AN ORDINAL-CONNECTION AXIOM AS A WEAK FORM OF GLOBAL CHOICE UNDER THE GCH

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Abstract. The minimal ordinal-connection axiom MOC was introduced by the first author in [2]. We observe that MOC is equivalent to a number of statements on the existence of certain hierarchies on the universe, and that under global choice, MOC is in fact equivalent to the GCH. Our main results then show that MOC corresponds to a weak version of global choice in models of the GCH: it can fail in models of the GCH without global choice, but also global choice can fail in models of MOC.

1. Introduction

Only some basic assignments of ordinals to sets, such as the cardinality and the rank, are granted by the axioms of ZFC, and the lack of a stronger connection between sets and ordinals can be seen to explain the deficiency of ZFC with respect to cardinal arithmetic. Ordinal-connection axioms were introduced by the first author in [2] with the purpose of expressing a strong connection between the universe of sets and ordinal numbers in simple terms. However, they can also be seen to isolate an important fragment of L-likeness, that is essentially the existence of a reasonably well-behaved *slow* rank function.[1] We will observe that the minimal ordinal-connection axiom MOC from [2], the definition of which we will recall in Section 2 below, is a weak form of global choice under the GCH, and that a local version of MOC, that was shown to imply the GCH in [2], is in fact equivalent to the GCH, thus in particular showing that the GCH can be seen as the assertion of the existence of well-behaved slow rank functions.

We will also observe that MOC is in fact equivalent to certain *hierarchy principles*. As an example, MOC is equivalent to the assertion that the universe can be written as a continuous increasing union $\bigcup_{\alpha\in\mathrm{Ord}} K_\alpha$ with the following properties:

- Each K_α is transitive and has size $|\alpha|$ in case $\alpha \geq \omega$.
- $K_\alpha = H(\alpha)$ whenever α is an infinite cardinal.

The central results of our paper investigate the role of MOC as a weak version of global choice under the GCH. Starting from a model of ZFC, we will provide models

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[1] A rank function is *slow* if the collection of objects of rank α only has size $|\alpha|$ for each infinite α. This slowness condition is clearly not satisfied by the usual rank function with respect to the von Neumann hierarchy.

with the following properties, letting GBc denote the axioms of Gödel-Bernays set theory without global choice, however with the axiom of choice included.

- GBc and the GCH hold while MOC fails.
- GBc and MOC hold, while global choice fails.

This contributes to the problem of having certain orderings on the universe of sets while not having a global wellorder. For example (see [3]), it seems to be open whether it is possible that there is a linear ordering of the universe of sets while there is no well-ordering of it.

2. Basic Definitions

We start by recalling the basic definitions from [2]. We will quite freely talk about classes as objects in the following. This should either be understood as referring to definable classes in the context of first order models, or as referring to the classes of models of the form $\langle M, \mathcal{C}\rangle$ of GBc, where M denotes the first order objects (i.e., sets) and $\mathcal{C}$ denotes the second order objects (i.e., classes) of such a model. Note that in each of these cases, every set is a class as well, that is, when we make statements about classes in the following, this is always meant to include the possibility that these classes in fact aren't proper classes, that is they are just sets. When we quantify over the variable α in the following, this is always to be understood as quantification over ordinals.

Definition 2.1. *Let K be a transitive class, and let $\rho\colon K \to \mathrm{Ord} \cap K$ be a class function. Then, ρ is an* ordinal-connection *for K if*

(C1) $\forall \alpha \in K\ \rho(\alpha) = \alpha$,
(C2) $\forall x, y \in K\ [x \in y \to \rho(x) < \rho(y)]$, *and*
(C3) $\forall \alpha \in K \setminus \omega\, \exists f\colon \alpha \to \{x \mid \rho(x) < \alpha\}$ *surjective.*

Definition 2.2. *If ρ is an ordinal-connection for K, we say that ρ is* minimal *if*

(C4) for every $x \in K$, if T is transitive such that $x \in T$, and r is an ordinal-connection for T, then $\rho(x) < r(x)^+$.[2]

We say that ρ is a (minimal) ordinal-connection in case it is a (minimal) ordinal-connection for V. $MOC(\rho)$ is the statement (with second order parameter ρ) that ρ is a minimal ordinal-connection. The minimal ordinal-connection axiom MOC is the (second order) statement that there exists a class that is a minimal ordinal-connection.

