A COUNTABLE SUPPORT ITERATION FOR THE TREE PROPERTY AT $\aleph_2$ AND RELATED PROPERTIES

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Abstract. We present easy proofs of three classic results by William Mitchell: Given an inaccessible cardinal κ, we present a simply defined countable support iteration P_κ of length κ of proper forcing notions, that satisfies the κ-cc, which forces that $\check{\kappa} = \aleph_2 = 2^{\aleph_0}$, and which has the following properties:

(1) If κ is inaccessible, then P_κ forces that there are no weak $\aleph_1$-Kurepa trees.
(2) If κ is Mahlo, then P_κ forces that there are no special $\aleph_2$-Aronszajn trees.
(3) If κ is weakly compact, then P_κ forces that $\aleph_2$ has the tree property, i.e. that there are no $\aleph_2$-Aronszajn trees.

In contrast to Mitchell's original results, our arguments do not generalize to larger cardinals.

1. The iteration and its basic properties

Let us recall the following standard definitions.

Definition 1.1.

- Given cardinals κ and λ, and a set X, let $\mathrm{Add}(\kappa, \lambda, X)$ denote the partial order of all partial functions p from $\kappa \times \lambda$ to X of size less than κ, ordered by extension.
- If κ^+ is an infinite successor cardinal, then a tree T of height κ^+ is *special* if there is a function $c\colon T \to \kappa$ with the property that for all $s, t \in T$ with $c(s) = c(t)$, we have that s and t are incompatible in T.
- A tree of height ω_1 is *almost special* if there is a function $c\colon T \to \omega$ with the property that for all $s, t, u \in T$ with $c(s) = c(t) = c(u)$ and $s \leq_T t, u$, we have that t and u are compatible in T.
- If T is a tree, we say that a set $B \subseteq [T]$ is *non-stationary* if there is an injection $i\colon B \to T$ with $i(b) \in b$ for all $b \in B$.

We will use the following result of Baumgartner:

Theorem 1.2 (Baumgartner). *If T is a tree of height ω_1 such that $[T]$ is non-stationary, then there is a ccc partial order ensuring that T is almost special in its generic extensions. If T has no cofinal branches, then it ensures that T is special.*

By specializing the disjoint sum of all trees of height ω_1 and with domain ω_1, this easily yields the following:

Corollary 1.3. *There is a ccc partial order ensuring that all ground model trees T of height and size ω_1 with $[T]$ non-stationary become almost special, and all ground model trees of height and size ω_1 with no cofinal branches become special in its generic extensions. We call this forcing the* specializing forcing.

Definition 1.4. Let κ be an inaccessible cardinal. The *tree property iteration of length κ* is the countable support iteration $\langle P_\alpha, \dot{Q}_\alpha \mid \alpha < \kappa \rangle$ of length κ with direct limit P_κ, where the $\dot{Q}_\alpha$'s are defined inductively as follows:

(i) If α is inaccessible, then $\dot{Q}_\alpha$ is trivial. If $\alpha = 3\bar{\alpha}$ is not inaccessible, then $\dot{Q}_\alpha$ is a canonical P_α-name for the forcing notion $\mathrm{Add}(\omega, \omega_2, 2)$ for adding ω_2 Cohen subsets of ω.

(ii) If $\alpha = 3\bar{\alpha} + 1$, then $\dot{Q}_\alpha$ is a canonical P_α-name for the forcing notion $\mathrm{Add}(\omega_1, 1, \mathcal{P}(\omega_1))$ for collapsing $\mathcal{P}(\omega_1)$ to become of size $\aleph_1$.

(iii) If $\alpha = 3\bar{\alpha} + 2$, then $\dot{Q}_\alpha$ is a canonical P_α-name for the specializing forcing.

The following lemma collects some basic properties of the above-defined iteration.

Lemma 1.5. (1) *If $\alpha < \kappa$, then $|P_\alpha| < \kappa$, and hence P_κ satisfies the κ-cc.*

(2) *For every $\alpha < \kappa$, P_α forces $\dot{Q}_\alpha$ to be proper, and hence each P_α for $\alpha \leq \kappa$ is proper.*

(3) *P_κ forces that $2^{\aleph_0} = \aleph_2 = \check{\kappa}$.*

Proof. (1) and (2) are standard, using that κ is inaccessible for (1). For (3), (ii) ensures that we are collapsing a cardinal to become of size $\aleph_1$ in κ-many steps of our iteration up to κ. Since there are only κ-many cardinals below κ in V, this means that in any P_κ-generic extension, there are no cardinals between ω_1 and κ, i.e. $\kappa = \omega_2$.

