RAMSEY-LIKE OPERATORS

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ABSTRACT. We introduce and investigate a uniform framework for large cardinal operators. This framework accommodates the Ramsey operator, and we will show that it also accommodates the subtle operator, the ineffability operator, and the pre-Ramsey operator. We use this framework to introduce several new large cardinal operators, which are closely connected to established large cardinal notions, such as weakly Ramsey and strongly Ramsey cardinals. As a test application for our framework, we show that a strong form of the results of James Baumgartner connecting ineffability to subtlety, and Ramseyness to pre-Ramseyness, generalizes to many further large cardinals and their related operators.

1. Introduction

In the set theoretic literature, some popular large cardinals have been connected to corresponding large cardinal ideals, and then also to operators on ideals, the earliest examples of the latter being the ineffability operator $\mathcal{I}$ due to James Baumgartner in [4], followed by the Ramsey operator $\mathcal{R}$ and the pre-Ramsey operator $\mathcal{R}_0$ that were introduced and extensively studied by Qi Feng in [8], while the subtle operator $\mathcal{I}_0$ was first made explicit in a recent paper by Brent Cody [6]. In the present paper, inspired by the large cardinal framework based on the existence of certain ultrafilters for small models of set theory that was introduced in [12], we introduce such a framework for large cardinal operators. We show that the four operators mentioned above fit into this framework, we provide some general results about these operators, and we use this framework to introduce a number of new large cardinal operators, which are closely related to established notions of Ramsey-like cardinals, via an abstract notion of *Ramsey-like operator*. As a test application for our generalized operators, we show that one of the key results of Baumgartner [3, 4] about the ineffable and the Ramsey operator, connecting them to the subtle operator and to the pre-Ramsey operator respectively, holds for our generalized operators (in a strong form, which is due to Cody for the Ramsey operator in [6]). Finally, we make some comments and ask some questions related to the notion of weak ineffability.

We will always require all ideals to be ideals on some regular and uncountable cardinal κ, and to be supersets of the bounded ideal on κ. For any ideal I, I^+

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denotes the collection of I-positive sets, that is, those subsets of κ which are not in I, while I^* denotes the filter that is dual to I, that is, the collection of complements of sets in I. We will often introduce ideals by defining the collection of their positive sets when this is more convenient. An *ideal operator* $\mathcal{O}$ is a map that takes ideals on regular uncountable cardinals κ as input, and outputs another (not necessarily strictly) larger ideal on the same cardinal κ, so that $\mathcal{O}(I) \subseteq \mathcal{O}(J)$ whenever $I \subseteq J$ are ideals on κ.

Let us start by introducing the two classical examples of ideal operators, the ineffability operator $\mathcal{I}$ and the Ramsey operator $\mathcal{R}$. If ideal operators are closely connected to notions of large cardinals, as for example is the case for the operators $\mathcal{I}$ and $\mathcal{R}$ (see Fact 1.2 below), then we also refer to them as *large cardinal operators*. The definition of the Ramsey operator that is provided below is not the original definition from [8], but a version that was shown to be equivalent in [6, Proposition 2.8]. Recall that for any set A, an *A-list* is a sequence $\langle a_x \mid x \in A\rangle$ such that $a_x \subseteq x$ for any $x \in A$.

Definition 1.1. Let I be an ideal on κ.

- Given a κ-list $\vec{a}$, we first define what we call a *local instance*, letting
$$\mathcal{I}^{\vec{a}}(I)^+ = \{x \subseteq \kappa \mid \exists H \in I^+ \ H \subseteq x \text{ is homogeneous for } \vec{a}\},$$
and we let $\mathcal{I}(I)^+ = \bigcap\{\mathcal{I}^{\vec{a}}(I)^+ \mid \vec{a} \text{ is a } \kappa\text{-list}\}$.
- Given a regressive function $c\colon [\kappa]^{<\omega} \to \kappa$, we define a local instance, letting
$$\mathcal{R}^c(I)^+ = \{x \subseteq \kappa \mid \exists H \in I^+ \ H \subseteq x \text{ is homogeneous for } c\},$$
and we let $\mathcal{R}(I)^+ = \bigcap\{\mathcal{R}^c(I)^+ \mid c\colon [x]^{<\omega} \to \kappa \text{ regressive}\}$.

Fact 1.2.

(1) If κ is weakly ineffable, then $\mathcal{I}([\kappa]^{<\kappa})$ is the weakly ineffable ideal on κ.
(2) If κ is ineffable, then $\mathcal{I}(\mathrm{NS}_\kappa)$ is the ineffable ideal on κ.
(3) If κ is Ramsey, then $\mathcal{R}([\kappa]^{<\kappa})$ is the Ramsey ideal on κ.
(4) If κ is ineffably Ramsey, then $\mathcal{R}(\mathrm{NS}_\kappa)$ is the ineffably Ramsey ideal on κ.

Proof. (1) holds by the very definition of weak ineffability (with Baumgartner's terminology being *almost ineffable*) in [3]. (2) holds by the very definition of ineffability in [3]. (3) and (4) follow from [6, Proposition 2.8]. □

In [12], three schemes were proposed that allow for the characterization of a large number of large cardinals up to measurability in a uniform way. In the present paper, we will focus on one of these schemes, that was called *Scheme B*. We say that M is a *weak κ-model* if $M \supseteq \kappa+1$ is of size κ and a model of ZFC^-, that is ZFC without the powerset axiom. A collection $U \subseteq M \cap \mathcal{P}(\kappa)$ is an *M-ultrafilter on κ* if $\langle M, U\rangle \models$ "*U is an ultrafilter on κ*".

Scheme B: An uncountable cardinal κ has the large cardinal property $\Phi(\kappa)$ if and only if for any $y \subseteq \kappa$, there is a transitive weak κ-model M with $y \in M$, and a uniform M-ultrafilter U on κ for which $\Psi(M, U)$ holds.

Ineffability and Ramseyness are two instances of this scheme, which can also be used to characterize not only large cardinals, but certain large subsets A of large cardinals (as was also extensively done in [12]) by a minor adaption – namely, by additionally asking whether $A \in U$. Regarding ineffability and Ramseyness, this yields characterizations of ineffable and Ramsey sets respectively. Item (1) below extends a result of Abramson, Harrington, Kleinberg and Zwicker [1, Corollary 1.3.1], and is due to Philipp Lücke and the author [12, Theorem 8.1]. Item (2) below is essentially due to William Mitchell [16], and was isolated by Ian Sharpe and Philip Welch in [18, Theorem 3.3] (see also [6, Theorem 2.10]). Recall that for a weak κ-model M, an M-ultrafilter U on κ is *κ-amenable for M* if whenever $\mathcal{A}\in M$ is a κ-sized collection of subsets of κ in M, then $\mathcal{A}\cap U\in M$.

Theorem 1.3. *(1) (Holy, Lücke) $x\subseteq\kappa$ is ineffable if and only if for every $y\subseteq\kappa$ there is a transitive weak κ-model M with $y\in M$ and an M-ultrafilter U on κ with $x\in U$ such that ΔU is stationary.*

(2) (Mitchell; Sharpe, Welch) $x\subseteq\kappa$ is Ramsey if and only if for every $y\subseteq\kappa$ there is a transitive weak κ-model M with $y\in M$ and a countably complete M-ultrafilter U on κ with $x\in U$ that is κ-amenable for M. Equivalently, we can additionally require U to be M-normal.

The goal of the first half of this paper is to extend these characterizations even further, namely to the corresponding large cardinal operators that are the ineffability operator $\mathcal{I}$ and the Ramsey operator $\mathcal{R}$,[1] and to show that these characterizations naturally induce related characterizations of the subtle operator and of the pre-Ramsey operator.

2. The ineffability operator

In this section, we show that we can characterize the ineffability operator $\mathcal{I}$ via the existence of certain ultrafilters for small collections of sets (this characterization will be needed in Section 12), and then we show that for input values I containing the nonstationary ideal, we can characterize $\mathcal{I}$ also via the existence of certain ultrafilters for small models of set theory. As a reference to the notion of *flipping property* that was introduced in [1], if $\vec{x}=\langle x_\xi\mid\xi<\kappa\rangle$ is a sequence of subsets of κ, let us say that a set $U=\{u_\xi\mid\xi<\kappa\}$ *flips* $\vec{x}$ if $u_\xi\in\{x_\xi,\kappa\setminus x_\xi\}$ for every $\xi<\kappa$. Assume that we have fixed U to be such a *flip* of a sequence $\vec{x}$. Then, we write ΔU to abbreviate $\Delta_{\xi<\kappa}u_\xi$.

Definition 2.1. For any ideal I on κ and $\mathcal{C}\in[\mathcal{P}(\kappa)]^\kappa$, we first define a local instance of a *version of the ineffability operator for collections of sets*, letting

- $x\in\mathcal{I}^{\mathcal{C}}_{coll}(I)^+$ if $x\in\mathcal{P}(\kappa)\setminus\mathcal{C}$, or for any κ-enumeration $\vec{c}$ of $\mathcal{C}$, there is a set U that flips $\vec{c}$ such that $x\in U$ and $\Delta U\in I^+$, and we let
- $\mathcal{I}_{coll}(I)^+=\bigcap_{\mathcal{C}\in[\mathcal{P}(\kappa)]^\kappa}\mathcal{I}^{\mathcal{C}}_{coll}(I)^+$,

The following result extends [1, Theorem 1.2.1 and Corollary 1.3.1].

Proposition 2.2. *Let I be an ideal on κ. Then, $\mathcal{I}(I)=\mathcal{I}_{coll}(I)$.*

Proof. Assume first that $\vec{a}$ is a κ-list, and that $x\in\mathcal{I}_{coll}(I)^+$. Define a sequence $\vec{r}$ by setting $r_0=x$, and for every $\xi<\kappa$, $r_{1+\xi}=\{\alpha\in x\mid\xi\in a_\alpha\}$. Making use of our assumption, we may pick a set $U=\{u_\xi\mid\xi<\kappa\}$ that flips $\vec{r}$ such that $x=u_0\in U$ and $\Delta_{\xi<\kappa}u_\xi\in I^+$, and therefore also $H:=\Delta_{\xi<\kappa}u_\xi\setminus\omega\in I^+$. Fix $\alpha<\beta$ in H and $\xi<\alpha$. Then, since also $1+\xi<\alpha$, both α and β are elements of $u_{1+\xi}$. Thus, if $r_{1+\xi}=u_{1+\xi}\in U$, we have $\xi\in a_\alpha\cap a_\beta$. Otherwise, both α and β are elements

[1]For $\mathcal{R}$, this was in fact already done by Sharpe and Welch in [18, Theorem 3.3].

of $\kappa \setminus r_{1+\xi}$, and hence $\xi \notin a_\alpha \cup a_\beta$. Together, this shows that $a_\alpha = a_\beta \cap \alpha$, and therefore that $H \in I^+$ is homogeneous for $\vec{a}$. Since $H \subseteq x$, we have $x \in \mathcal{I}^{\vec{a}}(I)^+$, as desired.

For the other direction, we assume that $x \in \mathcal{I}(I)^+$, and let $x \in \mathcal{C} \in [\mathcal{P}(\kappa)]^\kappa$. Pick an enumeration $\langle c_\xi \mid \xi < \kappa\rangle$ of $\mathcal{C}$, and let $\vec{a}$ be defined by setting, for every $\alpha \in x$, $a_\alpha = \{\xi < \alpha \mid \alpha \in c_\xi\}$. By our assumption, there is $H \subseteq x$ in I^+ that is homogeneous for $\vec{a}$. We may thus pick $A \subseteq \kappa$ such that $a_\alpha = A \cap \alpha$ for every $\alpha \in H$. Given $\xi < \kappa$, let $u_\xi = c_\xi$ if $\xi \in A$, and let $u_\xi = \kappa \setminus c_\xi$ otherwise. Let $U = \{u_\xi \mid \xi < \kappa\}$. $x \in \mathcal{I}_{coll}(I)^+$ is now a consequence of the following claim.

Claim 2.3. *$x \in U$, and $\Delta_{\xi<\kappa} u_\xi \in I^+$.*

Proof. We have $H \setminus (\xi+1) \subseteq c_\xi$ for all $\xi \in A$, and $H \cap c_\xi \subseteq \xi+1$ for $\xi \in \kappa \setminus A$. Hence, $H \setminus (\xi+1) \subseteq u_\xi$ for all $\xi < \kappa$, yielding that $x \in U$ and $H \subseteq \Delta_{\xi<\kappa} u_\xi \in I^+$. □

□

Definition 2.4. We define the *model version of the ineffability operator* as follows. For any $y \subseteq \kappa$ and any ideal I on κ, we first define a local instance, letting

- $x \in \mathcal{I}^y_{mod}(I)^+$ if there is a transitive weak κ-model M with $y \in M$, and an M-ultrafilter U on κ with $x \in U$, such that every diagonal intersection of U is in I^+ – we abbreviate this latter property of U and I by stating that $\Delta U \in I^+$.[2]
- Let $\mathcal{I}_{mod}(I)^+ = \bigcap_{y \subseteq \kappa} \mathcal{I}^y_{mod}(I)^+$.

The following result extends [12, Theorem 8.1].

Proposition 2.5. *If $\vec{a}$ is a κ-list, I is an ideal on κ, and $y \subseteq \kappa$ codes $\vec{a}$, then $\mathcal{I}^y_{mod}(I) \supseteq \mathcal{I}^{\vec{a}}(I)$, hence $\mathcal{I}_{mod}(I) \supseteq \mathcal{I}(I)$. If $I \supseteq \mathrm{NS}_\kappa$, then also $\mathcal{I}_{mod}(I) \subseteq \mathcal{I}(I)$. In particular thus, if $I \supseteq \mathrm{NS}_\kappa$, then $\mathcal{I}_{mod}(I) = \mathcal{I}(I)$.*

Proof. Let $\vec{a}$ be a κ-list, let $y \subseteq \kappa$ code $\vec{a}$, and let $x \in \mathcal{I}^y_{mod}(I)^+$. We may pick a transitive weak κ-model M witnessing that $x \in \mathcal{I}^y_{mod}(I)^+$, and an M-ultrafilter U on κ such that $x \in U$ and $\Delta U \in I^+$. Define, for every $\xi < \kappa$, $r_\xi = \{\alpha \in x \mid \xi \in a_\alpha\}$. Let $u_\xi = r_\xi$ if $r_\xi \in U$, and let $u_\xi = \kappa \setminus r_\xi$ otherwise, for every $\xi < \kappa$. Then, also $H := x \cap \Delta_{\xi<\kappa} u_\xi \in I^+$. Fix $\alpha < \beta$ in H and $\xi < \alpha$. Then, both α and β are elements of u_ξ. If $r_\xi = u_\xi \in U$, then $\xi \in a_\alpha \cap a_\beta$. Otherwise, both α and β are elements of $\kappa \setminus r_\xi$, and hence $\xi \notin a_\alpha \cup a_\beta$. Together, this shows that $a_\alpha = a_\beta \cap \alpha$, and therefore that $H \in I^+$ is homogeneous for $\vec{a}$. Since $H \subseteq x$, we have $x \in \mathcal{I}^{\vec{a}}(I)^+$, as desired.

