficient large cardinal consistency hypothesis.

Generalizing and Improving Wu's result

The consistency strength of the failure of $\square_\lambda$ for a regular uncountable cardinal λ is only that of a Mahlo cardinal. (Solovay) Thus in the light of Wu's result, the following seemed to be obvious questions (and were the motivating questions for our work on this subject):

Questions

- Can this result be generalized to larger regular cardinals?
- Can the large cardinal consistency assumption be reduced, ideally to that of a Mahlo cardinal?

As to the second question, we got pretty close, reducing it to a 2-Mahlo cardinal. As to the first, it is not known whether there is a small forcing to obtain Strong Condensation for ω_3 in the generic extension. That's why we have to use Local Club Condensation rather than the (essentially stronger) principle of Strong Condensation. Moreover, Strong Condensation is inconsistent for ω_1-Erdős cardinals, while Local Club Condensation is consistent with arbitrary large large cardinals (Friedman-Holy).

Square and Forcing

Theorem (Solovay)

If λ is regular and uncountable and $\kappa > \lambda$ is a Mahlo cardinal, then after performing a Lévy collapse so that κ becomes λ^+, $\Box_\lambda$ fails.

We want to collapse some large cardinal κ to become λ^+ while forcing LCC at λ^+ and then show that we can still verify the failure of $\Box_\lambda$ in the resulting model.

Theorem

Assume GCH *holds and $\lambda < \kappa$ are regular. There is a $<\lambda$-directed closed, κ-cc notion of forcing which ensures that $\kappa = \lambda^+$ and Local Club Condensation at κ hold in any generic extension.*

Rough idea of proof: We want to generically add $A \subseteq \kappa$ that witnesses Local Club Condensation at κ. Our desired forcing P will be an iteration of length κ with support of size $<\lambda$, where $P_{<\lambda}$ simply adds a Cohen subset of λ, and we take that to be $A \upharpoonright \lambda$.

Proof Sketch continued:

At stage $\alpha \in [\lambda, \kappa)$, we will be given (by a careful choice of bookkeeping function) a $P_{<\alpha}$-name for either 0 or 1, and we take that to be $A(\alpha)$. For $p \in P$, $p(\alpha)$ is of the form $p(\alpha) = (\gamma_\alpha, c_\alpha, f_\alpha)$ where

- $\gamma_\alpha < \lambda$,
- $c_\alpha \subseteq \gamma_\alpha$ is closed (with a maximal element) and
- $f_\alpha \colon \max(c_\alpha) \to \alpha$ is injective.

The extension relation is end-extension, and generically, our forcing will produce

- $A \subseteq \kappa$ such that $L_\kappa[A] = H_\kappa$.
- $C_\alpha \subseteq \lambda$ club for every $\alpha \in [\lambda, \kappa)$.
- $F_\alpha \colon \lambda \xrightarrow{onto} \alpha$ for every $\alpha \in [\lambda, \kappa)$.

By the last item, κ is seen to become λ^+ in the generic extension.

Proof Sketch continued:

Additional Coding Requirement:

$$\forall \delta \in c_\alpha \ A(\text{ot } f_\alpha[\delta]) = A(\alpha).$$

Idea for Local Club Condensation: For sufficiently many structures, ot $f_\alpha[\delta]$ is the image of α under the collapsing map of the structure, so for sufficiently many structures, the predicate A is preserved under their collapse.

Observe: The Coding Requirement gives an intricate connection between the Cohen subset of λ and the remaining part of a condition. In particular, tails of our iteration P will not be σ-closed.

Condensation and not Square

Theorem

Assume GCH*. If λ is regular and uncountable and $\kappa > \lambda$ is a 2-Mahlo cardinal, then there is a $<\lambda$-directed closed, κ-cc forcing that makes κ become λ^+ and ensures Local Club Condensation at κ to hold while $\Box_\lambda$ fails.*

A key ingredient in the proof (due to Liuzhen Wu and already used in his proof for ω_1) is the use of elementary substructures $M_0, M_1 \prec H_\theta$, with the key property that for some regular cardinal η,

$$\sup(M_0 \cap M_1 \cap \eta) < \sup(M_0 \cap \eta) = \sup(M_1 \cap \eta).$$

Wu constructed these models using a measurable cardinal, and it is not too hard to see that one can obtain such structures already if η is ω-Erdős. After giving this talk for the first time however, Boban Veličković hinted us at a construction of his that allows one to construct such models using no large cardinals at all.

The Model Construction

Adapting his construction, we were able to construct our desired models using only a Mahlo cardinal.

Lemma

If η is Mahlo, $\theta \geq \eta$ is regular, $\lambda < \eta$ is regular and $\mathcal{A}$ is a structure for a countable language with domain H_θ, then there is a pair of models M_0^* and M_1^* such that

1. *M_0^* and M_1^* are both substructures of $\mathcal{A}$.*
2. *M_0^* and M_1^* both have size λ.*
3. *$\lambda \subseteq M_0^*, M_1^*$.*
4. *Let $\bar{\delta} = \sup(\eta \cap M_0^* \cap M_1^*)$. Then $[M_0^* \cap V_{\bar{\delta}}]^{<\omega_1} \subseteq M_1^*$.*
5. *$\min(M_0^* \setminus \bar{\delta})$ has cofinality $\geq \lambda$*
6. *$\bar{\delta} < \delta := \sup(M_0^* \cap \eta) = \sup(M_1^* \cap \eta)$ and the latter have cofinality ω.*

Being able to show the above for η inaccessible would reduce our overall consistency assumption from 2-Mahlo to (the optimal) Mahlo.