(i) ensures that we are adding new subsets of ω in κ-many steps of our iteration up to κ, thus ensuring that $2^{\aleph_0} \geq \aleph_2$ in all P_κ-generic extensions. The reversed inequality follows by a standard counting of nice names argument, using that P_κ is a κ-cc partial order of size κ, and that κ is inaccessible.

□

2. No weak Kurepa trees

Definition 2.1. A tree of height and size ω_1 is a *weak Kurepa tree* if it has at least $\aleph_2$-many cofinal branches.

We will use the following results of Baumgartner:

Lemma 2.2 (Baumgartner)**.** *If T is a tree with levels of size less than $2^{\aleph_0}$ and P is a σ-closed notion of forcing, then P adds no new branches to T.*

Lemma 2.3 (Baumgartner)**.** *If T is a tree of height and size ω_1 and $B \subseteq [T]$, then B is non-stationary if and only if $|B| \leq \aleph_1$.*

Lemma 2.4 (Baumgartner)**.** *If T is an almost special tree of height ω_1, then $[T]$ is non-stationary.*

Theorem 2.5 (Mitchell)**.** *If κ is an inaccessible cardinal, then P_κ forces that there are no weak Kurepa trees.*

Proof. Assume that $p \in P_\kappa$, and that $\dot{T}$ is a P_κ-name such that p forces $\dot{T}$ to be a tree of height and size ω_1. By possibly passing to an isomorphic copy, we may assume p to also force that the domain of $\dot{T}$ is ω_1. Since P_κ now forces $\dot{T}$ to be an element of $H(\check{\kappa})$, we may further assume that $\dot{T} \in H(\kappa)$, using that $P_\kappa \subseteq H(\kappa)$ satisfies the κ-cc. But since P_κ is the direct limit of the P_α's for $\alpha < \kappa$, this implies that we find $\alpha < \kappa$ such that $\dot{T}$ is a P_α-name, and we may also assume that α is not inaccessible. By (i), $P_{3\alpha+1}$ forces $\neg$CH. Hence, by Lemma 2.2, $p \Vdash_{3\alpha+2} |[\dot{T}]| \leq \aleph_1$, and hence by Lemma 2.3, $p \Vdash_{3\alpha+2} [\dot{T}]$ is non-stationary. Then, in $P_{3\alpha+3}$, p forces that $\dot{T}$ is almost special. Since being almost special is upwards absolute between models with the same ω_1, it follows that also in P_κ, p forces $\dot{T}$ is almost special. Thus by Lemma 2.4, p forces in P_κ that $[\dot{T}]$ is nonstationary, and hence by Lemma 2.3, that $|[\dot{T}]| \leq \aleph_1$, i.e. p forces that $\dot{T}$ is not a weak Kurepa tree. This argument shows that there are no weak Kurepa trees in P_κ-generic extensions, as desired. □

3. No special $\aleph_2$-Aronszajn trees

Theorem 3.1 (Mitchell)**.** *If κ is a Mahlo cardinal, then P_κ forces that there are no special $\aleph_2$-Aronszajn trees.*

Proof. Assume for a contradiction that there is a condition $p \in P_\kappa$ and a P_κ-name $\dot{T}$ such that p forces $\dot{T}$ to be a special ω_2-Aronszajn tree. By passing to an isomorphic copy, we may assume p to also force that for every $\alpha < \omega_2$, $\dot{T}(\alpha) \subseteq \{\alpha\} \times \omega_1$. Since by Lemma 1.5, $p \Vdash \dot{T} \subseteq H(\check{\kappa})$, we may further assume that $\dot{T} \subseteq H(\kappa)$, using that $P_\kappa \subseteq H(\kappa)$ satisfies the κ-cc. Let θ be sufficiently large and regular, and let $M \prec H(\theta)$ be of size less than κ, with $\kappa, P_\kappa, p, \dot{T} \in M$, and with $\alpha = M \cap \kappa$ inaccessible, using that κ is Mahlo. Let $\bar{M}$ be the transitive collapse of M, and let $j\colon \bar{M} \to H(\theta)$ be the anticollapse embedding.