Note that the combination of (C2) and (C4) implies that if $x \in V_\omega$, then $\rho(x)$ agrees with the von Neumann rank of x. It is sometimes easier to think of an ordinal-connection in terms of a hierarchy. Whenever we talk about hierarchies, we will tacitly make the innocuous extra assumption that $\mathrm{Ord} \cap K$ is either a limit ordinal or equal to Ord itself.[3]

Definition 2.3. *A* hierarchy *for K is a continuous $\subseteq$-increasing class sequence $\vec{K} = \langle K_\alpha \mid \alpha \in \mathrm{Ord} \cap K\rangle$ such that $K = \bigcup_{\alpha \in \mathrm{Ord} \cap K} K_\alpha$.*

We say that $\vec{K}$ is an ordinal-connection hierarchy *for K if it is a hierarchy for K that satisfies the following additional properties:*

[2] If $r(x)$ happens to be finite, we let $r(x)^+ = r(x) + 1$.

[3] This will avoid having to notationally deal with an additional top level of our hierarchy in the case when $\mathrm{Ord} \cap K$ is a successor ordinal.

(H1) $\forall\alpha \in K\ K_\alpha \cap \mathrm{Ord} = \alpha$,
(H2) $\forall x, y \in K\ [x \in y \to \exists\alpha\ (x \in K_\alpha \wedge y \notin K_\alpha)]$, *and*
(H3) $\forall\alpha \in K\ [\alpha \geq \omega \to |K_\alpha| \leq |\alpha|]$.[4]

Note that property (H2) in particular implies that each K_α is transitive. If ρ is an ordinal-connection for K, we denote $\{x \mid \rho(x) < \alpha\}$ as $\rho_{<\alpha}$. Clearly, $\langle \rho_{<\alpha} \mid \alpha \in K\rangle$ is an ordinal-connection hierarchy for K. On the other hand, given a hierarchy $\vec{K} = \langle K_\alpha \mid \alpha \in K\rangle$, we can define $\rho(x)$ to be the least α such that $x \in K_{\alpha+1}$. We say that ρ is the rank function associated to the hierarchy of K_α's in this case. If $\vec{K}$ is an ordinal-connection hierarchy, then clearly, this ρ is an ordinal-connection for K. Note in particular that corresponding ordinal-connections and ordinal-connection hierarchies are interdefinable, and also that the axioms (C1), (C2) and (C3) directly correspond to the axioms (H1), (H2) and (H3) respectively. In Section 3 below, we will also introduce an analogue of the axiom (C4) in terms of hierarchies.

3. Some basic observations

In this section, we will make a number of simple observations, which we will make use of in the later sections of our paper.

We introduce an additional principle:

(C=)
- For every $n \in \omega$, $\rho_{<n} = V_n$, and
- for every infinite cardinal κ, $\rho_{<\kappa} = H(\kappa)$.

The following strengthens [2, Proposition 4.1].

Proposition 3.1. *If* $\rho\colon V \to \mathrm{Ord}$ *satisfies (C2), (C3) and (C4) with respect to* $K = V$*, then it satisfies (C=). In particular thus, this is the case if* ρ *is a minimal ordinal-connection.*

Proof. Assume that (C2), (C3) and (C4) hold. It is easy to check that for every $n \in \omega$, $\rho_{<n} = V_n$, and that $\rho_{<\omega} = H(\omega)$. Making use of continuity at limit cardinals, it thus suffices to verify (C=) for regular and uncountable cardinals κ. If $\rho(x) < \kappa$, then there is $\alpha < \kappa$ such that $\rho(x) < \alpha$, and by (C3), there is a surjective $f\colon \alpha \to \rho_{<\alpha}$. Making use of (C2), this clearly implies that $x \in H(\kappa)$.

If on the other hand, if $x \in H(\kappa)$, assume inductively that $\mathrm{trcl}(x) \subseteq \rho_{<\kappa}$. Since $|\mathrm{trcl}(x)| < \kappa$, it follows that there is some $\alpha < \kappa$ such that $\mathrm{trcl}(x) \subseteq \rho_{<\alpha}$. Let T be the transitive set $\mathrm{trcl}(\{x\}) \cup (\alpha + 1)$, and let $r\colon T \to T$ be defined by letting

- $r(x) = r(\alpha) = \alpha$,
- $r(\beta) = \beta$ for every $\beta < \alpha$, and
- $r(z) = \rho(z)$ for every $z \in \mathrm{trcl}(x)$.

r is easily seen to be an ordinal-connection for T. (C4) thus yields that $\rho(x) < r(x)^+ = \alpha^+ \leq \kappa$, as desired. □

The above proposition can also be reversed, altogether showing that over V, given (C2) and (C3), the axiom (C4) could equivalently be replaced by (C=).

