A SHORT FORCING ARGUMENT FOR THE PROPER FORCING AXIOM USING MAGIDOR'S CHARACTERIZATION OF SUPERCOMPACTNESS

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Abstract. We present a short proof of the consistency of the proper forcing axiom PFA starting from a supercompact cardinal, making use of Magidor's characterization of supercompactness in terms of small embeddings.

In a classical result of his, Menachem Magidor ([3, Theorem 1]) has shown supercompactness to be equivalent to the following property, which we take as our definition of supercompactness.

Definition 1. *A cardinal κ is* supercompact *if for every regular cardinal $\theta > \kappa$ and every $x \in H(\theta)$, there is a regular cardinal $\nu < \kappa$ and an elementary embedding $j\colon H(\nu) \to H(\theta)$ with $j(\operatorname{crit} j) = \kappa$ and $x \in \operatorname{range}(j)$.*

Making use of this characterization, together with the idea of iterating minimal counterexamples to the proper forcing axiom (instead of making use of a Laver function), which goes back to Arthur Apter ([1, Theorem 1]), allows for a very short proof of the relative consistency of the proper forcing axiom.

Definition 2. *Suppose that $\{P_\alpha \mid \alpha \in I\}$ is a set of forcing notions. The* lottery sum *of that set is the disjoint union of those forcing notions, together with a new weakest condition, that is above all $p \in P_\alpha$ for $\alpha \in I$. Note that in particular, the lottery sum of the empty set corresponds to the trivial forcing.*

Note that any lottery sum of proper notions of forcing is itself proper.

Definition 3.
1. *We say that a partial order P is a counterexample to* PFA *if P is proper and there exists a family $\mathcal{D}$ of $\aleph_1$ dense subsets of P, but no $\mathcal{D}$-generic filter on P.*
2. *The* minimal counterexample iteration for PFA of length κ *is the countable support iteration $\langle P_\alpha, \dot{Q}_\alpha \mid \alpha < \kappa\rangle$ that is defined inductively as follows: Given P_α, let $\dot{Q}_\alpha$ be a canonical P_α-name of hereditarily minimal size for the lottery sum of all counterexamples to* PFA *of hereditarily minimal size less than κ.*

Using the standard fact that for regular and uncountable κ, if $P \in H(\kappa)$ and $\Vdash_P \dot{x} \in H(\check{\kappa})$, then there is a name $\dot{y} \in H(\kappa)$ such that $\Vdash_P \dot{x} = \dot{y}$ (see for example [2, Fact 3.6]), the following is easily verified by induction:

Observation 4. *If κ is inaccessible, and $P_\kappa = \langle P_\alpha, \dot{Q}_\alpha \mid \alpha < \kappa\rangle$ is the minimal counterexample iteration for* PFA *of length κ, then $P_\alpha \in H(\kappa)$ for every $\alpha < \kappa$. Hence, $P_\kappa \subseteq H(\kappa)$.* □

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We will of course use Shelah's result that countable support iterations of proper notions of forcing are proper (see for example [2, Corollary 3.19]), and hence in particular P_κ is proper and thus preserves ω_1. We are now ready to provide an alternative proof for James Baumgartner's result that PFA can be obtained by forcing starting over a model with a supercompact cardinal.

Theorem 5. *Let κ be supercompact. Then, the minimal counterexample iteration P_κ for* PFA *of length κ forces that* PFA *holds.*

Proof. Assume for a contradiction that there is a P_κ-name $\dot{P}$ such that some $p \in P_\kappa$ forces that $\dot{P}$ is a hereditarily minimal counterexample to PFA. Using that P_κ preserves ω_1, let $\dot{\mathcal{D}} = \langle \dot{D}_\alpha \mid \alpha < \omega_1 \rangle$ be such that p forces that there is no $\dot{\mathcal{D}}$-generic filter over $\dot{P}$. Let θ be regular and sufficiently large. Using that κ is supercompact, let $\nu < \kappa$ be regular and let $j \colon H(\nu) \to H(\theta)$ be an elementary embedding with $j(\operatorname{crit} j) = \kappa$, and with $\dot{P}, \dot{\mathcal{D}}, P_\kappa, p$ all in the range of j. Since $\operatorname{crit} j$ is inaccessible by elementarity, letting $R_{\operatorname{crit} j}$ be the minimal counterexample iteration for PFA of length $\operatorname{crit} j$, it follows inductively that for $\alpha \le \operatorname{crit} j$, $P_\alpha = R_\alpha$: If at some stage $\alpha < \operatorname{crit} j$, we can find a counterexample to PFA in $H(\kappa)$, then by the elementarity of j, using that j fixes P_α, we can also find such a counterexample in $H(\operatorname{crit} j)$. But this means that $j(P_{\operatorname{crit} j}) = j(R_{\operatorname{crit} j}) = P_\kappa$. Since $p \in \operatorname{range} j \cap H(\kappa)$, we get $p \in H(\operatorname{crit} j)$ and therefore that $j(p) = p$. Thus, applying elementarity of j to our initial assumption,

$$p \Vdash_{\operatorname{crit} j} j^{-1}(\dot{P}) \text{ is a hereditarily minimal counterexample to PFA},$$

and since $j^{-1}(\dot{P}) \in \operatorname{dom} j = H(\nu)$, $j^{-1}(\dot{P})$ is also forced to be in $H(\kappa)$. But then, p forces $\dot{Q}_{\operatorname{crit} j}$ to be a lottery sum of forcing notions which include $j^{-1}(\dot{P})$. Since $\operatorname{dom} p \subseteq \operatorname{crit} j$, we can extend p to q by letting $q(\operatorname{crit} j)$ be the canonical $P_{\operatorname{crit} j}$-name for the weakest condition of $j^{-1}(\dot{P})$ in that lottery sum, i.e. by letting q *decide to force* with $j^{-1}(\dot{P})$ at stage $\operatorname{crit} j$.

Let G be P_κ-generic with $q \in G$. Since $j[G_{\operatorname{crit} j}] = G_{\operatorname{crit} j} \subseteq G$, we may apply Silver's lemma and lift j to $j^* \colon H(\nu)[G_{\operatorname{crit} j}] \to H(\theta)[G]$. In $V[G]$, we have a $(j^{-1}(\dot{P}))^{G_{\operatorname{crit} j}}$-generic filter $G(\operatorname{crit} j)$ over $V[G_{\operatorname{crit} j}]$. In particular, $G(\operatorname{crit} j)$ is $(j^{-1}(\dot{\mathcal{D}}))^{G_{\operatorname{crit} j}}$-generic, for the latter set is an element of $V[G_{\operatorname{crit} j}]$. Since $j^{-1}(\dot{\mathcal{D}}) = \langle j^{-1}(\dot{D}_\alpha) \mid \alpha < \omega_1 \rangle$, this implies that $j^*[G(\operatorname{crit} j)]$ meets $\dot{D}_\alpha^G$ for every $\alpha < \omega_1$, and we can find a $\dot{\mathcal{D}}^G$-generic filter on $\dot{P}^G$ in $V[G]$ by taking the upwards closure of $j^*[G(\operatorname{crit} j)]$ in $\dot{P}^G$. But this means that we have just shown $q \le p$ to force that $\dot{P}$ actually was no counterexample to PFA at all, yielding our desired contradiction. □

References

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