Now assume $I \supseteq \mathrm{NS}_\kappa$. Let $y \subseteq \kappa$ and $x \in \mathcal{I}(I)^+$. Let M be a transitive weak κ-model such that $x, y \in M$. Pick an enumeration $\vec{x} = \langle x_\xi \mid \xi < \kappa\rangle$ of all subsets of κ in M. Using Proposition 2.2, we obtain a flip U of $\vec{x}$ such that $x \in U$ and $\Delta U \in I^+$. In the light of the comments made in Footnote 2, it suffices to observe that U being an M-ultrafilter on κ easily follows from $\Delta U \in I^+ \subseteq \mathrm{NS}^+_\kappa$. □

[2] The meaning of $\Delta U \in I^+$ will thus depend on whether we have fixed an enumeration of U in the background, by having chosen U to flip a certain sequence. This should however not lead to any confusion. It is well-known that permuting the input of a diagonal intersection only changes its output by a non-stationary set (see [1, Lemma 1.3.3]). Hence, if $I \supseteq \mathrm{NS}_\kappa$, rather than requiring that every diagonal intersection of U be in I^+, it equivalently suffices to require one (arbitrary) diagonal intersection of U to be in I^+.

In particular, if κ is ineffable, then $\mathcal{I}_{mod}(\mathrm{NS}_\kappa) = \mathcal{I}_{coll}(\mathrm{NS}_\kappa) = \mathcal{I}(\mathrm{NS}_\kappa)$ is the ineffable ideal on κ. It seems to be open whether $\mathcal{I}_{mod}([\kappa]^{<\kappa})$ is the weakly ineffable ideal on κ. We will briefly discuss this together with some related open questions in Section 15.

3. A brief review of a notion of transfinite indescribability

Joan Bagaria [2] introduced a natural notion of Π^1_ξ formula for arbitrary ordinals ξ, that extends the hierarchy of Π^1_n-formulae for $n < \omega$. We will make use of this notion several times in the remainder of this paper, and we would like to shortly recall Bagaria's definitions in this section.

Definition 3.1 (Bagaria)**.** A formula is said to be $\Sigma^1_{\xi+1}$ if it is of the form $\exists X_0, \ldots, X_k\ \varphi(X_0, \ldots, X_k)$ for some Π^1_ξ-formula φ, and it is $\Pi^1_{\xi+1}$ if it is of the form $\forall X_0, \ldots, X_k\ \varphi(X_0, \ldots, X_k)$ for some Σ^1_ξ-formula φ, where all quantifiers displayed above are understood to be second order quantifiers.

If ξ is a limit ordinal, we say that a formula is Π^1_ξ if it is a conjunction of the form $\bigwedge_{\zeta<\xi} \varphi_\zeta$, where each φ_ζ is a Π^1_ζ-formula, and the infinite conjunction has only finitely-many free variables. It is Σ^1_ξ if it is a disjunction of the form $\bigvee_{\zeta<\xi} \varphi_\zeta$, where each φ_ζ is a Σ^1_ζ-formula, and the infinite disjunction has only finitely-many free variables.

A corresponding notion of Π^1_ξ-indescribability has been introduced by Bagaria, and independently, an equivalent notion had been introduced by Sharpe and Welch in [18, Definition 3.21].

Definition 3.2 (Bagaria)**.** Suppose that κ is a regular cardinal, and that $\xi < \kappa$ is an ordinal. A set $A \subseteq \kappa$ is Π^1_ξ-indescribable if for all $y \subseteq V_\kappa$ and every Π^1_ξ-sentence φ, if $\langle V_\kappa, \in, y\rangle \models \varphi$, then there is $\alpha \in A$ such that $\langle V_\alpha, \in, y \cap V_\alpha\rangle \models \varphi$.

We let $\Pi^1_\xi(\kappa)$ denote the Π^1_ξ-indescribable ideal – the collection of subsets of κ that are not Π^1_ξ-indescribable. We additionally let $\Pi^1_{-1}(\kappa) = [\kappa]^{<\kappa}$, and remark that $\Pi^1_0(\kappa) = \mathrm{NS}_\kappa$. The following will be of relevance in Section 14.

Lemma 3.3. [2, Proposition 4.4] *The statement $A \in \Pi^1_\beta(\kappa)^+$ is expressible as a $\Pi^1_{\beta+1}$-property of A over V_κ.*

We will want to make use of the following result of Cody, that generalizes a classical result of Baumgartner from [3].

Lemma 3.4. [6, Lemma 2.20] *If $A \subseteq \kappa$, $\beta < \kappa$, and every A-list has a homogeneous set in $Q \subseteq \bigcap_{\xi\in\{-1\}\cup\beta} \Pi^1_\xi(\kappa)^+$, then A is $\Pi^1_{\beta+1}$-indescribable.*

The following minor generalization of folklore results will be of relevance in combination with Lemma 3.4.

Lemma 3.5. *Assume that M is a weak κ-model, U is an M-ultrafilter on κ that contains all club subsets of κ in M, is κ-amenable for M, and $A \in U$. Then, every A-list in M has a homogeneous set in U.*

Proof. In [10, Lemma 3.6], Gitman shows that under the assumptions of our lemma, if $f\colon [A]^2 \to 2$ is in M, then there is $H \subseteq A$ in U such that H is homogeneous for f.[3] Now one can use an easy adaption of Kunen's argument that ineffability can

[3] In fact, Gitman makes the additional assumption that the ultrapower of M by U is well-founded, however this assumption is never made use of in her proof.

be characterized either in terms of the existence of homogeneous sets for colourings or for lists [15, Theorem 4]: Given an A-list $\vec{a}$, this argument allows us to obtain a function $f\colon [A]^2 \to 2$ that is definable from $\vec{a}$ (and hence an element of M) such that whenever $H \subseteq A$ is homogeneous for f, one can find a club subset C of κ that is definable from $\vec{a}$ and H (and thus an element of M), such that $H \cap C$ is homogeneous for $\vec{a}$.[4] Note that since $C \in U$, we also have $H \cap C \in U$. □

4. The iterated ineffability operator

Iterated operators are defined in a natural way: For any ideal I and ideal operator $\mathcal{O}$, let $\mathcal{O}^0(I) = I$, let $\mathcal{O}^{\alpha+1}(I) = \mathcal{O}(\mathcal{O}^\alpha(I))$ for any ordinal α, and let $\mathcal{O}^\alpha(I) = \bigcup_{\beta<\alpha} \mathcal{O}^\beta(I)$ in case α is a limit ordinal.

In [5, Remark 3.24], it is mentioned that there is a refinement of the ineffability hierarchy via indescribability that is highly analogous to the refinement of the Ramsey hierarchy via indescribability that is studied in detail in [6]. We will need a result on the ineffability hierarchy (namely, Lemma 4.2 below) later on in our paper, and for the benifit of our readers, we would like to provide a proof of this result.[5] Given his comments in [5], Cody was certainly aware of the possibility of this easy adaption of his material from [6], and hence the results in this section should be credited to him. They are the adaptions of [6, Lemma 3.1 and Lemma 3.2] to the ineffability operator.

Lemma 4.1 (Cody)**.** *Let $\alpha < \kappa$ and $\beta \in \{-1\} \cup \kappa$. Suppose $S \in \mathcal{I}^\alpha(\Pi^1_\beta(\kappa))^+$, and for each $\xi \in S$, let $S_\xi \in \mathcal{I}^\alpha(\Pi^1_\beta(\xi))^+$. Then, $\bigcup_{\xi \in S} S_\xi \in \mathcal{I}^\alpha(\Pi^1_\beta(\kappa))^+$.*

Proof. By induction on α. If $\alpha = 0$ or α is a limit ordinal, the argument is exactly as in [6, Lemma 3.1]. If α is a successor ordinal, fix a $\bigcup_{\xi \in S} S_\xi$-list $\vec{a}$. For each $\xi \in S$, there is some $H_\xi \subseteq S_\xi$ in $\mathcal{I}^{\alpha-1}(\Pi^1_\beta(\xi))^+$ that is homogeneous for $\vec{a}\restriction S_\xi$. Since $S \in \mathcal{I}^\alpha(\Pi^1_\beta(\kappa))^+$, there is a homogeneous set $H \subseteq S$ in $\mathcal{I}^{\alpha-1}(\Pi^1_\beta(\kappa))^+$ for the S-list $\langle H_\xi \mid \xi \in S\rangle$. By our inductive hypothesis, $\bigcup_{\xi \in H} H_\xi \in \mathcal{I}^{\alpha-1}(\Pi^1_\beta(\kappa))^+$, but clearly, $\bigcup_{\xi \in H} H_\xi$ is homogeneous for $\vec{a}$, and we are done. □

Lemma 4.2 (Cody)**.** *If $\kappa \in \mathcal{I}^\alpha(\Pi^1_\beta(\kappa))^+$, $\alpha < \kappa$ and $\beta \in \{-1\} \cup \kappa$, then*

$$S = \{\xi < \kappa \mid \xi \in \mathcal{I}^\alpha(\Pi^1_\beta(\xi))\} \in \mathcal{I}^\alpha(\Pi^1_\beta(\kappa))^+.$$

Proof. Assume that κ is the least counterexample to the statement of the lemma – that is, for some fixed α and β, κ is least such that $\kappa \in \mathcal{I}^\alpha(\Pi^1_\beta(\kappa))^+$, while $S = \{\xi < \kappa \mid \xi \in \mathcal{I}^\alpha(\Pi^1_\beta(\xi))\} \in \mathcal{I}^\alpha(\Pi^1_\beta(\kappa))$. Then, $\kappa \setminus S \in \mathcal{I}^\alpha(\Pi^1_\beta(\kappa))^+$. For each $\zeta \in \kappa \setminus S$, by the minimality of κ, $S \cap \zeta \in \mathcal{I}^\alpha(\Pi^1_\beta(\zeta))^+$. Thus, by Lemma 4.1, $S = \bigcup_{\zeta \in \kappa \setminus S} S \cap \zeta \in \mathcal{I}^\alpha(\Pi^1_\beta(\kappa))^+$, contradicting our assumption on κ, as desired. □

5. The Ramsey operator

In this section, we want to present an argument showing that one can also characterize the Ramsey operator via the existence of certain ultrafilters for small models of set theory. This characterization is due to Sharpe and Welch [18, Theorem

[4] This argument can also be found within the proof of [12, Lemma 11.3].

[5] In fact, we only need the below result for the ineffability operator rather than its iterations, however since treating also its iterations provides almost no additional effort, we would like to provide this more general result.

3.3], however we would like to present a somewhat different proof based on the presentation of the proof of a somewhat less general result from [9], and we will need to take a closer look at some of these arguments in order to be able to adapt them later on in Section 9. The operator that we introduce below is implicit in the statement of [18, Theorem 3.3], and we want to call it the model version of the Ramsey operator.

Definition 5.1. For any ideal I on κ, and $y \subseteq \kappa$, we first define a local instance, letting

- $x \in \mathcal{R}^y_{mod}(I)^+$ if there is a transitive weak κ-model M with $y \in M$, and an M-normal M-ultrafilter U on κ with $x \in U$ that is κ-amenable for M, such that every countable intersection of elements of U is in I^+, and we let
- $\mathcal{R}_{mod}(I)^+ = \bigcap_{y \subseteq \kappa} \mathcal{R}^y_{mod}(I)^+$.

The goal of this section will be to present an argument showing that the operators $\mathcal{R}$ and $\mathcal{R}_{mod}$ are equal to each other. The first direction is an easy generalization of well-known results (see for example [10, Theorem 3.10]).

Lemma 5.2. *If $c\colon [\kappa]^{<\omega} \to \kappa$ is a regressive function, I is an ideal on κ, and $y \subseteq \kappa$ codes c, then $\mathcal{R}^y_{mod}(I) \supseteq \mathcal{R}^c(I)$. In particular, $\mathcal{R}_{mod}(I) \supseteq \mathcal{R}(I)$.*

Proof. Assume that $x \in \mathcal{R}^y_{mod}(I)^+$, and let $c\colon [\kappa]^{<\omega} \to \kappa$ be a regressive function that is coded by $y \subseteq \kappa$. Pick a transitive weak κ-model M with $y \in M$, and an M-ultrafilter U on κ witnessing that $x \in \mathcal{R}^y_{mod}(I)^+$. Using that $c \in M$, and following a line of well-known arguments, as for example in the proof of [6, Theorem 2.10], for every $n \in \omega$, we find a set $H_n \in U$ that is homogeneous for $c \restriction [x]^n$. But then, by the properties of U, we have $H := \bigcap_{n \in \omega} H_n \in I^+$ homogeneous for c, showing that $x \in \mathcal{R}(I)^+$. □

The other direction will be substantially more work, and we will need some preparatory results first. Let us start by recalling a standard definition.

Definition 5.3. Suppose κ is a cardinal and $\mathcal{A} = \langle L_\kappa[A], A\rangle$ with $A \subseteq \kappa$. Then, $J \subseteq \kappa$ is a set of *good indiscernibles* for $\mathcal{A}$ if for all $\gamma \in J$, the following hold.

- $\langle L_\gamma[A], A\rangle \prec \langle L_\kappa[A], A\rangle$,
- γ is a cardinal, and
- $J \setminus \gamma$ is a set of indiscernibles for $\langle L_\kappa[A], A, \xi\rangle_{\xi<\gamma}$.[6]

We will rely on the following. A proof of Item Lemma 5.4(1) below can be found within the proof of [9, Lemma 2.43],[7] and Lemma 5.4(2) is obvious from the details provided in that proof as well. The same argument is essentially contained in the proof of [18, Lemma 2.9]. In the present section, we will only need Lemma 5.4(1), but Lemma 5.4(2) will be of good use in Section 9 later on.