Observation 3.2. *If* $\rho\colon V \to \mathrm{Ord}$ *satisfies (C2), (C3) and (C=) with respect to* $K = V$*, then it satisfies (C4).*

[4]Clearly, from (H1) we obtain $|K_\alpha| \geq |\alpha|$, so that together with (H3), we obtain $|K_\alpha| = |\alpha|$ for any infinite α.

Proof. Assume that $x \in K$, T is transitive such that $x \in T$, and r is an ordinal-connection for T. The case when $x \in V_\omega$ is easily checked, so let us assume that $x \notin V_\omega$. Let κ^+ be least such that $x \in H(\kappa^+)$. This implies that $r(x) \geq \kappa$: If $r(x) < \kappa$, then the first part of the argument from the proof of Proposition 3.1, which uses (C2) and (C3), shows that $x \in H(\kappa)$, contradicting our minimality assumption. But since $\rho_{<\kappa^+} = H(\kappa^+)$ by (C=), it follows that $\rho(x) < \kappa^+ \leq r(x)^+$, yielding (C4) to hold. □

The axiom (C=), and thus also (C4), clearly has a counterpart for ordinal-connection hierarchies.

Definition 3.3. *If $\vec{K} = \langle K_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ is an ordinal-connection hierarchy (for V), we say that $\vec{K}$ is* minimal *if it satisfies the following.*

(H4)
- *For every $n \in \omega$, $K_n = V_n$, and*
- *For every infinite cardinal κ, $K_\kappa = H(\kappa)$.*

Note that under the axioms (H2) and (H3), the axiom (H4) directly corresponds to the axiom (C4) as a consequence of Proposition 3.1 and Observation 3.2.

Proposition 3.4. *Assume that there is a hierarchy $\vec{K} = \langle K_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ (for V) that satisfies (H3) and (H4) with respect to $K = V$. Then, there is a minimal ordinal-connection that is definable from $\vec{K}$ (over V).*

Proof. Consider the hierarchy $\vec{K}^* = \langle K^*_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ obtained by letting K^*_α be the transitive closure of K_α for each ordinal α. This yields a hierarchy of transitive levels K^*_α which still has properties (H3) and (H4): Regarding (H3), note that by (H4) for $\vec{K}$, we have $K_\alpha \subseteq H(\alpha^+)$ for every ordinal α, hence if K_α has size at most $|\alpha|$ by (H3), then this is also the case for its transitive closure.

Now, we refine $\vec{K}^*$ in a continuous way, replacing each successor level K^*_α by less than α^+-many levels, corresponding to the ranks (in the von Neumann hierarchy) of elements of K^*_α: That is, instead of K^*_α, we first have a level consisting of the union of the earlier K^*_β's in order to ensure continuity, then the next level will additionally have the rank-minimal new elements of K^*_α (if there are any), then the next level will additionally have the new elements of K^*_α of the next-largest rank (if there are any) etc. Note that by (H3), we will this way have exhausted K^*_α after less than α^+-many steps. Let $\langle K^{**}_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ be the hierarchy thus obtained, which satisfies (H2) by construction, which clearly still satisfies (H3), and which still satisfies (H4) by the above. Now we let $K^{***}_\alpha = K^{**}_\alpha \cup \alpha$ for every ordinal α. It is straightforward to check that $\langle K^{***}_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ is a minimal ordinal-connection hierarchy for K that is definable from σ (over V), and thus so is its corresponding ordinal-connection. □

Remark: In the above, we isolated a weak hierarchy principle that implies MOC. Going in the other direction, one could also try to find strong hierarchy principles that follow from MOC. In addition to the axioms (H1), (H2), (H3) and (H4), one could for example require that $K_{\alpha+1}$ contains all subsets of K_α that can be obtained by application of the Gödel operations (see [5, Definiton 13.6]) to elements of K_α, thus obtaining a very structured hierarchy witnessing MOC, and supporting the idea of MOC expressing an important fragment of L-likeness.

We close this section with two more remarks regarding [2]. First, note that Proposition 3.1 immediately yields the result from [2, Section 3] that GCH is a consequence of MOC.