Claim 1. *$j(\alpha) = \kappa$, $H(\alpha) \cup \{p, P_\alpha\} \subseteq \bar{M}$, $j \restriction H(\alpha) = id$, $j(P_\alpha) = P_\kappa$, $j(p) = p$.*

Proof. Clearly, $j^{-1}(\kappa) = \alpha$. Since $M \cap \kappa = \alpha$ is an inaccessible cardinal, it follows that $H(\kappa) \cap M = H(\alpha)$, that $p \in H(\alpha)$, and hence that $j \restriction H(\alpha) = id$, and that $H(\alpha) \subseteq \bar{M}$. Since P_α is definable from α over $H(\alpha)$, it follows that $P_\alpha \in \bar{M}$, and by elementarity of j, using that $j(\alpha) = \kappa$ and that the definition of the tree property iteration is sufficiently absolute, it follows that $j(P_\alpha) = P_\kappa$. ☐

Let $\dot{\bar{T}}$ be a P_α-name such that $j(\dot{\bar{T}}) = \dot{T}$.

Claim 2. *$p \Vdash_\kappa \dot{\bar{T}} = \dot{T} \restriction \check{\alpha}$.*

Proof. We show that for every $q \leq_\kappa p$ and every pair $\langle \beta, \xi \rangle \in \alpha \times \omega_1$, q forces that $\langle \check{\beta}, \check{\xi} \rangle \in \dot{\bar{T}}$ if and only if q forces that $\langle \check{\beta}, \check{\xi} \rangle \in \dot{T} \restriction \check{\alpha}$, which is clearly sufficient to show that p forces the domains of $\dot{\bar{T}}$ and of $\dot{T} \restriction \check{\alpha}$ to agree. We leave the analogous result with respect to the orderings of those trees to the interested reader.

Assume first q forces that $\langle \check{\beta}, \check{\xi} \rangle \in \dot{\bar{T}}$. Since this statement is absolute and $\dot{\bar{T}}$ is a P_α-name, this is already forced by $q \restriction \alpha \in H(\alpha)$. Then, by the elementarity of j, $j(q \restriction \alpha) = q \restriction \alpha \Vdash \langle \check{\beta}, \check{\xi} \rangle \in \dot{T}$. Since $q \leq q \restriction \alpha$ and by our assumptions on $\dot{T}$, it thus follows that $q \Vdash_\kappa \langle \check{\beta}, \check{\xi} \rangle \in \dot{T} \restriction \check{\alpha}$.

Now assume q forces that $\langle \check{\beta}, \check{\xi} \rangle \in \dot{T} \restriction \check{\alpha} \subseteq \dot{T}$, and let $r \leq_\kappa q$. Then, by elementarity of j, and since $r \leq r \restriction \alpha$, there is a condition $\bar{r} \leq r \restriction \alpha$ in P_α forcing that $\langle \check{\beta}, \check{\xi} \rangle \in \dot{\bar{T}}$. But then, the greatest lower bound r^* of $\bar{r}$ and r in P_κ is stronger than r, and still forces that statement. We thus showed that there is a dense set of conditions below q forcing that $\langle \check{\beta}, \check{\xi} \rangle \in \dot{\bar{T}}$, yielding that $q \Vdash \langle \check{\beta}, \check{\xi} \rangle \in \dot{\bar{T}}$, as desired. ☐

By the elementarity of j, $\bar{M} \models p \Vdash_\alpha \dot{\bar{T}}$ is a special ω_2-Aronszajn tree. Since this is sufficiently absolute, this forcing statement also holds true in our universe V. Since $\dot{Q}_\alpha$ is trivial, p also forces this statement in $P_{\alpha+1}$. Note that by Lemma 1.5, $2^{\aleph_0} = \aleph_2$ after forcing with $P_{\alpha+1}$. Since $\dot{Q}_{\alpha+1}$ is forced to be σ-closed, it thus follows by Lemma 2.2 that $p \Vdash_{\alpha+2} \dot{\bar{T}}$ has no cofinal branches. Pick a $P_{\alpha+2}$-name $\langle \dot{\alpha}_i \mid i < \omega_1 \rangle$ for a strictly increasing continuous sequence of ordinals below α that is cofinal in α, and let $\dot{S}$ be a $P_{\alpha+2}$-name such that $p \Vdash_{\alpha+2} \dot{S} = \bigcup_{i<\omega_1} \dot{\bar{T}}(\dot{\alpha}_i)$. Then $p \Vdash_{\alpha+2} \dot{S}$ is a tree of height and size ω_1 without cofinal branches. Then, $p \Vdash_{\alpha+3} \dot{S}$ is special. But then, by the upwards absoluteness of being special, p also forces in P_κ that $\dot{S}$, and hence also $\dot{\bar{T}}$ have no cofinal branches. However by Claim 2, any P_κ-name for a node of $\dot{T}$ on level $\check{\alpha}$ yields a P_κ-name for a cofinal branch through $\dot{\bar{T}}$, namely the name for the set of $\dot{T}$-predecessors of that node, which is clearly a contradiction. ☐