Lemma 5.4. *Let κ be an inaccessible cardinal, and let $A \subseteq \kappa$.*

(1) There is a club $C \subseteq \kappa$ and a regressive function $h\colon [C]^{<\omega} \to \kappa$ such that any κ-sized homogeneous set for h is a set of good indiscernibles for $\langle L_\kappa[A], A\rangle$.[8]

[6] That is, for all $n < \omega$, all increasing n-sequences from $J \setminus \gamma$ satisfy the same first order formulas over $L_\kappa[A]$ using ordinals $\xi < \gamma$ as parameters and using A as a second order predicate.

[7] In her Lemma 2.43, Gitman assumes that κ is a Ramsey cardinal, and thus that certain homogeneous sets for colourings do exist. But the assumption of Ramseyness is otherwise not needed, and her proof is easily seen to verify the below lemma.

[8] In fact, it would suffice to require the homogeneous set to be of limit order type.

(2) For any inaccessible $\alpha \in C$, $C \cap \alpha$ and $h \upharpoonright \alpha$ also have the above properties with respect to $A \cap \alpha$.

Proof. Since (2) is not mentioned anywhere in the literature, we would like to present the definition of C and h given κ and A, following the proof of [9, Lemma 2.43], and then observe that this relationship is preserved under restrictions to inaccessible elements of C, thus yielding (2). For the complete proof of (1), the interested reader should consult [9].

In her proof of [9, Lemma 2.43], Gitman makes use of an arbitrary bijection $f\colon \kappa\times\kappa \to \kappa\setminus\{\emptyset\}$, and for the sake of simplicity, we may take f to be defined using Gödel pairing, by setting $f(\langle\alpha,\beta\rangle) = 1 + \prec\alpha,\beta\succ$, yielding in particular that every cardinal is closed under f. The club C may then simply be defined as the set of all uncountable cardinals $\alpha < \kappa$ for which $\langle L_\alpha[A], A\rangle$ is an elementary substructure of $\langle L_\kappa[A], A\rangle$. We use f to define a coding function $g\colon [\kappa]^{<\omega} \to \kappa$ in the following way. We let $g\upharpoonright\kappa$ be the identity on κ. Given $g\upharpoonright[\kappa]^{<n}$ for some $n < \omega$ with $n > 1$, and given $\vec\alpha = \langle\alpha_0,\ldots,\alpha_{n-1}\rangle \in [\kappa]^n$, let $g(\vec\alpha) = f(\alpha_0, g(\langle\alpha_1,\ldots,\alpha_{n-1}\rangle))$.

Next, we fix an enumeration $\langle\varphi_m \mid m \in \omega\rangle$ of all formulas in the first order $\in$-language using the predicate A, and consider the following condition (*) on ordered tuples $\vec\alpha = \langle\alpha_1,\ldots,\alpha_{2n}\rangle$ of length $2n$ for some $n \in \omega$ of elements of κ:

(*) $\exists\delta_1 < \ldots < \delta_k < \alpha_1$ and $m \in \omega$ such that:

$$\langle L_\kappa[A], A\rangle \not\models \varphi_m(\vec\delta, \alpha_1,\ldots,\alpha_n) \leftrightarrow \varphi_m(\vec\delta, \alpha_{n+1},\ldots,\alpha_{2n}).$$

If $\vec\alpha$ satisfies (*), let $w(\vec\alpha)$ be the least $\lambda = g(m,\vec\delta)$ so that m and $\vec\delta$ witness (*) for $\vec\alpha$, and let $w(\vec\alpha) = \emptyset$ otherwise. Now we define $h\colon [C]^{<\omega} \to \kappa$ by setting $h(\vec\alpha) = w(\vec\alpha)$ if $\vec\alpha$ is of even length, and setting $h(\vec\alpha) = \emptyset$ for $\vec\alpha$ of odd length.

The remainder of the argument for (1), namely that C and h have the desired properties stated there, is fairly straightforward, and proceeds by showing that h has to take value 0 on any κ-sized homogeneous set for h, since any other value quickly leads to a contradiction. The interested reader may find the remaining details for this argument in [9].

Let us observe that (2) holds true: First note that any inaccessible cardinal $\alpha \in C$ is a limit point of C, and therefore $C \cap \alpha$ is a club subset of α. But now, using that $\langle L_\alpha[A], A\rangle \prec \langle L_\kappa[A], A\rangle$, it clearly follows that $w\upharpoonright[\alpha]^{<\omega}$ as obtained above is the same as the function $\bar w$ that we would have obtained starting with α and $A \cap \alpha$ rather than with κ and with A. But then, it is immediate that if we restrict our function h to $[\alpha]^{<\omega}$, then this is the same as the function $\bar h$ that we would obtain from $\bar w$, thus yielding the statement of (2). □

We also need the following characterization of the Ramsey operator (this is actually the original definition of the Ramsey operator in [8]):

Lemma 5.5. [6, Proposition 2.8] *For any ideal I, $\mathcal{R}(I)^+ = \{x \subseteq \kappa \mid \forall c\colon [x]^{<\omega} \to \kappa$ regressive $\forall C \subseteq \kappa$ club $\exists H \in I^+\ H \subseteq x \cap C$ is homogeneous for $c\}$.*

We are now ready to proceed with the main argument of this section, following the basic line of argument of [6, Theorem 2.10], which relies mostly on the arguments from [10, Section 4].

Theorem 5.6. *Let κ be an inaccessible cardinal. For any ideal I on κ,*

$$\mathcal{R}_{mod}(I) = \mathcal{R}(I).$$

Proof. Having shown Lemma 5.2, it only remains to show that $\mathcal{R}(I) \supseteq \mathcal{R}_{mod}(I)$. Assume thus that $x \in \mathcal{R}(I)^+$, and that $y \subseteq \kappa$, and let $A \subseteq \kappa$ code both x and y. Making use of Lemma 5.4, let C be a club subset of κ and let $h\colon [C]^{<\omega} \to \kappa$ be a regressive function such that any κ-sized homogeneous set for h is a set of good indiscernibles for $\mathcal{A} := \langle L_\kappa[A], A\rangle$. Making use of Lemma 5.5, let $H \subseteq x \cap C$ be homogeneous for h, with $H \in I^+$. Hence, H is a set of good indiscernibles for $\mathcal{A}$.

Now we proceed almost exactly as in [10, Section 4], and construct M and U witnessing that $x \in \mathcal{R}_{mod}(I)^+$. There are only three differences to the argument required:

(1) Our homogeneous set H is in I^+ rather than just unbounded in κ, but this simply carries through the argument without requiring any modification.
(2) We need to show that $x \in U$, but this will be an easy consequence of having chosen $H \subseteq x$.
(3) We asked for U to be M-normal, which was omitted in [10].[9] We have to show that the M-ultrafilter U that is constructed through the arguments of [10, Section 4] is actually already M-normal.

For the convenience of our readers, we would like to present a mostly self-contained outline of the proof, only refering to [10] for some short intermediate results. Another reason for providing this presentation is that in [10], the justification for one of the main points of the argument, namely that the final filter U is κ-amenable for M, is missing – Gitman only declares this to be *easy*, which to us, after figuring out the actual argument following a helpful conversation with Gitman, does perhaps not seem quite justified, for it seems to be one of the more intricate parts of the proof. Using quite different notation, slightly more detail concerning this argument is provided in [18], however some points also seem to only be touched there somewhat briefly, in particular the analogue of Lemma 5.12 below seems to be missing. Finally, and perhaps most importantly, we will need to refer to the proof of the present theorem in some detail within the proof of Theorem 9.1 later on, so it will be very convenient for the reader to at least have its essential structure available here for reference. Let us thus continue with the argument, which will closely follow the line of argument in (and also the notation from) [10, Section 4] for its most parts.

For every $\gamma \in H$ and $n \in \omega$, let $\vec{\gamma}_n$ denote the increasing sequence $\langle \gamma_1, \ldots, \gamma_n\rangle$ of the first n elements of H (strictly) above γ, and let $\tilde{\mathcal{M}}^n_\gamma = \langle \tilde{M}^n_\gamma, A \cap \tilde{M}^n_\gamma\rangle$ be the Skolem closure of $(\gamma+1) \cup \vec{\gamma}_n$ in $\mathcal{A}$, using the definable Skolem functions of $\mathcal{A}$. Since κ is inaccessible, $L_\kappa[A] \models \mathrm{ZFC}$, and hence in $L_\kappa[A]$, for every λ, $H(\lambda)$ exists and is a model of ZFC^-. Since $\tilde{\mathcal{M}}^n_\gamma \prec \mathcal{A}$ and $\gamma \in \tilde{M}^n_\gamma$, we have $H(\gamma^+)^{\mathcal{A}} \in \tilde{M}^n_\gamma$. Let $M^n_\gamma = \tilde{M}^n_\gamma \cap H(\gamma^+)^{\mathcal{A}}$, and let $\mathcal{M}^n_\gamma = \langle M^n_\gamma, A \cap M^n_\gamma\rangle$.

Lemma 5.7. [10, Lemma 4.2.1] *Each M^n_γ is a transitive model of* ZFC^-.

Lemma 5.8. [10, Lemma 4.2.2] *For every $\gamma \in H$ and $n \in \omega$, $\mathcal{M}^n_\gamma \prec \mathcal{M}^{n+1}_\gamma$.*

If $a \in \tilde{M}^n_\gamma$, then $a = S(\xi_0, \ldots, \xi_m, \gamma, \vec{\gamma}_n)$, where S is a definable Skolem function of $\mathcal{A}$, and $\xi_i \in \gamma$ for $i \le m$. Given $\gamma < \delta$ in H and $n < \omega$, define $\tilde{f}^n_{\gamma\delta}\colon \tilde{M}^n_\gamma \to \tilde{M}^n_\delta$ by setting $\tilde{f}^n_{\gamma\delta}(a) = S(\xi_0, \ldots, \xi_m, \delta, \vec{\delta}_n)$ in case $a = S(\xi_0, \ldots, \xi_m, \gamma, \vec{\gamma}_n)$. Using that

[9] We could have omitted the requirement of M-normality of U as well in our definition of $\mathcal{R}_{mod}(I)$, and this would yield yet another (known) characterization of the Ramsey operator.

$H \setminus \gamma$ is a set of indiscernibles for $\langle L_\kappa[A], A, \xi\rangle_{\xi\in\gamma}$, and that $\tilde{\mathcal{M}}^n_\gamma$ and $\tilde{\mathcal{M}}^n_\delta$ are both elementary substructures of $\mathcal{A}$, it easily follows that $\tilde{f}^n_{\gamma\delta}\colon \tilde{\mathcal{M}}^n_\gamma \to \tilde{\mathcal{M}}^n_\delta$ is a well-defined elementary embedding. Since $\tilde{f}^n_{\gamma\delta}(\xi) = \xi$ for all $\xi < \gamma$ and $\tilde{f}^n_{\gamma\delta}(\gamma) = \delta$, γ is the critical point of $\tilde{f}^n_{\gamma\delta}$. Moreover, different such embeddings commute, that is if $\gamma < \delta < \epsilon$ are in H, then $\tilde{f}_{\gamma\epsilon} = \tilde{f}_{\delta\epsilon} \circ \tilde{f}_{\gamma\delta}$.

Lemma 5.9. [10, Lemma 4.2.3] *For $\gamma < \delta$ in H and $n < \omega$, $f^n_{\gamma\delta} := \tilde{f}^n_{\gamma\delta}\restriction M^n_\gamma\colon \mathcal{M}^n_\gamma \to \mathcal{M}^n_\delta$ is an elementary embedding.*

For $\gamma \in H$, define $U^n_\gamma = \{X \in \mathcal{P}(\gamma)^{M^n_\gamma} \mid \gamma \in f^n_{\gamma\delta}(X) \text{ for some } \delta > \gamma \text{ in } H\}$. Equivalently, we could have used "for all $\delta > \gamma$" in this definition. The only nontrivial observation in the next lemma is that $U^n_\gamma \in M^{n+2}_\gamma$.

Lemma 5.10. [10, Lemma 4.2.5] *For any $\gamma \in H$ and $n < \omega$, $U^n_\gamma \in M^{n+2}_\gamma$ is an M^n_γ-normal M^n_γ-ultrafilter on γ.*

It is easy to check that for any $\gamma < \delta$ in H and $n < \omega$, $f^n_{\gamma\delta} = f^{n+1}_{\gamma\delta}\restriction M^n_\gamma$ and $U^n_\gamma = U^{n+1}_\gamma \cap M^n_\gamma$. Thus, let $M_\gamma = \bigcup_{n\in\omega} M^n_\gamma$, $U_\gamma = \bigcup_{n\in\omega} U^n_\gamma$ and $f_{\gamma\delta} = \bigcup_{n\in\omega} f^n_{\gamma\delta}$. Let $\mathcal{M}_\gamma = \langle M_\gamma, A \cap M_\gamma\rangle$. Using elementarity, it follows that M_γ is a transitive model of ZFC$^-$, and it is easy to check that $f_{\gamma\delta}$ is an elementary embedding from $\mathcal{M}_\gamma$ to $\mathcal{M}_\delta$ mapping its critical point γ to δ, and that U_γ is an M_γ-normal M_γ-ultrafilter on γ.

Lemma 5.11. [10, Lemma 4.2.6] *U_γ is γ-amenable for M_γ.*

We will need the following, which is not mentioned in [10], in order to be able to verify our final filter U to be κ-amenable for M:

Lemma 5.12. *For $\gamma < \delta$ in H and $n < \omega$, $f_{\gamma\delta}(U^n_\gamma) = U^n_\delta$.*

Proof. Fix $n < \omega$. By the argument for [10, Lemma 4.2.5], there is a first order formula φ, such that for any $\gamma \in H$, U^n_γ is definable in $\tilde{M}^{n+2}_\gamma$ using the formula φ and using γ and the first $n+1$ elements $\vec{\gamma}_{n+1}$ of H above γ as parameters. This implies that $\tilde{f}^{n+2}_{\gamma\delta}(U^n_\gamma) = U^n_\delta$. Since $U^n_\gamma \in M_\gamma$ by Lemma 5.10, this implies our desired statement. □

Now, for every $\gamma \in H$, consider the structure $\langle M_\gamma, \in, A \cap M_\gamma, U_\gamma\rangle$ which extends $\mathcal{M}_\gamma$ by the predicate for the M_γ-ultrafilter U_γ on γ. If $\gamma < \delta$ are in H, we have an elementary embedding $f_{\gamma\delta}\colon \mathcal{M}_\gamma \to \mathcal{M}_\delta$ with critical point γ, such that also $X \in U_\gamma \iff f_{\gamma\delta}(X) \in U_\delta$. This is thus a directed system of embeddings between these structures, and we let $\langle B, E, A', W\rangle$ be its direct limit. Elements of B are functions t with domains $\{\xi \in H \mid \xi \geq \alpha\}$ for some $\alpha \in H$ satisfying that

(1) for $\gamma \in \operatorname{dom} t$, $t(\gamma) \in M_\gamma$,
(2) for $\gamma < \delta \in \operatorname{dom} t$, $t(\delta) = f_{\gamma\delta}(t(\gamma))$, and
(3) there is no $\xi \in H \cap \alpha$ for which there is $a \in M_\xi$ such that $f_{\xi\alpha}(a) = t(\alpha)$.