Corollary 3.5. *MOC implies the* GCH*.*

Proof. For any infinite cardinal κ, by Proposition 3.1 and by (C3), we have

$$\kappa^+ = |\rho_{<\kappa^+}| = |H(\kappa^+)| = 2^\kappa.$$

□

Our above results also yield a strengthening of [2, Proposition 5.1],[5] and provides us with a large number of examples of models of MOC and their minimal ordinal-connections.

Proposition 3.6. *If $L[A]$ is such that for every regular uncountable cardinal κ we have $L[A]_\kappa = H(\kappa)^{L[A]}$, then the $L[A]$-rank λ^A is a minimal ordinal-connection in $L[A]$.*

Proof. All the required properties of λ^A other than (C4) are very easy to verify. But note that (H4) for the hierarchy corresponding to λ^A is immediate from our assumption, and thus so is (C4) for λ^A by Observation 3.2. □

This result could clearly be extended to a large class of models with a suitable fine structural hierarchy.

4. Provability from GCH

The following local version of MOC was also introduced in [2].

Definition 4.1. *The* local minimal ordinal-connection axiom *$LMOC$ is the statement that for every α there is an ordinal $\beta > \alpha$ such that there is a minimal ordinal-connection $\rho\colon V_\beta \to \beta$.*

It is shown in [2, Proposition 6.4] that $LMOC$ implies the GCH. We show that this implication can also be reversed, that is $LMOC$ is a theorem of ZFC plus GCH.

Theorem 4.2. ZFC *plus* GCH *implies $LMOC$.*

Proof. Given an ordinal α, let $\beta > \alpha$ be an uncountable limit cardinal such that $\beta = \beth_\beta$, and thus also $V_\beta = H(\beta)$. Now, let $\vec{K} = \langle K_\gamma \mid \gamma < \beta\rangle$ be a hierarchy with union $H(\beta)$ that satisfies (H3) and (H4). Such $\vec{K}$ can easily be chosen using the GCH. Now we proceed exactly as in the argument for the proof of Proposition 3.4 to construct a minimal ordinal-connection hierarchy, however not for V as we did there, but for $H(\beta)$. □

This can now easily be used to show that MOC is equivalent to GCH over GBC (where the latter notably includes the assertion of the existence of a global wellorder, or equivalently, the axiom of global choice).

Theorem 4.3. *GBC plus* GCH *implies MOC, in fact, given a global wellorder, we can definably construct a minimal ordinal-connection over any model of* ZFC *plus* GCH*.*

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[5]Note that Acceptability for $L[A]$, which was used as assumption in [2, Proposition 5.1], implies that for every regular uncountable cardinal κ, we have $L[A]_\kappa = H(\kappa)^{L[A]}$.

Proof. Simply use the global wellorder to glue together (pieces of) witnesses for Theorem 4.2. □

5. A failure of *MOC*

By Theorem 4.3, *MOC* is a consequence of global choice for models of *GBc* that satisfy the GCH. In [1], William Easton has shown that there are models of *GBc* in which global choice fails. We strengthen this to show that there are models of *GBc* plus GCH in which *MOC* fails.

Theorem 5.1. *Any model of* ZFC *has an extension, that is obtained as the first order part of a class forcing extension, which satisfies GBc plus* GCH*, and in which MOC fails.*

Proof. Since we may first obtain the GCH by class forcing, we may as well assume it to already hold in the ground model V. Now let $\mathbb{P}$ be the Easton support class forcing product $\prod_{\gamma \text{ regular}} \mathrm{Add}(\gamma,\gamma^+)$, adding γ^+ new Cohen subsets of γ for every infinite regular cardinal γ. Let G be $\mathbb{P}$-generic. $\mathbb{P}$ is a tame notion of class forcing that preserves all cardinals, all cofinalities and the GCH, and in particular, the first order part $V[G]$ of its generic extension is a model of ZFC. If we thus consider the second order model $V[G]$ with only its definable classes as second order objects, we clearly obtain a model of *GBc* plus GCH, and we are thus left with showing that no class function ρ witnessing *MOC* for $V[G]$ is definable (using parameters, that is) over $V[G]$.