4. No Aronszajn trees

Theorem 4.1 (Mitchell)**.** *If κ is a weakly compact cardinal, then P_κ forces that there are no $\aleph_2$-Aronszajn trees.*

Proof. Assume for a contradiction that there is a condition $p \in P_\kappa$ and a P_κ-name $\dot{T}$ such that p forces $\dot{T}$ to be an ω_2-Aronszajn tree. As in the proof of Theorem 3.1, we may assume that p forces that for every $\alpha < \omega_2$, $\dot{T}(\alpha) \subseteq \{\alpha\} \times \omega_1$, and we may take $\dot{T}$ to be a nice name of the form

$$\dot{T} = \{\{\langle \check{\beta}, \check{\xi} \rangle\} \times A_{\beta,\xi} \mid \beta < \kappa, \xi < \omega_1\} \subseteq H(\kappa),$$

for certain (possibly empty) antichains $A_{\beta,\xi}$ of P_κ. An analogous argument applies to the ordering relation of $\dot{T}$. Viewing the name $\dot{T}$ as a binary relation between pairs $\langle \beta, \xi \rangle$ of ordinals less than κ and conditions in P_κ, using that P_κ is κ-cc, let C be the club subset of κ consisting of all cardinals α which are *closure points* of $\dot{T}$, in the sense that all pairs $\langle \beta, \xi \rangle$ of ordinals less than α are related only to conditions in P_α. Let $\alpha \in C$ be inaccessible and greater than the supremum of the support of p, such that in $P_\alpha = P_\kappa \cap H(\alpha)$, p forces the name $\dot{T} \cap H(\alpha)$ to denominate an ω_2-Aronszajn tree, using that the corresponding statement about p, $\dot{T}$ and P_κ is a Π^1_1-statement over $H(\kappa)$, and that κ is weakly compact.

Claim 3. *Let $\dot{\dot{T}} = \dot{T} \cap H(\alpha)$. Then, p forces that $\dot{\dot{T}} = \dot{T} \upharpoonright \check{\alpha}$.*

Proof. We show that p forces the domains of the trees $\dot{\dot{T}}$ and of $\dot{T} \upharpoonright \check{\alpha}$ to agree, and again leave the analogous argument for the orderings of those trees to the interested reader. Since $\dot{\dot{T}} \subseteq \dot{T}$, it is immediate that p forces that $\dot{\dot{T}} \subseteq \dot{T}$. But also, every element of $\dot{\dot{T}}$ is forced by p to be on some level below $\check{\alpha}$, i.e. $p \Vdash \dot{\dot{T}} \subseteq \dot{T} \upharpoonright \check{\alpha}$.

Now if $q \leq_\kappa p$ forces that $\langle \check{\beta}, \check{\xi} \rangle \in \dot{T} \upharpoonright \check{\alpha}$, then $\beta < \alpha$, and using that $\alpha \in C$, it follows that there is a condition $a \in A_{\beta,\xi} \subseteq P_\alpha$ with $q \leq a$. But then,

$$\langle \langle \check{\beta}, \check{\xi} \rangle, a \rangle \in \dot{T} \cap H(\alpha) = \dot{\dot{T}},$$

yielding that $q \leq a \Vdash \langle \check{\beta}, \check{\xi} \rangle \in \dot{\dot{T}}$, showing that $p \Vdash \dot{T} \upharpoonright \check{\alpha} \subseteq \dot{\dot{T}}$, as desired. □

Now, the remaining proof proceeds exactly as in the proof of Theorem 3.1. □

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