Note that any $t \in B$ is determined once $t(\xi)$ is known for any $\xi \in \operatorname{dom} t$.

Lemma 5.13. [10, Lemma 4.2.7] *The relation E on B is well-founded.*

We may therefore let $\langle M, \in, A^*, U\rangle$ be the Mostowski collapse of $\langle B, E, A', W\rangle$.

Lemma 5.14. [10, Lemma 4.2.8] *$\kappa \in M$.*

For any $\gamma \in H$, let $j_\gamma \colon M_\gamma \to M$ be defined such that for any $a \in M_\gamma$, $j_\gamma(a)$ is the collapse of the (unique) function $t \in B$ for which $t(\gamma) = a$. Then, j_γ is easily seen to be an elementary embedding of $\mathcal{M}_\gamma$ into $\langle M, \in, A^*\rangle$, and to also be elementary for atomic formulas in the language with the predicate for the ultrafilter. Observe that $j_\gamma(\xi) = \xi$ for all $\xi < \gamma$, $j_\gamma(\gamma) = \kappa$, and hence that $\mathrm{crit}(j_\gamma) = \gamma$. Moreover, if $\gamma < \delta$ are elements of H, then $j_\delta \circ f_{\gamma\delta} = j_\gamma$.

The proof of the following lemma is not contained in [10]: M-normality had not been considered, and the verification of κ-amenability is somewhat strangely missing there.

Lemma 5.15. *U is an M-normal M-ultrafilter on κ that is κ-amenable for M.*

Proof. It is easy to check that U is an M-ultrafilter on κ. For the M-normality of U, we show that every regressive function f on a set $x \in U$ is constant on a set in U. Let $\gamma \in H$ be such that there are g and y in M_γ with $f_\gamma(g) = f$ and $f_\gamma(y) = x$. By elementarity for atomic formulas using the predicate for the ultrafilter, $y \in U_\gamma$. By the M_γ-normality of U_γ, g is constant on a set $h \in U_\gamma$. It thus follows that f is constant on the set $f_\gamma(h) \in U$.

For the κ-amenability of U, let $\vec{x}$ be a κ-sequence of elements of $\mathcal{P}(\kappa)$ in M. Let $\gamma \in H$ be such that there is $\vec{a}$ in M_γ for which $j_\gamma(\vec{a}) = \vec{x}$. Using that $M_\gamma = \bigcup_{n<\omega} M_\gamma^n$, we may fix $n < \omega$ for which $\vec{a} \in M_\gamma^n$. Then, by Lemma 5.10,

$$b = \{\alpha < \gamma \mid a_\alpha \in U_\gamma\} = \{\alpha < \gamma \mid a_\alpha \in U_\gamma^n\} \in M_\gamma.$$

But then, making use of Lemma 5.12, for every $\delta > \gamma$ in H, we have

$$f_{\gamma\delta}(b) = \{\alpha < \delta \mid f_{\gamma\delta}(\vec{a})_\alpha \in U_\delta^n\} = \{\alpha < \delta \mid f_{\gamma\delta}(\vec{a})_\alpha \in U_\delta\}.$$

By the properties of the direct limit, it thus follows that $j_\gamma(b) = \{\alpha < \kappa \mid x_\alpha \in U\}$, showing that U is κ-amenable for M. □

Lemma 5.16. [10, Lemma 4.2.10] *For every $X \subseteq \kappa$, $X \in U$ if and only if there is $\alpha \in H$ such that $\{\xi \in H \mid \xi > \alpha\} \subseteq X$.*

As an easy consequence, one then obtains the following, that we would like to provide the short proof of for the convenience of our readers:

Lemma 5.17. [10, Lemma 4.2.11] *U is countably complete.*

Proof. Let $\langle A_n \mid n < \omega\rangle$ be a sequence of elements of U. For each $n < \omega$, there is $\gamma_n \in H$ for which $X_n = \{\xi \in H \mid \xi > \gamma_n\} \subseteq A_n$. Thus,

$$\emptyset \neq \bigcap_{n<\omega} X_n \subseteq \bigcap_{n<\omega} A_n.$$

□

Lemma 5.18. [10, Lemma 4.2.12] *$A^* \restriction \kappa = A$, and hence $A \in M$.*

This implies that both x and y are elements of M. Since we have chosen H to be a subset of x, it follows by Lemma 5.16 that $x \in U$, which concludes our argument. □

In particular, if κ is a Ramsey cardinal, then $\mathcal{R}_{mod}([\kappa]^{<\kappa}) = \mathcal{R}([\kappa]^{<\kappa})$ is the Ramsey ideal on κ. If κ is an ineffably Ramsey cardinal, then $\mathcal{R}_{mod}(\mathrm{NS}_\kappa) = \mathcal{R}(\mathrm{NS}_\kappa)$ is the ineffably Ramsey ideal on κ.

6. Pre-Operators

Building on Baumgartner's notion of a *pre-Ramsey cardinal* [4], in his [8], Feng introduced the pre-Ramsey operator, which behaves with respect to the Ramsey operator as does the subtle operator with respect to the ineffability operator, and it is this notion that motivates the naming of our notion of *pre-operators*. We want to introduce some simple and natural additional terminology, that will allow us to define pre-operators in a uniform way. Our ideal operators are all defined via local instances that are parametrized by certain objects. Given a cardinal κ, we would like to refer to the collection of all such objects on κ as the *object type at* κ of such an operator $\mathcal{O}$, and denote this by $\mathcal{T}(\mathcal{O},\kappa)$. The object type $\mathcal{T}(\mathcal{I},\kappa)$ of the ineffability operator at κ is the collection of all κ-lists, the object type $\mathcal{T}(\mathcal{R},\kappa)$ of the Ramsey operator at κ is the collection of all regressive functions $c\colon[\kappa]^{<\omega}\to\kappa$, and the object type of our model based operators at κ is simply the powerset of κ.

Definition 6.1. Each object type $\mathcal{T}$ at κ comes with an associated restriction operator, which, given some $y\in\mathcal{T}$ and some $\alpha<\kappa$, outputs its natural restriction $y\restriction\alpha$ to α.

- If $\mathcal{T}=\mathcal{P}(\kappa)$ and $y\in\mathcal{T}$, then $y\restriction\alpha=y\cap\alpha$.
- If $\mathcal{T}$ is the collection of all κ-lists and $y\in\mathcal{T}$, then $y\restriction\alpha$ is the restriction of y to the domain α, i.e. the initial segment of length α of the κ-sequence y.
- If $\mathcal{T}$ is the collection of all functions $c\colon[\kappa]^{<\omega}\to 2$ and $y\in\mathcal{T}$, then $y\restriction\alpha$ is the restriction of y to the domain $[\alpha]^{<\omega}$.

Each ideal operator $\mathcal{O}$ with local instances $\mathcal{O}^y$ has what we would like to call its associated *pre-operator* $\mathcal{O}_0$. To define such an operator, we start with a sequence

$$\vec{I}=\langle I_\alpha\mid\alpha\leq\kappa\text{ is a regular uncountable cardinal}\rangle$$

for some inaccessible cardinal κ, such that each I_α is an ideal on α. We will refer to such a sequence as a *sequence of ideals* in the following.

Definition 6.2. Given an ideal operator $\mathcal{O}$ together with its local instances $\mathcal{O}^y$, we define its associated pre-operator $\mathcal{O}_0$ as follows. Given a sequence $\vec{I}$ of ideals, let

$$\mathcal{O}_0(\vec{I})^+=\{x\subseteq\kappa\mid\forall y\in\mathcal{T}(\mathcal{O},\kappa)\,\forall C\subseteq\kappa\text{ club }\exists\alpha\in x\ x\cap C\cap\alpha\in\mathcal{O}^{y\restriction\alpha}(I_\alpha)^+\},$$

where α is understood to range over regular uncountable cardinals. If the I_α's are uniformly definable from α, we also write $\mathcal{O}_0(I_\kappa)$ rather than $\mathcal{O}_0(\vec{I})$. We say that I_κ *induces the sequence of ideals* $\vec{I}$ in this case.[10]

7. The subtle operator

The subtle operator is the usual name for what could be called the pre-ineffable operator, which is implicit in [3], and explicit in [6]: it is the pre-operator $\mathcal{I}_0$ defined via the ineffability operator $\mathcal{I}$ (or rather, its local instances $\mathcal{I}^y$), and we let $\mathcal{I}_{mod\,0}=(\mathcal{I}_{mod})_0$ be the *model version* of this operator, defined via the local instances of the model version of the ineffability operator. Note that while the operators $\mathcal{I}$ and $\mathcal{I}_{mod}$ agree on ideals that contain the nonstationary ideal by Proposition 2.5, their local

[10] Being precise here, note that it is actually rather the *definition of* I_κ that induces the sequence of ideals $\vec{I}$. In the following, we will only be concerned with natural examples, for example when I_κ is NS_κ or some indescribability ideal on κ.

instances do not seem to do so, and it is therefore not immediate that $\mathcal{I}_0$ and $\mathcal{I}_{mod\,0}$ actually agree on these ideals. We will however show this to be the case below. If κ is a subtle cardinal, then $\mathcal{I}_0(\mathrm{NS}_\kappa)$ is the subtle ideal on κ (see [3, Theorem 5.1]), which is classically defined as follows:

Definition 7.1. $x \subseteq \kappa$ is *subtle* if for every x-list $\vec{a}$ and every club $C \subseteq \kappa$, there are $\alpha < \beta$ in C such that $a_\alpha = a_\beta \cap \alpha$. The *subtle ideal* on κ is the collection of all subsets of κ that are not subtle.

Let us start with a simple observation.

Observation 7.2. *If $I_\kappa \supseteq \mathrm{NS}_\kappa$ is an ideal on κ that induces a sequence $\vec{I}$ of ideals, then $\mathcal{I}_{mod\,0}(\vec{I}) \supseteq \mathcal{I}_0(\vec{I})$.*

Proof. For every κ-list $\vec{a}$, there is $y \subseteq \kappa$ coding $\vec{a}$, and any reasonable choice of coding will have the property that for every cardinal $\alpha < \kappa$, $y \cap \alpha$ codes $\vec{a}\restriction\alpha$. By Proposition 2.5 thus, for every $\alpha < \kappa$, $\mathcal{I}_{mod}^{y\restriction\alpha}(I_\alpha) \supseteq \mathcal{I}^{\vec{a}\restriction\alpha}(I_\alpha)$, and hence the observation immediately follows from the definition of the operators $\mathcal{I}_{mod\,0}$ and $\mathcal{I}_0$. □

By a careful adaptation of the arguments for Proposition 2.5 (2), it is in fact possible to verify equality.

Theorem 7.3. *If $I_\kappa \supseteq \mathrm{NS}_\kappa$ is an ideal on κ that induces a sequence $\vec{I}$ of ideals, then $\mathcal{I}_{mod\,0}(\vec{I}) = \mathcal{I}_0(\vec{I})$.*

Proof. Having Observation 7.2 available, it only remains to show that $\mathcal{I}_{mod\,0}(\vec{I}) \subseteq \mathcal{I}_0(\vec{I})$. We may also assume that κ is a subtle cardinal, for otherwise $\mathcal{I}_0(\vec{I}) = \mathcal{P}(\kappa)$, and we are thus done. Assume that $x \in \mathcal{I}_0(\vec{I})^+$. We want to show that $x \in \mathcal{I}_{mod\,0}(\vec{I})^+$. Let $y \subseteq \kappa$ and let C be a club subset of κ. We need to find $\alpha \in x$ such that $x \cap C \cap \alpha \in \mathcal{I}_{mod}^{y\cap\alpha}(I_\alpha)^+$.

Fix a set of Skolem functions for $H(\kappa^+)$, and let M be the Skolem hull of $(\kappa+1) \cup \{x, y, C\}$ in $H(\kappa^+)$. Pick an enumeration $\langle x_\xi \mid \xi < \kappa\rangle$ of all subsets of κ in M, and let $\vec{a}$ be the κ-list defined by setting $a_\beta = \{\xi < \beta \mid \beta \in x_\xi\}$ for every $\beta < \kappa$. Let D be the club set of cardinals γ below κ such that if M_γ denotes the Skolem hull of $\gamma \cup \{\kappa, x, y, C\}$ in M, then

(1) $M_\gamma \cap \kappa = \gamma$, and
(2) $\langle x_\xi \mid \xi < \gamma\rangle$ enumerates all subsets of κ in M_γ.

Making use of our assumption that $x \in \mathcal{I}_0(\vec{I})^+$, there is $\alpha \in x$ such that

$$x \cap C \cap D \cap \alpha \in I^{\vec{a}\restriction\alpha}(I_\alpha)^+.$$

Let $\bar{M}$ be the transitive collapse of M_α. Then, $\bar{M}$ is a transitive weak α-model with $x\cap\alpha, y\cap\alpha, C\cap\alpha \in \bar{M}$, and by (1) and (2) above, $\langle x_\xi \cap \alpha \mid \xi < \alpha\rangle$ enumerates all subsets of α in $\bar{M}$. Moreover, $\vec{a}\restriction\alpha$ satisfies that $a_\beta = \{\xi < \beta \mid \beta \in x_\xi \cap \alpha\}$ for every $\beta < \alpha$. We now proceed exactly as in the proof of Proposition 2.5: By our choice of α, there is $H \subseteq x \cap C \cap D \cap \alpha$ in I_α^+ that is homogeneous for $\vec{a}\restriction\alpha$. We may thus pick $A \subseteq \alpha$ such that $a_\beta = A \cap \beta$ for every $\beta \in H$. Given $\xi < \alpha$, let $u_\xi = x_\xi \cap \alpha$ if $\xi \in A$, and let $u_\xi = \alpha \setminus x_\xi$ otherwise. Let $U = \{u_\xi \mid \xi < \alpha\}$. By the corresponding version at α of the claim within the proof of Proposition 2.5, U is an $\bar{M}$-ultrafilter on α with $x \cap C \cap \alpha \in U$, such that $\Delta_{\xi<\alpha} u_\xi \in I_\alpha^+$. This shows that $x \cap C \cap \alpha \in \mathcal{I}_{mod}^{y\cap\alpha}(I_\alpha)^+$, as desired. □

8. A small embedding characterization of subtlety using an anti-correctness property

In this short section, we want to place a sidenote that doesn't really use the techniques developed in this paper, but is somewhat closely related to them, and gives a strong hint towards a possible negative answer for an open question [13, Question 8.5]. In that paper, we provided so-called *small embedding characterizations* for many types of large cardinals, including subtle cardinals. These characterizations state that there exists an embedding $j\colon M \to H(\theta)$ with $j(\operatorname{crit} j) = \kappa$ and such that certain additional properties hold true. All of these characterizations except for the one for subtle cardinals were based on what we called *correctness properties*, that is properties that were either provable in V or in M, and that were ascertained to also hold in M or V respectively by our characterization. In [13, Question 8.5], it is asked whether subtle cardinals have a small embedding characterization that is based on a correctness property. We want to show here that for subtlety, we can in fact provide a natural small embedding characterization that is rather based on an *anti-correctness property*, i.e. a property that at least in some cases is provably non-absolute between M and V.