Assume for a contradiction that such ρ exists, definably over $V[G]$. Suppose $\psi(x,y,z)$ is a formula defining the property that $\rho(x)<\rho(y)$ when using the parameter z, i.e.,

$$V[G]\models\forall x,y\ [\psi(x,y,z)\iff\rho(x)<\rho(y)],$$

and let $\dot z$ be a $\mathbb{P}$-name for z. Suppose that $p\in G$ forces this property of $\dot z$. Let γ be a regular cardinal above the supports of all conditions appearing in the name $\dot z$, and let g_α for $\alpha<\gamma^+$ denote the Cohen subsets of γ added at stage γ of our product $\mathbb{P}$. By $MOC(\gamma)$ and by Proposition 3.1, $\rho(g_\alpha)<\gamma^+$ for every $\alpha<\gamma^+$. Since $\rho_{<(\rho(g_0)+1)}$ only has cardinality γ by Property (C3), there must be some $\delta<\gamma^+$ such that $\rho(g_0)<\rho(g_\alpha)$ whenever $\alpha\geq\delta$. For every $\alpha<\gamma^+$, let $\dot g_\alpha$ be a canonical $\mathbb{P}$-name for g_α. Pick $q\leq p$ in G such that

$$q\Vdash\forall\alpha\geq\check\delta\ \psi(\dot g_0,\dot g_\alpha,\dot z).$$

For any $\alpha<\gamma^+$, Let π_α denote the automorphism of $\mathbb{P}$ that swaps the information on g_0 with the information on g_α of any condition in $\mathbb{P}$ and is the identity otherwise, that is it only swaps the coordinates 0 and α in the Cohen product at stage γ of our product. We also use π_α to denote its natural extension to $\mathbb{P}$-names.[6] Note that for any α, $\pi_\alpha(\dot g_0)=\dot g_\alpha$, $\pi_\alpha(\dot g_\alpha)=\dot g_0$, and $\pi_\alpha(\dot z)=\dot z$, where the latter follows from our choice of γ.

Claim 5.2. *There is some $\delta\leq\alpha<\gamma^+$ so that $\pi_\alpha(q)\in G$.*

Applying π_α to the above forcing statement about q, we obtain that

$$\pi_\alpha(q)\Vdash\psi(\dot g_\alpha,\dot g_0,\dot z).$$

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[6]Recall that for any name σ and automorphism π, the name $\pi(\sigma)$ is obtained by recursively applying π to all conditions appearing in (the transitive closure of) the name σ.

If $\pi_\alpha(q) \in G$, this means that $V[G] \models \psi(g_\alpha, g_0, z)$, and hence that $\rho(g_\alpha) < \rho(g_0)$, contradicting that $\alpha \geq \delta$. It remains to verify the above claim.

Proof of the Claim. It suffices to show that there is a dense set of conditions s below q for which there exists an $\alpha \geq \delta$ such that $s \leq \pi_\alpha(q)$. Having done so, we simply pick some such condition $s \in G$, which we can do for G meets any set that is dense below any of its elements. Then clearly, for the corresponding α, $\pi_\alpha(q) \in G$, as desired.

Thus, let $r \leq q$, and pick $\alpha \in [\delta, \gamma^+)$ so that r has no information about the α^{th} Cohen subset of γ to be added at stage γ of the product $\mathbb{P}$. It follows that $\pi_\alpha(q)$ is compatible to r, and we may thus pick some s below both $\pi_\alpha(q)$ and r, which thus will be as desired. □

□

With a little extra care, the above argument also shows the following, yielding second order models in which *MOC* fails, and for which the classes are not just the definable classes.

Theorem 5.3. *For any second order model $\mathcal{M} = \langle M, \mathcal{C}\rangle$ of GBc, there is a class forcing notion $\mathbb{P}$ such that whenever G is $\mathbb{P}$-generic over $\mathcal{M}$, then the model $\langle M[G], \mathcal{D}\rangle$ satisfies GBc plus* GCH*, however fails to satisfy MOC, where $\mathcal{D}$ denotes the classes that are definable over $M[G]$ using class parameters from $\mathcal{C}$.*

Proof. Proceeding almost exactly as in the proof of Theorem 5.1, the only thing to note is that we may pick a canonical name $\check{C}$ for some class parameter C from our ground model, so that only the trivial condition of our notion of forcing $\mathbb{P}$ appears within the name $\check{C}$, and that this clearly implies that for any automorphism π of $\mathbb{P}$, we have $\pi(\check{C}) = \check{C}$, allowing the argument by contradiction to proceed as in the proof of Theorem 5.1. □

6. *MOC* without global choice

In this section, we show that it is consistent to have models of *GBc* plus *MOC* in which global choice fails. Parts of the argument here are to some extent inspired by a (somewhat unnecessarily complicated) argument that Joel Hamkins gave on MathOverflow ([4]) to argue for the consistency of a failure of a weak form of global choice over models of *GBc*.