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Proof: Let $P = P(\lambda, \kappa)$ denote the forcing to obtain $\kappa = \lambda^+$ and LCC at κ. Assume $\dot{C} = \langle \dot{C}_\eta \mid \eta < \kappa \rangle$ is a P-name for a $\Box_\lambda$-sequence in a P-generic extension. Using that P is κ-cc and conditions have *bounded support*, there is a club of $\eta < \kappa$ such that $\dot{C} \restriction \eta$ is a $P(\lambda, \eta)$-name. By the large cardinal properties of κ, we may choose such η which is Mahlo. As η is regular after forcing with $P(\lambda, \eta)$, it follows that $\dot{C}_\eta$ cannot have a $P(\lambda, \eta)$-name, as otherwise its evaluation would have to have order-type $\eta > \lambda$, contradicting that $\dot{C}$ is a P-name for a $\Box_\lambda$-sequence. Pick $t_0 \perp t_1$ in P and $\xi < \eta$ with $t_0 \restriction \eta = t_1 \restriction \eta$ such that t_0 and t_1 disagree about whether $\xi \in \dot{C}_\eta$. Let M_0^* and M_1^* be elementary substructures of $(H_\theta, \in, \eta, \lambda, \xi, t_0, t_1, \dot{C}_\eta, \ldots)$ as provided by the lemma.

Let $M_0 \prec (M_0^*, \in, \ldots)$ be countable with $\sup(M_0 \cap \eta) = \delta$ $(= \sup(M_0^* \cap \eta))$ and let $s_0 \leq t_0$ be (M_0, P)-complete. Using that $[M_0^* \cap V_{\bar\delta}]^{<\omega_1} \subseteq M_1^*$, we obtain $s_0 \restriction \bar\delta \in M_1^*$. Let $M_1 \prec (M_1^*, \in, s_0 \restriction \bar\delta, \ldots)$ be countable with $\sup(M_1 \cap \eta) = \delta$. It is easy to see that $s_0 \restriction \bar\delta$ and t_1 are compatible, using that $s_0 \leq t_0$ and $t_0 \restriction \eta = t_1 \restriction \eta$. Let s_1 be stronger than both and (M_1, P)-complete. $s_1 \restriction \bar\delta \leq s_0 \restriction \bar\delta$, $\operatorname{supp}(s_0) \cap [\bar\delta, \eta) \subseteq M_0^*$, $\operatorname{supp}(s_1) \cap [\bar\delta, \eta) \subseteq M_1^*$ and M_0^* and M_1^* are disjoint in the interval $[\bar\delta, \eta)$. Therefore $s_0 \restriction \eta$ and $s_1 \restriction \eta$ are compatible.

Both s_i force that $\delta = \sup(M_i \cap \eta) \in \operatorname{Lim}(\dot C_\eta)$. Hence using that $\dot C$ is a P-name for a $\square_\lambda$-sequence, both s_i force that

$$\xi \in \dot C_\eta \iff \xi \in \dot C_\delta.$$

This is a contradiction as $\dot C_\delta$ has a $P(\lambda, \eta)$-name and $(s_0 \restriction \eta) \parallel (s_1 \restriction \eta)$, hence s_0 and s_1 cannot disagree about whether $\xi \in \dot C_\delta$. $\square$

Definition (Chang's Conjecture)

$(\alpha, \beta) \twoheadrightarrow (\gamma, \delta)$: For every countable language $\mathcal{L}$ with a unary predicate $A \in \mathcal{L}$ and every $\mathcal{L}$-structure $\mathcal{M} = (M, A^{\mathcal{M}}, \dots)$ with $|M| = \alpha$ and $|A^{\mathcal{M}}| = \beta$, there exists a substructure $\mathcal{N}$ of $\mathcal{M}$ s.t. $|N| = \gamma$ and $|A^{\mathcal{N}}| = \delta$. Let $\mathrm{CC}(\kappa)$ say that for every infinite $\lambda < \kappa$, $(\kappa, \lambda) \twoheadrightarrow (\omega_1, \omega)$.

Theorem

- LCC *at* ω_2 *refutes* $\mathrm{CC}(\omega_2)$*.*
- *Strong Condensation for* κ *refutes* $\mathrm{CC}(\kappa)$ *for any* $\kappa \geq \omega_2$*.*
- *Assume* GCH*. If* κ *is* ω_1*-Erdős and* $\omega_2 \leq \lambda < \kappa$ *is regular, we may force to make* $\kappa = \lambda^+$*, preserve all cardinals* $\leq\lambda$ *and obtain* $\mathrm{CC}(\kappa)$ *and Local Club Condensation at* κ*.*
- *The proof of this last result is a straightforward adaptation of a result by James Baumgartner, showing that the usual Lévy collapse of an* ω_1*-Erdős cardinal to* λ^+ *forces* $\mathrm{CC}(\lambda^+)$*, again replacing the Lévy collapse by the Local Club Condensation Collapse forcing.*