Definition 8.1. [13, Definition 1.1] Given cardinals $\kappa < \theta$, we say that a nontrivial elementary embedding $j\colon M \to H(\theta)$ is a *small embedding* for κ if $M \in H(\theta)$ is transitive, and $j(\operatorname{crit} j) = \kappa$ holds.

We next make the immediate observation that the property $\kappa \in \mathcal{I}_0(\mathrm{NS}_\kappa)^+$ can be rewritten to yield a small embedding characterization of the subtlety of κ that is different to the one provided in [13, Lemma 5.2].

Observation 8.2. *A cardinal κ is subtle if for every cardinal $\theta > \kappa$, every κ-list $\vec{a}$ and every club $C \subseteq \kappa$, there is a small embedding $j\colon M \to H(\theta)$ for κ such that $\vec{a}, C \in \operatorname{range} j$ and $C \cap \operatorname{crit} j \in \mathcal{I}^{\vec{a}\restriction\alpha}(\mathrm{NS}_{\operatorname{crit} j})^+$.*

However, the property used to characterize subtlety in the above is easily seen to be an anti-correctness property in many circumstances:

Observation 8.3. *Assume that κ is subtle, but not ineffable. Then, there are a κ-list $\vec{a}$ and a club subset C of κ such that for every cardinal $\theta > \kappa$ and every small embedding $j\colon M \to H(\theta)$ for κ with $\vec{a}$ and C both in the range of j, letting $\bar{C} = C \cap \operatorname{crit} j = j^{-1}(C)$ and $\bar{a} = \vec{a}\restriction\operatorname{crit} j = j^{-1}(\vec{a})$, M thinks that $\bar{C} \in \mathcal{I}^{\bar{a}}(\mathrm{NS}_{\operatorname{crit} j})$.*

Proof. If this weren't the case, then κ would be ineffable by the elementarity of the small embeddings. □

9. The pre-Ramsey operator

The pre-Ramsey operator $\mathcal{R}_0$ is the pre-operator defined with respect to the Ramsey operator $\mathcal{R}$ (or rather, its local instances $\mathcal{R}^y$). A cardinal κ is a *pre-Ramsey* cardinal if $\kappa \in \mathcal{R}_0([\kappa]^{<\kappa})^+$. If κ is a pre-Ramsey cardinal, then $\mathcal{R}_0([\kappa]^{<\kappa})$ is the pre-Ramsey ideal on κ (see [3]). We let $\mathcal{R}_{mod\,0} = (\mathcal{R}_{mod})_0$ be the *model version* of the pre-Ramsey operator, defined via the local instances of the model version of the Ramsey operator. Note that as for the ineffability operator and its model version, while the Ramsey operator and its model version agree, their local instances do not seem to do so, and it is therefore not immediate that the operators $\mathcal{R}_0$ and $\mathcal{R}_{mod\,0}$ actually agree. We will however show this to be the case below. The (second part of the) proof of the following theorem proceeds by a careful adaptation of the arguments for Theorem 5.6.

Theorem 9.1. *For any sequence $\vec{I}$ of ideals, $\mathcal{R}_{mod\,0}(\vec{I}) = \mathcal{R}_0(\vec{I})$.*

Proof. For every regressive function $f\colon [\kappa]^{<\omega} \to \kappa$, there is $y \subseteq \kappa$ coding f, and any reasonable choice of coding will have the property that for every cardinal $\alpha < \kappa$, $y \cap \alpha$ codes $f\restriction\alpha$. By Lemma 5.2 thus, for every $\alpha < \kappa$, $\mathcal{R}^{y\restriction\alpha}_{mod}(I_\alpha) \supseteq \mathcal{R}^{\vec{a}\restriction\alpha}(I_\alpha)$. Therefore, $\mathcal{R}_{mod\,0}(\vec{I}) \supseteq \mathcal{R}_0(\vec{I})$ immediately follows from the definition of the operators $\mathcal{R}_{mod\,0}$ and $\mathcal{R}_0$.

It thus remains to show that $\mathcal{R}_{mod\,0}(\vec{I}) \subseteq \mathcal{R}_0(\vec{I})$. We may also assume that κ is a pre-Ramsey cardinal, for otherwise $\mathcal{R}_0(\vec{I}) = \mathcal{P}(\kappa)$, and we are thus done. Assume that $x \in \mathcal{R}_0(\vec{I})^+$. We want to show that $x \in \mathcal{R}_{mod\,0}(\vec{I})^+$. Let $y \subseteq \kappa$ and let C be a club subset of κ. We need to find $\alpha \in x$ such that $x \cap C \cap \alpha \in \mathcal{R}^{y\cap\alpha}_{mod}(I_\alpha)^+$.

Let $A \subseteq \kappa$ code x on the even ordinals, and y on the odd ordinals. Let $h\colon [\kappa]^{<\omega} \to \kappa$ be a regressive function and let $D \subseteq \kappa$ be the club set of cardinals obtained from an application of Lemma 5.4 (1) for $A \subseteq \kappa$. Making use of our assumption that $x \in \mathcal{R}_0(\vec{I})^+$, there is $\alpha \in x$ such that $x \cap C \cap D \cap \alpha \in \mathcal{R}^{h\restriction\alpha}(I_\alpha)^+$.

Let $\mathcal{A} = \langle L_\alpha[A], A \cap \alpha\rangle$, which is an elementary substructure of $\langle L_\kappa[A], A\rangle$, since $\alpha \in D$. By Lemma 5.4 (2), $D \cap \alpha$ and $h\restriction\alpha$ witness that Lemma 5.4 (1) holds for $A \cap \alpha$. By our choice of α and by Lemma 5.5, there is $H \subseteq x \cap C \cap D \cap \alpha$ in I^+_α that is homogeneous for $h\restriction\alpha$, and by Lemma 5.4 (1), H is a set of good indiscernibles for $\mathcal{A}$. We now proceed exactly as in the proof of Theorem 5.6, constructing a weak α-model M with $y \in M$ and an M-normal M-ultrafilter U on α that is κ-amenable for M and countably complete with $x \cap C \cap D \cap \alpha \in U$, thus showing that $x \cap C \cap D \cap \alpha \in \mathcal{R}^{y\cap\alpha}_{mod}(I_\alpha)^+$. Since $x \cap C \cap D \cap \alpha \subseteq x \cap C \cap \alpha$, and the latter set is easily (somewhat cumbersome, but straightforward, by proceeding along the model construction in the proof of Theorem 5.6) checked to be an element of M, and thus also of U, this implies that $x \cap C \cap \alpha \in \mathcal{R}^{y\cap\alpha}_{mod}(I_\alpha)^+$, as desired. □

10. An abstract notion of ideal operator

In this section, we present an abstract notion of ideal operator, which has both (the model versions of) the ineffability operator and the Ramsey operator as special instances, and which – unlike (the original versions of) the ineffability operator and the Ramsey operator – can easily be used to produce further interesting instances of ideal operators. We also provide a first few basic results for such operators.

Definition 10.1. Let $\Psi(M, U)$ and $\Omega(U, I)$ be parameter-free first order formulae such that ZFC proves that for any ideal I on a regular uncountable cardinal κ, any transitive weak κ-model M and any M-ultrafilter U on κ,

- $\Omega(U, I)$ implies that $U \subseteq I^+$, and
- for any ideal J on κ, $[I \supseteq J \wedge \Omega(U, I)] \to \Omega(U, J)$.

Let us say that a pair of formulas $\langle\Psi, \Omega\rangle$ satisfying the above is *regular*.

We define an ideal operator $\mathfrak{O}\Psi\Omega$ as follows. For any ideal I on κ and $y \subseteq \kappa$, we first define a local instance by letting

- $x \in \mathfrak{O}\Psi\Omega^y(I)^+$ if there exists a transitive weak κ-model M with $y \in M$ and an M-ultrafilter U on κ with $x \in U$ such that $\Psi(M, U)$ and $\Omega(U, I)$ hold, and we let
- $\mathfrak{O}\Psi\Omega(I)^+ = \bigcap_{y\subseteq\kappa} \mathfrak{O}\Psi\Omega^y(I)^+$.

We will observe in Proposition 10.2 below that regularity of $\langle\Psi,\Omega\rangle$ implies that $\mathfrak{O}\Psi\Omega$ is indeed an ideal operator. Let us check how the operators $\mathcal{I}_{mod}$ and $\mathcal{R}_{mod}$ fit into the above scheme:

- If $\Psi(M,U)$ is trivial, and $\Omega(U,I)$ denotes the property that $\Delta U{\in}I^+$, then $\mathfrak{O}\Psi\Omega$ is the model version $\mathcal{I}_{mod}$ of the ineffability operator.
- If $\Psi(M,U)$ denotes the property that U is M-normal and κ-amenable for M, and $\Omega(U,I)$ denotes the property that every countable intersection of elements of U is in I^+, then $\mathfrak{O}\Psi\Omega$ is (the model version $\mathcal{R}_{mod}$ of) the Ramsey operator.

Proposition 10.2. *Assume that $\langle\Psi,\Omega\rangle$ is regular, and that $I\supseteq J$ are ideals on κ. Then, the following hold.*

- *$\mathfrak{O}\Psi\Omega(I)\supseteq I$ is an ideal on κ.*
- *$\mathfrak{O}\Psi\Omega(I)\supseteq\mathfrak{O}\Psi\Omega(J)$.*
- *If for any transitive weak κ-model M and any M-ultrafilter U on κ, the conjunction $\Psi(M,U)\wedge\Omega(U,I)$ implies that U is M-normal, then $\mathfrak{O}\Psi\Omega(I)$ is normal.*
- *In particular, if $I\supseteq\mathrm{NS}_\kappa$, then $\Delta U\in I^+$ implies that U is M-normal.*
- *If $\langle\Psi',\Omega'\rangle$ is regular as well, and $\Psi'(M,U)\wedge\Omega'(U,I)$ implies $\Psi(M,U)\wedge\Omega(U,I)$ for any transitive weak κ-model M and any M-ultrafilter U on κ, then $\mathfrak{O}\Psi'\Omega'(I)\supseteq\mathfrak{O}\Psi\Omega(I)$.*

Proof. Assume that I is an ideal on a cardinal κ, that $A\in\mathfrak{O}\Psi\Omega(I)$, and that $B\subseteq A$. We want to show that also $B\in\mathfrak{O}\Psi\Omega(I)$. Let $y\subseteq\kappa$ be such that y codes A and $A\in\mathfrak{O}\Psi\Omega^y(I)$. Now if M is a transitive weak κ-model with $y\in M$ and U is an M-ultrafilter on κ such that both $\Psi(M,U)$ and $\Omega(U,I)$ hold, then $A\in M$, however $A\notin U$, and hence also $B\subseteq A$ is not an element of U, showing that $B\in\mathfrak{O}\Psi\Omega^y(I)\subseteq\mathfrak{O}\Psi\Omega(I)$, as desired.

Now assume that both A and B are in $\mathfrak{O}\Psi\Omega(I)$. We want to show that also $A\cup B\in\mathfrak{O}\Psi\Omega(I)$. Let y_A and y_B be such that $A\in\mathfrak{O}\Psi\Omega^{y_A}(I)$ and $B\in\mathfrak{O}\Psi\Omega^{y_B}(I)$. Let $y\subseteq\kappa$ code all of A, B, y_A and y_B. Now if M is a transitive weak κ-model with $y\in M$ and U is an M-ultrafilter on κ such that both $\Psi(M,U)$ and $\Omega(U,I)$ hold, it follows that both A and B are in $(\mathcal{P}(\kappa)\cap M)\setminus U$, and hence also that $A\cup B$ is not in U, showing that $A\cup B\in\mathfrak{O}\Psi\Omega^y(I)\subseteq\mathfrak{O}\Psi\Omega(I)$, as desired.

Clearly, using that $\Omega(U,I)$ implies that $U\subseteq I^+$ by the requirements from Definition 10.1, if $x\in\mathfrak{O}\Psi\Omega(I)^+$, then $x\in I^+$, thus finishing the argument for the first item of the proposition.

The monotonicity statement in the second item is immediate from our monotonicity requirement on Ω from Definition 10.1.

For the third item, assume that $\Psi(M,U)\wedge\Omega(U,I)$ implies that U is M-normal. Let $A\in\mathfrak{O}\Psi\Omega(I)^+$, and let $f\colon A\to\kappa$ be a regressive function. Assume for a contradiction that $f^{-1}(\{\alpha\})\in\mathfrak{O}\Psi\Omega(I)$ for every $\alpha<\kappa$. We may thus pick a sequence $\vec{y}=\langle y_\alpha\mid\alpha<\kappa\rangle$ such that $f^{-1}(\{\alpha\})\in\mathfrak{O}\Psi\Omega^{y_\alpha}(I)$ for every $\alpha<\kappa$. Let $y\subseteq\kappa$ code both f and $\vec{y}$. Let M be a transitive weak κ-model with $y\in M$, and let U be an M-ultrafilter on κ with $A\in U$ such that both $\Psi(M,U)$ and $\Omega(U,I)$ hold. Since $y_\alpha\in M$ for every $\alpha<\kappa$, it follows that for no $\alpha<\kappa$ we have $f^{-1}(\{\alpha\})\in U$. On the other hand, since U is M-normal, there is some $B\in U$ that is homogeneous for f, which is clearly contradicting the above, as desired.

For the fourth item, assume that $I \supseteq \mathrm{NS}_\kappa$ and $\Delta U \in I^+$, however that U is not M-normal. Then, there is a κ-sequence of elements of M with a diagonal intersection that is not in U, and hence the complement of this diagonal intersection is in U. But now, using this, ΔU is nonstationary, which contradicts that $\Delta U \in I^+$.