Theorem 6.1. *Any model V of* ZFC *has a class forcing extension $\langle V[G], \mathcal{C}\rangle$ with an inner model of the form $\langle V[G], \mathcal{D}\rangle$ for some $\mathcal{D} \subseteq \mathcal{C}$, which satisfies GBc plus MOC, and in which global choice fails.*

Proof. Since we may first obtain the GCH together with the existence of a global well-order by class forcing, we may as well assume these to already hold in the ground model, which would now be a second order model of the form $\langle V, \mathcal{B}\rangle$. We may moreover assume that this global wellorder $\prec$ has the property that for every infinite cardinal κ, the set of the first κ-many elements in the ordering corresponds exactly to $H(\kappa)$. Let $\langle x_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ be the enumeration of V corresponding to $\prec$. Let $\mathbb{P}$ be the ordinal length Easton support product which at every regular infinite cardinal γ adds $\gamma \cdot 2$ new Cohen subsets of γ by applying the forcing $\mathrm{Add}(\gamma, \gamma \cdot 2)$, and which is trivial otherwise. By standard arguments, this notion of class forcing is tame, preserves all cardinals and cofinalities, and also preserves the GCH. Moreover, it has the property that whenever γ is an infinite regular cardinal, then every element of $H(\gamma)$ in a $\mathbb{P}$-generic extension has a $\mathbb{P}$-name in $H(\gamma)$. Let G be $\mathbb{P}$-generic. In particular then, $\langle V[G], \mathcal{B}[G]\rangle$ satisfies GBC. Our aim is to find some $\mathcal{D} \subseteq \mathcal{B}[G]$ so that $\langle V[G], \mathcal{D}\rangle$ satisfies GBc plus MOC, however so that global choice fails in that model.

Working in the ground model V, let Δ_γ be a collection of cardinality γ of permutations of $\gamma \cdot 2$ such that for every ordinal $\bar\gamma < \gamma$, there is some $\mu = \mu^\gamma_{\bar\gamma} \in \Delta_\gamma$ with the following properties:

- $\mu[\gamma] = [\gamma, \gamma \cdot 2)$ and $\mu[[\gamma, \gamma \cdot 2)] = \gamma$.
- $\mu[\bar\gamma] \subseteq [\gamma + \bar\gamma, \gamma \cdot 2)$.

For any $\mu \in \Delta_\gamma$, let π_μ be the automorphism of $\mathbb{P}_{\gamma+1}$ which permutes the factors of the product $\mathrm{Add}(\gamma, \gamma \cdot 2)$ at stage γ of our iteration, that is it permutes the indices of the Cohen subsets of γ to be added by that forcing, according to the permutation μ. We will also use π_μ to denote its natural extension to $\mathbb{P}$, and to arbitrary $\mathbb{P}$-names.

We now inductively construct a sequence of names $\langle \dot K_\alpha \mid \alpha \in \mathrm{Ord}\rangle$, the evaluation $\langle K_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ with any generic filter of which will be a hierarchy satisfying (H3) and (H4), and thus by Proposition 3.4 witnessing MOC for our $\mathbb{P}$-generic extension. For $\alpha \le \omega$, we let $\dot K_\alpha$ be (the name in the trivial forcing for) V_α. If α is a regular uncountable cardinal, we let $\dot K_\alpha \subseteq H(\alpha)$ be a canonical $\mathbb{P}_\alpha$-name for $H(\alpha)$ of size α. For singular limit ordinals α, we let $\dot K_\alpha = \bigcup_{\beta<\alpha} \dot K_\beta$. Having picked $\dot K_\alpha$, let $\dot K_{\alpha+1} \supseteq \dot K_\alpha$ be the $\prec$-least $\mathbb{P}_{|\alpha|+1}$-name with the following properties:[7]

- $|\dot K_{\alpha+1}| = |\alpha|$ and $\dot K_{\alpha+1} \subseteq H(\alpha^+)$.[8]
- If x_α happens to be a $\mathbb{P}_{|\alpha|+1}$-name for an element of $H(\alpha^+)$, then
$$\langle x_\alpha, 1_{\mathbb{P}_{|\alpha|+1}}\rangle \in \dot K_{\alpha+1}.$$[9]
- For any regular $\delta \le |\alpha|$ and any $\mu \in \Delta_\delta$, the name $\dot K_{\alpha+1}$ is closed under both $\pi = \pi_\mu$ and $\pi = \pi_\mu^{-1}$, in the sense that whenever $\langle \sigma, p\rangle \in \dot K_{\alpha+1}$, then also $\langle \pi(\sigma), \pi(p)\rangle \in \dot K_{\alpha+1}$.

Claim 6.2. *Such names $\dot K_{\alpha+1}$ exist.*

Proof. In fact, there is not much to prove here – the second property is clearly easy to achieve, and the third property is so as well, for it is only at most $|\alpha|$-many automorphisms π under which we need to close off $\dot K_{\alpha+1}$. Now if we let $\dot K_{\alpha+1}$ be $\subseteq$-smallest possible with these properties, it will clearly satisfy the first property above, using that this property holds for $\dot K_\alpha$ inductively. □

Let $\vec K = \langle K_\gamma \mid \gamma \in \mathrm{Ord}\rangle$ denote the sequence $\langle \dot K_\gamma^G \mid \gamma \in \mathrm{Ord}\rangle$ obtained from G. This sequence is clearly a hierarchy for $V[G]$, satisfying (H3) and (H4) over $V[G]$. We consider the second order model $\langle V[G], \mathcal{D}\rangle$ where $\mathcal{D}$ is the collection of all second order objects that are definable over $V[G]$ using the sequence $\vec K$ as (class) parameter. We clearly obtain a model of GBc, and we obtain MOC by Proposition 3.4.

[7]Note that none of the below properties refers to forcing statements, but to properties of the actual name $\dot K_{\alpha+1}$ in the ground model V.

[8]This will imply that $K_{\alpha+1}$ is an α-size subset of $H(\alpha^+)$ in $\mathbb{P}$-generic extensions.

[9]This will imply that $\bigcup_{\beta<\alpha^+} K_\beta = H(\alpha^+)$ in $\mathbb{P}$-generic extensions.

We are thus left with showing that global choice fails in this model, that is, that there is no global wellorder of $V[G]$ that is definable using $\vec{K}$. The crucial property that we obtained is provided by the following claim.

Claim 6.3. *For any regular infinite cardinal γ and any $\mu \in \Delta_\gamma$, $\vec{K}$ is invariant under π_μ, that is if G is $\mathbb{P}$-generic and $G^* = \pi_\mu[G]$, then both generics give rise to the same sequence $\vec{K}$ in their $\mathbb{P}$-generic extensions.*

Proof. Let γ and $\mu \in \Delta_\gamma$ be as in the statement of the claim. It clearly suffices to show that for any ordinal α, $\dot{K}_{\alpha+1}$ is invariant under π_μ and its inverse, that is $\pi_\mu[\dot{K}_{\alpha+1}] = \dot{K}_{\alpha+1} = \pi_\mu^{-1}[\dot{K}_{\alpha+1}]$. Now $\dot{K}_{\alpha+1}$ is a $\mathbb{P}_{|\alpha|+1}$-name, and is thus unaffected by π_μ in case $\gamma > \alpha$. If $\gamma \leq \alpha$ however, then this is immediate from the closure under π_μ and under π_μ^{-1} that we required back when we picked $\dot{K}_{\alpha+1}$ in the above. □

Assume for a contradiction that a global wellorder $\prec^*$ of $V[G]$ exists definably over $V[G]$ using $\vec{K}$ as class parameter, and some set parameter z. Suppose $\psi(x, y, z, \vec{K})$ is a formula defining the property that $x \prec^* y$, i.e.,

$$V[G] \models \forall x, y\ [\psi(x, y, z, \vec{K}) \iff x \prec^* y],$$

and let $\dot{z}$ be a $\mathbb{P}$-name for z. Suppose that $p \in G$ forces this property of $\dot{z}$. Let γ be a regular cardinal above the supports of all conditions appearing in the name $\dot{z}$. Working in $V[G]$, let $A = \{g_\alpha \mid \alpha < \gamma\}$ denote the set of the first γ-many Cohen subsets of γ that were added by our forcing, and let $B = \{g_{\gamma+\alpha} \mid \alpha < \gamma\}$ be the second bunch. Let us assume without loss of generality that $A \prec^* B$. Note that we can choose canonical $\mathbb{P}$-names $\dot{A}$ and $\dot{B}$ for both A and B. Pick $q \leq p$ in G such that

$$q \Vdash \psi(\dot{A}, \dot{B}, \dot{z}, \dot{\vec{K}}).$$

Claim 6.4. *There is an automorphism $\pi \in \Delta_\gamma$ such that*

- $\pi(\dot{A}) = \dot{B}$, $\pi(\dot{B}) = \dot{A}$, $\pi(\dot{z}) = \dot{z}$, $\pi(\dot{\vec{K}}) = \dot{\vec{K}}$, *and*
- $\pi(q) \in G$.