The statement of the fifth item is immediate from the definitions involved. □

Definition 10.3. Let $\langle\Psi, \Omega\rangle$ be a pair of formulas, and let $\mathcal{O}$ be an ideal operator.

- The pair $\langle\Psi, \Omega\rangle$ is *ineffable* in case ZFC proves that for any ideal I on a regular uncountable cardinal κ, any transitive weak κ-model M and any M-ultrafilter U on κ, $\Psi(M, U) \wedge \Omega(U, I)$ implies that for every $A \in U$, every A-list $\vec{a} \in M$ has a homogeneous set in I^+.
- The operator $\mathcal{O}$ is *ineffable* in case ZFC proves that for any ideal I on a regular uncountable cardinal κ, whenever $A \in \mathcal{O}(I)^+$ and $\vec{a}$ is an A-list, then $\vec{a}$ has a homogeneous set in I^+.

Note that by the above, the ineffability operator $\mathcal{I}$ is ineffable. But also, if $\mathcal{O}$ can be characterized in the form $\mathcal{O} = \mathfrak{O}\Psi\Omega$ for some regular ineffable pair of formulas $\langle\Psi, \Omega\rangle$, then $\mathcal{O}$ is ineffable.

Observation 10.4. *Let $\langle\Psi, \Omega\rangle$ be regular, and let $\mathcal{O}$ be the operator $\mathfrak{O}\Psi\Omega$. Then,*

- *If $\Psi(M, U)$ ZFC-provably implies that U is κ-amenable for M and contains all club subsets of κ in M as elements, then $\mathcal{O}$ is ineffable.*
- *If $\Omega(U, I)$ ZFC-provably implies that $\Delta U \in I^+$, then $\mathcal{O}$ is ineffable.*
- *If $\mathcal{O}$ is ineffable, then for any ideal I on a regular uncountable cardinal κ, $\mathcal{O}(I) \supseteq \mathcal{I}(I) \supseteq \mathrm{NS}_\kappa$.*
- *If ZFC proves that for any ideal I on a regular and uncountable cardinal κ, $\mathcal{O}(I) \supseteq \mathcal{I}(I)$, then $\mathcal{O}$ is ineffable.*

Proof. The first item follows from Lemma 3.5, and the second item follows from Proposition 2.5. The third and fourth item are both immediate from the definition of the ineffability operator $\mathcal{I}$, and the fact that $\mathcal{I}(I) \supseteq \mathcal{I}([\kappa]^{<\kappa}) \supseteq \mathrm{NS}_\kappa$, where the easy argument for the latter inclusion can be found within the proof of [3, Theorem 2.3]. □

In particular, the first item above implies that the operator $\mathcal{R} = \mathcal{R}_{mod}$ is ineffable. The following important observation is now immediate from Lemma 3.4.

Corollary 10.5. *If $\mathcal{O}$ is ineffable, $I \supseteq \bigcup_{\xi\in\{-1\}\cup\beta} \Pi^1_\xi(\kappa)$ is an ideal on κ, and $\beta < \kappa$ is an ordinal, then*

$$\mathcal{O}(I) \supseteq \Pi^1_{\beta+1}(\kappa).$$

11. A review of some material from [12]

In Section 12 below, we would like to introduce a number of new large cardinal operators. In order to be able to do so, we will first need to review some material from [12].[11] We also present a proof for a slight strengthening of [12, Lemma 9.13(2)], which we will make use of in Section 12, and the original proof of which in [12] has a slight flaw.

[11] In [12], we consider the general case of Σ_0-correct models of ZFC$^-$ containing some κ as an element, while

We first need to present some material regarding correspondences between the existence of certain ultrafilters and certain elementary embeddings. These are minor generalizations of standard results, transferred to the context of (possibly non-transitive) weak κ-models, including the case of possibly non-wellfounded ultrapowers. We will restrict our attention to Σ_0-correct weak κ-models M, that is we additionally require that M is Σ_0-elementary in V.[12] Since all the weak κ-models that we consider later on in our paper are either transitive or elementary substructures of some $H(\theta)$, they will always meet this requirement.

In the following, we let $j : M \longrightarrow \langle N, \epsilon_N\rangle$ always denote an elementary embedding between $\langle M, \in\rangle$ and $\langle N, \epsilon_N\rangle$, whose domain M is a (possibly non-transitive) Σ_0-correct weak κ-model.

Definition 11.1 (Jump)**.** Given $j : M \longrightarrow \langle N, \epsilon_N\rangle$ and an ordinal $\alpha \in M$, we say that j *jumps at* α if there exists an N-ordinal γ with $\gamma \,\epsilon_N\, j(\alpha)$ and $j(\beta)\,\epsilon_N\,\gamma$ for all $\beta \in M \cap \alpha$.

Note that, in the above situation, for every N-ordinal γ, there is at most one ordinal α in M such that γ witnesses that j jumps at α. Moreover, elementarity directly implies that elementary embeddings only jump at limit ordinals.

Definition 11.2 (Critical Point)**.** Given $j : M \longrightarrow \langle N, \epsilon_N\rangle$, if there exists an ordinal $\alpha \in M$ such that j jumps at α, then we denote the minimal such ordinal by $\operatorname{crit} j$, the *critical point of* j.

The following property implies the existence of a canonical representative for κ in the target model of our elementary embedding.

Definition 11.3 (κ-embedding)**.** Given $j : M \longrightarrow \langle N, \epsilon_N\rangle$ that jumps at κ, the embedding j is a *κ-embedding* if there exists an ϵ_N-minimal N-ordinal γ witnessing that j jumps at κ. We denote this ordinal by κ^N.[13]

Proposition 11.4. [12, Proposition 2.9] *Given $j : M \longrightarrow \langle N, \epsilon_N\rangle$ with $\operatorname{crit} j = \kappa$, the following statements are equivalent:*

(1) j is a κ-embedding.
(2) The ordinal $\operatorname{otp}(\mathrm{M} \cap \kappa)$ is an element of the transitive collapse of the well-founded part of $\langle N, \epsilon_N\rangle$.

The following definition generalizes the notion of a κ-powerset preserving elementary embedding, that is usually defined for embeddings between transitive weak κ-models, to the context of κ-embeddings. Since we may identify κ and κ^N when $j \colon M \to \langle N, \epsilon_N\rangle$ is a κ-embedding by Proposition 11.4, it is essentially just stating that the domain and the target model of the embedding have the same powerset of κ.

Definition 11.5 (κ-powerset preservation)**.** Given a κ-embedding $j : M \longrightarrow \langle N, \epsilon_N\rangle$ with $\operatorname{crit} j = \kappa$, the embedding j is *κ-powerset preserving* if

$$\forall y \in N\ \exists x \in M\ \left[\langle N, \epsilon_N\rangle \models \text{“} y \subseteq \kappa^N \text{”} \ \longrightarrow\ x = \{\alpha < \kappa \mid j(\alpha)\,\epsilon_N\, y\}\right].$$

[12]The results regarding these correspondences in [12] are somewhat more general than the versions that we wil present here, for we do not make the assumption that $\kappa + 1 \subseteq M$ in [12].

[13]If $\operatorname{crit} j = \kappa$, then Proposition 11.4 shows that κ^N is the unique N-ordinal on which the ϵ_N-relation has order-type $M \cap \kappa$. Otherwise, κ^N might also depend on the embedding j, which we nevertheless suppress in our notation.

We will now present further material from [12, Section 3], showing that we can interchangeably talk about ultrafilters or about elementary embeddings, also in our generalized context. If M is a Σ_0-correct weak κ-model, κ is a cardinal of M, and U is an M-ultrafilter on κ, then we can use the Σ_0-correctness of M[14] to define the induced ultrapower embedding $j_U : M \longrightarrow \langle \mathrm{Ult}(M,U), \epsilon_U \rangle$ as usual: define an equivalence relation $\equiv_U$ on the class of all functions $f : \kappa \longrightarrow M$ contained in M by setting $f \equiv_U g$ if and only if $\{\alpha < \kappa \mid f(\alpha) = g(\alpha)\} \in U$, let $\mathrm{Ult}(M,U)$ consist of all sets $[f]_U$ of rank-minimal elements of $\equiv_U$-equivalence classes, define $[f]_U \, \epsilon_U [g]_U$ to hold if and only if $\{\alpha < \kappa \mid f(\alpha) \in g(\alpha)\} \in U$ and set $j_U(x) = [c_x]_U$, where $c_x \in M$ denotes the constant function with domain κ and value x. It is easy to check that the assumption that $M \models \mathrm{ZFC}^-$ implies that Łos' Theorem still holds true in our setting, i.e. we have

$$\mathrm{Ult}(M,U) \models \varphi([f_0]_U, \ldots, [f_{n-1}]_U)$$

$$\Longleftrightarrow$$

$$\langle M, U \rangle \models \text{“}\exists x \in U \ \forall \alpha \in x \ \varphi(f_0(\alpha), \ldots, f_{n-1}(\alpha))\text{”}$$

for every first order ϵ-formula $\varphi(v_0, \ldots, v_{n-1})$ and all functions $f_0, \ldots, f_{n-1} : \kappa \longrightarrow M$ in M.

Given an elementary embedding $j : M \longrightarrow \langle N, \epsilon_N \rangle$ that jumps at κ, let γ be a witness for this, and let

$$U_j^\gamma \ = \ \{A \in M \cap \mathcal{P}(\kappa) \mid \gamma \, \epsilon_N \, j(A)\}$$

denote the M-ultrafilter induced by γ and by j. Since γ is not in the range of j, the filter U_j^γ is non-principal. If j is a κ-embedding and $\gamma = \kappa^N$, then we call $U_j = U_j^\gamma$ the *canonical* M-ultrafilter induced by j, or simply *the* M-ultrafilter induced by j.

Given a cardinal κ, a property $\Psi(M,U)$ of Σ_0-correct weak κ-models M and M-ultrafilters U on κ *κ-corresponds* to a property $\Theta(M,j)$ of such models M and elementary embeddings $j : M \longrightarrow \langle N, \epsilon_N \rangle$ if the following statements hold:

- If $\Psi(M,U)$ holds for an M-ultrafilter U on κ, then $\Theta(M, j_U)$ holds.
- If $\Theta(M,j)$ holds for an elementary embedding $j : M \longrightarrow \langle N, \epsilon_N \rangle$, then j is a κ-embedding and $\Psi(M, U_j)$ holds.

Proposition 11.6. [12, Corollary 3.3 and Corollary 3.7] *Given $A \subseteq \kappa$, “U is an M-ultrafilter on κ that contains A as an element, and U is M-normal and κ-amenable for M” κ-corresponds to “$\mathrm{crit}\, j = \kappa$ and j is a κ-powerset preserving κ-embedding with $\kappa^N \epsilon_N j(A)$”.* □

Lemma 11.7. [12, Lemma 3.5] *Let κ be an inaccessible cardinal, let $b : \kappa \longrightarrow \mathrm{V}_\kappa$ be a bijection, let M be a Σ_0-correct weak κ-model with $b \in M$ and let $j : M \longrightarrow \langle N, \epsilon_N \rangle$ be a κ-powerset preserving κ-embedding with* $\mathrm{crit}\, j = \kappa$*.*

(1) The map

$$j_* : M \cap \mathrm{V}_{\kappa+1} \longrightarrow \langle \{y \in N \mid y \, \epsilon_N \, \mathrm{V}^N_{\kappa^N+1}\}, \epsilon_N \rangle; \ x \mapsto (j(x) \cap V_{\kappa^N})^N$$

is an ϵ-isomorphism extending $j \restriction (M \cap \mathrm{V}_\kappa)$.

[14] Note that, given a Σ_0-correct ZFC^--model M and functions $f, g : \kappa \longrightarrow M$ in M, then the set $\{\alpha < \kappa \mid f(\alpha) = g(\alpha)\}$ and $\{\alpha < \kappa \mid f(\alpha) \in g(\alpha)\}$ are both contained in M and satisfy the same defining properties in it.

(2) There is an ϵ-isomorphism

$$j^* : \mathrm{H}(\kappa^+)^M \longrightarrow \langle \{y \in N \mid y \,\epsilon_N\, \mathrm{H}((\kappa^N)^+)^N\}, \epsilon_N \rangle$$

extending j_*.

In the context of Lemma 11.7, we may (and will) thus identify elements of $H((\kappa^N)^+)^N$ with the corresponding elements of $H(\kappa^+)^M$ via j^*.

We next review the formal notions of Ramsey-like cardinals and Ramsey-like sets from [12, Section 9].

Definition 11.8. [12, Definition 9.4] Let $\kappa < \theta$ be uncountable regular cardinals, let $\alpha \leq \kappa$ be an infinite regular cardinal, let A be an unbounded subset of κ and let $\Phi(v_0, v_1)$ be a first order $\in$-formula.

- A is Φ^κ_α*-Ramsey* if for every $x \subseteq \kappa$, there is a transitive weak κ-model M closed under $<\alpha$-sequences and a uniform, κ-amenable M-normal M-ultrafilter U on κ such that $x \in M$, $A \in U$ and $\Phi(M, U)$ holds.
- A is Φ^θ_α*-Ramsey* if for every $x \in H(\theta)$, there is a weak κ-model $M \prec H(\theta)$ closed under $<\alpha$-sequences and a uniform, κ-amenable M-normal M-ultrafilter U on κ such that $x \in M$, $A \in U$ and $\Phi(M, U)$ holds.
- A is $\Phi^\forall_\alpha$*-Ramsey* if it is Φ^θ_α-Ramsey for every regular cardinal $\theta > \kappa$.
- If $\vartheta \in \{\kappa, \theta, \forall\}$, then κ is a Φ^ϑ_α*-Ramsey cardinal* if κ is Φ^ϑ_α-Ramsey as a subset of itself.

Let us remark that by Theorem 1.3(2), a cardinal κ is Ramsey if and only if it is $\mathbf{cc}^\kappa_\omega$-Ramsey, where $\mathbf{cc}(M, U)$ denotes the property that U is countably complete, that is, any countable intersection of elements of U is nonempty. Other properties that we will be interested in are the following:

- $\mathbf{T}(M, U)$ denotes the (trivial) property that $U = U$.
- $\mathbf{wf}(M, U)$ denotes the property that the ultrapower of M by U is well-founded.