Given such an automorphism, and applying it to the above forcing statement about q, we obtain that

$$\pi(q) \Vdash \psi(\dot{B}, \dot{A}, \dot{z}, \dot{\vec{K}}).$$

But since $\pi(q) \in G$, this means that $V[G] \models \psi(B, A, z, \vec{K})$, and hence that $B \prec^* A$, which is a contradiction. It remains to verify the above claim.

Proof of the Claim. We show that there is a dense set of conditions s below q for which there is an automorphism $\pi \in \Delta_\gamma$ with the first of the desired properties, and such that $s \leq \pi(q)$. Having done so, we simply pick some such condition $s \in G$. Then clearly, $\pi(q) \in G$, and thus the automorphism π corresponding to s will be as desired.

Thus, let $r \leq q$. We find $\bar{\gamma} < \gamma$ such that the information that r (and hence also q) has about the Cohen subsets of γ is *bounded* by $\bar{\gamma}$, that is r specifies at most the first $\bar{\gamma}$-many bits of each of the Cohen subsets of γ with indices below $\bar{\gamma}$ and of those with indices in the interval $[\gamma, \gamma + \bar{\gamma})$, and doesn't specify any other bits of the Cohen subsets of γ. We have to find an automorphism π with the properties specified in the first item above, and such that $\pi(q)$ is compatible to r, for we may then pick some s below both $\pi(q)$ and r, which thus will be as desired. But $\pi = \pi_\mu$ has all the desired properties when we pick $\mu = \mu^{\gamma}_{\bar\gamma}$: Clearly $\pi(\dot A) = \dot B$, $\pi(\dot B) = \dot A$, and we have already argued that $\pi(\vec K) = \vec K$ in Claim 6.3. Since π is nontrivial only at stage γ of our product, and by the choice of γ, it follows that $\pi(\dot z) = \dot z$. Finally, by the choice of $\bar\gamma$, we have $\pi(q)$ compatible to r. □

□

7. Preservation of *MOC*

In this final section, we want to provide an easy sample result showing that *MOC* is a reasonably robust set theoretic principle, for it is preserved by certain well-behaved set forcing notions. Note that *MOC* cannot be preserved by set forcing in general, for it implies the GCH (see Corollary 3.5).

Theorem 7.1. *For any infinite regular cardinal κ, MOC is preserved under $<\kappa$-closed, κ^+-cc forcing of size κ^+.*

Proof. Assume that *MOC* holds, as witnessed by $\vec K = \langle K_\alpha \mid \alpha \in \mathrm{Ord}\rangle$, and that $\mathbb{P}$ is a $<\kappa$-closed, κ^+-cc forcing notion of size κ^+. By passing to an isomorphic copy of $\mathbb{P}$, we may as well assume that $\mathbb{P} \subseteq H(\kappa^+)$. Let G be $\mathbb{P}$-generic. We want to find a minimal ordinal-connection hierarchy $\vec M = \langle M_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ for $V[G]$. Using that $\mathbb{P}$ is $<\kappa$-closed, we may simply let $M_\alpha = K_\alpha$ for $\alpha < \kappa$. For $\alpha \geq \kappa$, we let $M_\alpha = \{\dot x^G \mid \dot x \in K_\alpha\}$. Using Proposition 3.4, it suffices to show that $\vec M$ satisfies (H3) and (H4) with respect to $V[G]$. Property (H3) for $\vec M$ is immediate from Property (H3) for $\vec K$. Property (H4) for $\vec M$ follows easily as well, noting that by our assumptions on $\mathbb{P}$, forcing with $\mathbb{P}$ preserves all cardinals, and whenever $\lambda \geq \kappa$, every element of $H(\lambda^+)^{V[G]}$ has a $\mathbb{P}$-name in $H(\lambda^+)$. □

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