The following lemma is a strengthening of [12, Lemma 9.13(2)]. In that lemma, it is assumed that κ is a $\Phi^{\kappa^+}_\alpha$-Ramsey cardinal rather than just a Φ^κ_α-Ramsey cardinal, as we assume here. We will actually need this stronger version of this lemma in Section 12 below. Moreover, the proof of [12, Lemma 9.13(2)] that is provided in [12] does not quite seem to work, and a minor modification, that we provide below, is needed.

Lemma 11.9. *Assume that $\alpha \leq \kappa$ are regular infinite cardinals, and let $\Phi(M, U)$ be a first order formula using only parameters from V_κ, such that $\Phi(X, U \cap X)$ holds whenever $X \subseteq M$ and $\Phi(M, U)$ holds. For arbitrary regular $\gamma \leq \kappa$, let $\alpha(\gamma) = \alpha$ in case $\alpha < \kappa$, and let $\alpha(\gamma) = \gamma$ in case $\alpha = \kappa$. Assume that κ is a Φ^κ_α-Ramsey cardinal such that Φ is absolute between V and arbitrary transitive weak κ-models containing V_κ. For any regular cardinal γ, let $X^\Phi_\gamma = \{\nu < \gamma \mid \nu$ is not $\Phi^{\nu^+}_{\alpha(\nu)}$-Ramsey$\}$. Then,*

$$X^\Phi_\kappa \textit{ is a } \Phi^\kappa_\alpha\textit{-Ramsey subset of } \kappa.$$

Proof. Assume that κ is the least $\Phi^\gamma_{\alpha(\gamma)}$-Ramsey cardinal γ with the property that X^Φ_γ is not a $\Phi^\gamma_{\alpha(\gamma)}$-Ramsey subset of γ. Let this property of X^Φ_κ be witnessed by $x \subseteq \kappa$. Using that κ is Φ^κ_α-Ramsey, there is a weak κ-model M with $x \in M \supseteq V_\kappa$ that is closed under $<\alpha$-sequences, and an M-normal uniform M-ultrafilter U on κ that is κ-amenable for M with $X^\Phi_\kappa \notin U$ and such that $\Phi(M, U)$ holds.

Claim 11.10. *X^Φ_κ is not a Φ^κ_α-Ramsey subset of κ in M.*

Proof. We claim that this is witnessed by $x \in M$. Working in M, we thus have to show that $X^\Phi_\kappa \notin u$ for every transitive weak κ-model m that is closed under $<\alpha$-sequences and any uniform κ-amenable M-normal M-ultrafilter u on κ such that $\Phi(m,u)$ holds. But note that since M is closed under $<\alpha$-sequences, and by our assumptions on Φ, the above properties of m and of u are absolute between M and V. But this means that the claim immediately follows from our assumption that X^Φ_κ is not a Φ^κ_α-Ramsey subset of κ (in V). $\square$

Using Proposition 11.6, let $j = j_U \colon \langle M, \in\rangle \to \langle N, \epsilon_N\rangle$ be the κ-powerset preserving κ-embedding with critical point κ induced by U, which thus satisfies $\kappa^N \notin j(X^\Phi_\kappa)$. Using Lemma 11.7, we identify $H((\kappa^N)^+)^N$ with $H(\kappa^+)^M$, and in particular, we identify κ^N with κ. Therefore, κ is a $\Phi^{\kappa^+}_\alpha$-Ramsey cardinal below $j(\kappa)$ in $\langle N, \epsilon_N\rangle$. Since M and N have the same subsets of κ, it follows that κ is a Φ^κ_α-Ramsey cardinal in M.[15] Then clearly, in $M \supseteq V_\kappa$, κ is also the least $\Phi^\gamma_{\alpha(\gamma)}$-Ramsey cardinal γ with the property that X^Φ_γ is not a $\Phi^\gamma_{\alpha(\gamma)}$-Ramsey subset of γ. But then, by elementarity, in $\langle N, \epsilon_N\rangle$, $j(\kappa)$ is the least $\Phi^\gamma_{j(\alpha(\gamma))}$-Ramsey cardinal with the property that X^Φ_γ is not a $\Phi^\gamma_{j(\alpha(\gamma))}$-Ramsey subset of γ. It follows that X^Φ_κ is Φ^κ_α-Ramsey in $\langle N, \epsilon_N\rangle$. Since M and N have the same subsets of κ however, this contradicts Claim 11.10. $\square$

12. New large cardinal operators

Definition 11.8 easily lends itself to provide a notion of what we would like to call Ramsey-like subset operators. We will need some variants of Definition 10.1.

Definition 12.1. Let $\langle\Psi,\Omega\rangle$ be regular. We define an ideal operator $\mathfrak{O}^\oplus\Psi\Omega$ as follows. For any ideal I on κ and $y \subseteq \kappa$, we first define a local instance by letting

- $x \in \mathfrak{O}^\oplus\Psi\Omega^y(I)^+$ if there exists a transitive weak κ-model $M \prec H(\kappa^+)$ with $y \in M$ and an M-ultrafilter U on κ with $x \in U$ such that $\Psi(M,U)$ and $\Omega(U,I)$ hold, and we let
- $\mathfrak{O}\Psi\Omega(I)^\oplus = \bigcap_{y\subseteq\kappa} \mathfrak{O}^\oplus\Psi\Omega^y(I)^+$.

We also define an ideal operator $\mathfrak{O}^\forall\Psi\Omega$ as follows. For any ideal I on κ and $y \subseteq \kappa$, we first define a local instance by letting

- $x \in \mathfrak{O}^\forall\Psi\Omega^y(I)^+$ if for every regular cardinal $\theta > \kappa$, there exists a transitive weak κ-model $M \prec H(\theta)$ with $y \in M$ and an M-ultrafilter U on κ with $x \in U$ such that $\Psi(M,U)$ and $\Omega(U,I)$ hold, and we let
- $\mathfrak{O}^\forall\Psi\Omega(I)^+ = \bigcap_{y\subseteq\kappa} \mathfrak{O}^\forall\Psi\Omega^y(I)^+$.

Definition 12.2. Let $\Phi(v_0,v_1)$ be a first order $\in$-formula. Let $\Psi(M,U)$ be the statement that U is M-normal and κ-amenable for M (where $\kappa = \bigcup U$) and that $\Phi(M,U)$ holds, and let $\Omega(U,I)$ be the statement that $U \subseteq I^+$.

- We let $\Phi = \mathfrak{O}\Psi\Omega$ denote the *Φ-Ramsey subset operator*.
- We let $\Phi^\oplus = \mathfrak{O}^\oplus\Psi\Omega$ denote the *$\Phi^\oplus$-Ramsey subset operator*.

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[15] Here is where our proof essentially differs from that of [12, Lemma 9.13(2)] provided in [12]. That proof assumes that we can somehow conclude κ to be $\Phi^{\kappa^+}_\alpha$-Ramsey in M, which however seems very problematic, for we don't even have $\kappa^+ \in M$, not in the situation of our current proof here, and neiter in the context of the proof of [12, Lemma 9.13(2)] that is provided in [12].

- We let $\Phi^\forall = \mathfrak{O}^\forall\Psi\Omega$ denote the $\Phi^\forall$-*Ramsey subset operator*.

Let $\Psi_{\text{cl}}(M,U)$ be the conjunction of $\Psi(M,U)$ with the assertion that M is closed under $<\kappa$-sequences.

- We let $\Phi_{\text{cl}} = \mathfrak{O}\Psi_{\text{cl}}\Omega$ denote the Φ_{cl}-Ramsey subset operator.

Note that by the very definitions of the properties involved, $A \subseteq \kappa$ is Φ^κ_ω-Ramsey / $\Phi^{\kappa^+}_\omega$-Ramsey / $\Phi^\forall_\omega$-Ramsey / Φ^κ_κ-Ramsey if and only if $A \in \Phi([\kappa]^{<\kappa})^+$ / $A \in \Phi^\oplus([\kappa]^{<\kappa})^+$ / $A \in \Phi^\forall([\kappa]^{<\kappa})^+$ / $A \in \Phi_{\text{cl}}([\kappa]^{<\kappa})$ respectively, and that taking $A = \kappa$, this relates Φ^κ_ω-Ramsey / $\Phi^{\kappa^+}_\omega$-Ramsey / $\Phi^\forall_\omega$-Ramsey / Φ^κ_κ-Ramsey cardinals with their corresponding large cardinal operators. In the present section, we want to focus on three instances of Definition 12.2:[16] The $\mathbf{T}^\kappa_\omega$-Ramsey subset operator $\mathbf{T}$, the $\mathbf{wf}^\kappa_\omega$-Ramsey subset operator $\mathbf{wf}$, and the $\mathbf{T}^\kappa_\kappa$-Ramsey subset operator $\mathbf{T}_{\text{cl}}$.[17]

- The notion of $\mathbf{T}^\kappa_\omega$-Ramsey cardinal was first considered in [12, Section 10].
- The notion of $\mathbf{wf}^\kappa_\omega$-Ramsey cardinal corresponds exactly to the notion of *weakly Ramsey cardinal* that was introduced in [10, Definition 1.2].
- The notion of $\mathbf{T}^\kappa_\kappa$-Ramsey cardinal corresponds exactly to the notion of *strongly Ramsey cardinal* that was introduced in [10, Definition 1.4].

For the convenience of our readers, let us remark at this point already that regarding the cardinals involved in this discussion, we have the following known chain of strict implications regarding their consistency strength, which will also be a consequence of our below results.[18]

Strongly Ramsey $\to$ Ramsey $\to$ Weakly Ramsey $\to$ $\mathbf{T}^\kappa_\omega$-Ramsey $\to$ Ineffable.

Clearly, $\mathcal{I}(I) \subseteq \mathbf{T}(I) \subseteq \mathbf{wf}(I) \subseteq \mathcal{R}(I) \subseteq \mathbf{T}_{\text{cl}}(I)$ for any ideal I: The first inclusion follows from the fact that the operator $\mathbf{T}$ is ineffable by Observation 10.4, and the remaining inclusions follow from the final statement of Proposition 10.2. We will see below that the above chain of inclusions is a chain of strict inclusions in case $I = [\kappa]^{<\kappa}$, and that this also holds with respect to certain other ideals for some of these inclusions. In particular, this shows that the large cardinal operators $\mathcal{I}$, $\mathbf{T}$, $\mathbf{wf}$, $\mathcal{R}$ and $\mathbf{T}_{\text{cl}}$ are actually different from each other.

Remark 12.3. Let us remark that we cannot expect any of the above subset relations to be proper for arbitrary ideals I, since for example if κ is a completely Ramsey cardinal (that is, there is an ordinal α so that $\kappa \notin \mathcal{R}^\alpha([\kappa]^{<\kappa}) = \mathcal{R}^{\alpha+1}([\kappa]^{<\kappa})$), and I is the completely Ramsey ideal on κ (that is $I = \mathcal{R}^\alpha([\kappa]^{<\kappa})$ for such an α), then $I \subseteq \mathcal{I}(I) \subseteq \mathbf{T}(I) \subseteq \mathbf{wf}(I) \subseteq \mathcal{R}(I) = I$, and hence all of these ideals have to be equal.[19]

[16]Recall that $\mathbf{T}(M,U)$ denotes the (trivial) property that $U = U$, and that $\mathbf{wf}(M,U)$ denotes the property that the ultrapower of M by U is well-founded.

[17]Note that if a weak κ-model M is closed under $<\kappa$-sequences and an M-ultrafilter U is M-normal, then U is countably closed, yielding $\mathbf{T}^\kappa_\kappa$-Ramsey cardinals, $\mathbf{wf}^\kappa_\kappa$-Ramsey cardinals and $\mathbf{cc}^\kappa_\kappa$-Ramsey cardinals and their corresponding large cardinal operators to be equivalent notions.

[18]All but the final implication are easily seen to also be valid direct implications, however even strongly Ramsey cardinals need not be ineffable, for being strongly Ramsey is a Π^1_2-property (this is immediate from the results of Section 13 below), and ineffable cardinals are Π^1_2-indescribable by Lemma 3.4.

[19]Completely Ramsey cardinals were introduced by Feng in [8, Section 3]. By [8, Theorem 3.1], any measurable cardinal is completely Ramsey.

It was shown in [12, Lemma 10.1], using different notation, that $\mathcal{I}([\kappa]^{<\kappa}) \subsetneq \mathbf{T}([\kappa]^{<\kappa})$, as witnessed by $\{\xi<\kappa \mid \xi \in \mathcal{I}([\xi]^{<\xi})\} \in \mathbf{T}([\kappa]^{<\kappa}) \setminus \mathcal{I}([\kappa]^{<\kappa})$.[20] We want to argue that this result can be extended to the nonstationary ideal, essentially by the same argument, which we would like to provide here for the convenience of our readers.

Proposition 12.4. *If $I \in \{[\kappa]^{<\kappa}, \mathrm{NS}_\kappa\}$ and $\kappa \in \mathcal{I}(I)^+$, then*

$$\mathcal{I}(I) \subsetneq \mathbf{T}(I).$$

Proof. The inclusion itself has already been discussed above, and we are only left to verify its properness. Since the bounded ideal was already treated in [12], let us assume for simplicity of notation that $I = \mathrm{NS}_\kappa$ (the proof for $I = [\kappa]^{<\kappa}$ is essentially the same). By Lemma 4.2, $X = \{\xi < \kappa \mid \xi \in \mathcal{I}(\mathrm{NS}_\xi)\} \notin \mathcal{I}(\mathrm{NS}_\kappa)$. We will thus be done if we can show that $X \in \mathbf{T}(\mathrm{NS}_\kappa)$. Assume for a contradiction that this is not the case. Then, there is a transitive weak κ-model M with $b \in M$ for some bijection $b\colon \kappa \to V_\kappa$, and there is a κ-amenable, M-normal M-ultrafilter $U \subseteq \mathrm{NS}_\kappa^+$ on κ such that $X \in U$. Note that since $V_\kappa \subseteq M$, the set X satisfies the same definition in M that it satisfies in V. Using Proposition 11.6, let $j = j_U\colon \langle M, \in\rangle \to \langle N, \in_N\rangle$ be the κ-powerset preserving κ-embedding with critical point κ induced by U. Using the identification provided by Lemma 11.7, and since $X \in U$, this means that $N \models \kappa \in_N j(X)$, and hence that $N \models \kappa \in_N \mathcal{I}(\mathrm{NS}_\kappa)$.

On the other hand, since being stationary in κ is downwards absolute from V to $M \supseteq V_\kappa$,[21] it follows that every element of U is a stationary subset of κ in M. But then, given a collection $\mathcal{A} \in ([\mathcal{P}(\kappa)]^\kappa)^M$, $U \cap \mathcal{A} \in M$ by the κ-amenability of U, and $\Delta(U \cap \mathcal{A}) \in U$ by the M-normality of U. Since $U \subseteq \mathrm{NS}_\kappa^+$, this shows that $M \models \kappa \in \mathcal{I}_{coll}(\mathrm{NS}_\kappa)^+ = \mathcal{I}(\mathrm{NS}_\kappa)^+$, where the equality follows by Proposition 2.2.[22] Using that M and N have the same subsets of κ, it follows that also $N \models \kappa \in_N \mathcal{I}(\mathrm{NS}_\kappa)^+$, which is clearly a contradiction. □

We next want to separate $\mathbf{T}$ from $\mathbf{wf}$ in a strong sense. In order to do so, we will also need to make use of the operators $\mathbf{T}^\oplus$, $\mathbf{T}^\forall$, $\mathbf{wf}^\oplus$ and $\mathbf{wf}^\forall$.

- Let us recall [4, Page 101] that $A \subseteq \kappa$ is *completely ineffable* if there is an ordinal α so that $A \notin \mathcal{I}^\alpha([\kappa]^{<\kappa}) = \mathcal{I}^{\alpha+1}([\kappa]^{<\kappa})$.[23] In [12, Theorem 11.4], it is shown that κ is completely ineffable if and only if κ is $\mathbf{T}^\forall_\omega$-Ramsey.
- The notion of ω-Ramsey cardinal was introduced by Philipp Schlicht and the author in [14, Definition 5.1], and in our terminology, ω-Ramsey cardinals are exactly the $\mathbf{wf}^\forall_\omega$-Ramsey cardinals.[24]
- The notion of $\mathbf{T}^{\kappa^+}_\omega$-Ramsey cardinal was first considered in [12, Section 10].

[20] Note that this latter statement implies in particular that below any $\mathbf{T}^\kappa_\omega$-Ramsey cardinal, there are unboundedly many ineffable cardinals, and thus that $\mathbf{T}^\kappa_\omega$-Ramseyness is of strictly higher consistency strength than ineffability. Analogous remarks, which we will omit to make, apply to the other large cardinals and their related ideal operators that we will discuss in the remainder of this section.

[21] This use of downwards absoluteness is the reason why we don't know how to generalize this argument to ideals other than the bounded and the nonstationary ideals.

[22] In fact, we need to observe that Proposition 2.2 is a theorem of ZFC^-. Another subtletly is that NS_κ^M is not an element of M, it is only definable over M, which however clearly suffices.

[23] In fact, Baumgartner's original definition of complete ineffability uses NS_κ instead of $[\kappa]^{<\kappa}$, but since $\mathcal{I}([\kappa]^{<\kappa}) \supseteq \mathrm{NS}_\kappa$, as is shown within the proof of [3, Theorem 2.3], this clearly yields an equivalent notion.

[24] The equivalence with the original definition from [14] is immediate from Proposition 11.6.

- The notion of *super weakly Ramsey cardinal* was introduced by Philipp Schlicht and the author in [14, Definition 4.5], and in our terminology, they are exactly the $\mathbf{wf}_\omega^{\kappa^+}$-Ramsey cardinals κ.

Let us once again provide some comments on the strength of the large cardinal notions involved in the current discussion. $\mathbf{T}_\omega^{\kappa^+}$-Ramsey cardinals have a strictly higher conistency strength than $\mathbf{T}_\omega^{\kappa}$-Ramsey cardinals (this is a particular instance of [12, Lemma 9.13(1)]), and a strictly lower consistency strength than completely ineffable cardinals (this is a particular instance of [12, Lemma 9.14], since completely ineffable cardinals are exactly the $\mathbf{T}_\omega^{\forall}$-Ramsey cardinals). Summarizing, we get the following chain of strict implications with respect to consistency strength:[25]

$$\text{Completely ineffable} \to \mathbf{T}_\omega^{\kappa^+}\text{-Ramsey} \to \mathbf{T}_\omega^{\kappa}\text{-Ramsey}.$$

Particular instances of the same two lemmas from [12], using the property **wf** rather than **T**, imply the following chain of strict implications with respect to consistency strength:[26]

$$\omega\text{-Ramsey} \to \text{super weakly Ramsey} \to \text{weakly Ramsey}.$$

In order to connect this chain of implications with the previous one, let us remark that [10, Theorem 3.7] weakly Ramsey cardinals are of strictly higher consistency strength than completely ineffable cardinals.

We will need the following, which in particular yields an easy argument that completely ineffable cardinals κ are Π^1_β-indescribable for every $\beta < \kappa$.

Lemma 12.5. *If κ is inaccessible, $\theta > \kappa$ is a regular cardinal, $M \prec H(\theta)$ is a weak κ-model with $V_\kappa \subseteq M$, and U is an M-normal M-ultrafilter on κ that is κ-amenable for M, then, for every $\beta < \kappa$, $U \subseteq \Pi^1_\beta(\kappa)^+$. In particular, for every $\beta < \kappa$,*

- $\mathbf{T}^{\oplus}(\Pi^1_\beta(\kappa)) = \mathbf{T}^{\oplus}([\kappa]^{<\kappa})$,
- $\mathbf{T}^{\forall}(\Pi^1_\beta(\kappa)) = \mathbf{T}^{\forall}([\kappa]^{<\kappa})$,
- $\mathbf{wf}^{\oplus}(\Pi^1_\beta(\kappa)) = \mathbf{wf}^{\oplus}([\kappa]^{<\kappa})$, *and*
- $\mathbf{wf}^{\forall}(\Pi^1_\beta(\kappa)) = \mathbf{wf}^{\forall}([\kappa]^{<\kappa})$.

Proof. For the first statement of the lemma, assume for a contradiction that for some $\beta < \kappa$, an M-ultrafilter U as in the statement of the lemma contains a set X that is Π^1_β-describable. Thus, there is a Π^1_β-formula φ and $Q \subseteq V_\kappa$ such that $\langle V_\kappa, Q\rangle \models \varphi$, however, for every $\alpha \in X$, $\langle V_\alpha, Q \cap V_\alpha\rangle \models \neg\varphi$. Since $V_\kappa \subseteq M$, the code for such a formula φ is an element of M. Thus, by elementarity, and since $V_\kappa \in M$ by the inaccessibility of κ, the above statement with respect to this particular φ holds true also in M. But then, in the ultrapower of M by U, by Proposition 11.6, using the identification from Lemma 11.7, and using that $X \in U$, $\langle V_\kappa, Q\rangle \models \neg\varphi$. This however contradicts that U is κ-amenable for M, and hence that M and the ultrapower of M by U contain the same subsets of κ by Proposition 11.6.

The statements about $\mathbf{T}^{\oplus}$, $\mathbf{T}^{\forall}$, $\mathbf{wf}^{\oplus}$ and $\mathbf{wf}^{\forall}$ are immediate from the above. □

[25] These are also valid (strict) direct implications, as is immediate from the definitions, using again that completely ineffable cardinals are exactly the $\mathbf{T}_\omega^{\forall}$-Ramsey cardinals.

[26] They are also valid (strict) direct implications, as is immediate from the definitions.

Note that by the very definitions of the operators involved, for any ideal I on κ, we trivially have $\mathbf{T}(I) \subseteq \mathbf{T}^{\oplus}(I) \subseteq \mathbf{T}^{\forall}(I)$, and $\mathbf{wf}(I) \subseteq \mathbf{wf}^{\oplus}(I) \subseteq \mathbf{wf}^{\forall}(I)$. As particular instances of Lemma 11.9, we obtain the following.

Lemma 12.6. *For any $\beta < \kappa$, we have the following.*

(a) If $\kappa \in \mathbf{T}(\Pi^1_\beta(\kappa))^+$, then $\{\gamma < \kappa \mid \gamma \in \mathbf{T}^{\oplus}(\Pi^1_\beta(\gamma))\} \notin \mathbf{T}(\Pi^1_\beta(\kappa))$.

(b) If $\kappa \in \mathbf{wf}(\Pi^1_\beta(\kappa))^+$, then $\{\gamma < \kappa \mid \gamma \in \mathbf{wf}^{\oplus}(\Pi^1_\beta(\gamma))\} \notin \mathbf{wf}(\Pi^1_\beta(\kappa))$.

Proof. For (a), let $\Phi(M, U)$ be the statement that $U \subseteq \Pi^1_\beta(\kappa)^+$, and for (b), let $\Phi(M, U)$ be the conjunction of that statment with $\mathbf{wf}(M, U)$. Both formulas clearly satisfy the requirements of the lemma. Chasing the relevant definitions, it is straightforward to see that for any $X \subseteq \kappa$, X is a Φ^κ_ω-Ramsey subset of κ just in case (a) $X \notin \mathbf{T}(\Pi^1_\beta(\kappa))$ or (b) $X \notin \mathbf{wf}(\Pi^1_\beta(\kappa))$ respectively, and that for any γ, γ is $\Phi^{\gamma^+}_\omega$-Ramsey just in case (a) $\gamma \in \mathbf{T}^{\oplus}(\Pi^1_\beta(\gamma))^+$ or (b) $\gamma \in \mathbf{wf}^{\oplus}(\Pi^1_\beta(\gamma))^+$ respectively. □

We are finally ready to conclude our desired strong separation result. The case when $\beta = -1$ in the below is due to Philipp Lücke and the author [12, Lemma 10.1 and Lemma 12.1].

Theorem 12.7. *If $\kappa \in \mathbf{T}(\Pi^1_\beta(\kappa))^+$ and $\beta \in \{-1\} \cup \kappa$, then*

$$\mathbf{T}(\Pi^1_\beta(\kappa)) \subsetneq \mathbf{wf}(\Pi^1_\beta(\kappa)).$$

Proof. Clearly, by the very definitions involved, $\mathbf{T}(I) \subseteq \mathbf{wf}(I)$ for any ideal I on κ, and it only remains to verify inequality in the above. Since this is trivial otherwise, we may as well assume that $\kappa \in \mathbf{wf}(\Pi^1_\beta(\kappa))^+$. By Lemma 12.6(a),

$$X = \{\alpha < \kappa \mid \alpha \in \mathbf{T}^{\oplus}(\Pi^1_\beta(\alpha))\} \notin \mathbf{T}(\Pi^1_\beta(\kappa)).$$

On the other hand, the proof of Gitman's [10, Theorem 3.7] shows (see also [12, Theorem 1.5]) that if κ is a weakly Ramsey cardinal, then

$$Y = \{\alpha < \kappa \mid \alpha \text{ is not completely ineffable}\} \in \mathbf{wf}([\kappa]^{<\kappa}).$$

Since $\mathbf{wf}([\kappa]^{<\kappa}) \subseteq \mathbf{wf}(\Pi^1_\beta(\kappa))$, we will be done if we show that $X \subseteq Y$, which amounts to showing that completely ineffable cardinals α satisfy $\alpha \in \mathbf{T}^{\oplus}(\Pi^1_\beta(\alpha))^+$. But clearly, $\mathbf{T}^{\oplus}(\Pi^1_\beta(\alpha)) \subseteq \mathbf{T}^{\forall}(\Pi^1_\beta(\alpha)) = \mathbf{T}^{\forall}([\alpha]^{<\alpha})$ by Lemma 12.5, and thus we are done. □

Finally, a combination of known results provides a weak separation result for the operators $\mathbf{wf}$ and $\mathcal{R}$.

Proposition 12.8. *If $\kappa \in \mathbf{wf}([\kappa]^{<\kappa})^+$, then $\mathbf{wf}([\kappa]^{<\kappa}) \subsetneq \mathcal{R}([\kappa]^{<\kappa})$.*

Proof. The inclusion is trivial, and we only need to verify its properness. By [12, Corollary 9.16], the set $\{\alpha < \kappa \mid \alpha \in \mathbf{wf}([\alpha]^{<\alpha})\}$ is not an element of $\mathbf{wf}([\kappa]^{<\kappa})$, and the argument for the proof of [11, Theorem 4.1] shows that this set is an element of $\mathcal{R}([\kappa]^{<\kappa})$, noting that weakly Ramsey cardinals are exactly the 1-iterable cardinals from that paper, and that Ramsey cardinals are α-iterable for any α. □

If the answer to the following question were positive, using ω-Ramseyness rather than complete ineffability and using Lemma 12.6(b) rather than Lemma 12.6(a) in the proof of Theorem 12.7, we could analogously separate the operators $\mathbf{wf}$ and $\mathcal{R}$ in a strong sense. This question is a particular instance of [12, Question 17.4].

Question 12.9. Assume that κ is a Ramsey cardinal. Does it follow that the set of ω-Ramsey cardinals below κ is a Ramsey subset of κ, i.e., an element of $\mathcal{R}([\kappa]^{<\kappa})^+$?

Another possibility to separate the operators $\mathbf{wf}$ and $\mathcal{R}$ at least on NS_κ would be to show that if $\kappa \in \mathbf{wf}(\mathrm{NS}_\kappa)^+$, then the set $\{\alpha < \kappa \mid \alpha \in \mathbf{wf}(\mathrm{NS}_\alpha)\}$ is not an element of the ideal $\mathbf{wf}(\mathrm{NS}_\kappa)$, for it is not too hard to see that the argument for [11, Theorem 4.1] can be adapted to show this set to be an element of the ideal $\mathcal{R}(\mathrm{NS}_\kappa)$. We thus ask the following:

Question 12.10. Assume that $\kappa \in \mathbf{wf}(\mathrm{NS}_\kappa)^+$. Does it follow that

$$\{\alpha < \kappa \mid \alpha \in \mathbf{wf}(\mathrm{NS}_\alpha)\} \notin \mathbf{wf}(\mathrm{NS}_\kappa)?$$

By results from [8] and [12], it finally follows that $\mathcal{R}$ and $\mathbf{T}_{\mathrm{cl}}$ act differently, at least on the bounded and the nonstationary ideal.

Proposition 12.11. *If $I \in \{[\kappa]^{<\kappa}, \mathrm{NS}_\kappa\}$ and $\kappa \in \mathcal{R}(I)^+$, then*

$$\mathcal{R}(I) \subsetneq \mathbf{T}_{cl}(I).$$

Proof. The inclusion itself has already been discussed above, and we are only left to verify its properness. Let $\vec{I}$ be the sequence of ideals induced by I. By [8, Theorem 4.5], $S = \{\xi < \kappa \mid \xi \in \mathcal{R}(I_\xi)\} \notin \mathcal{R}(I)$. However, $S \in \mathbf{T}_{\mathrm{cl}}(I)$ by [12, Lemma 14.2]. □

13. Some coding apparatus

In order to be able to present a sample result for our generalized operators in Section 14, we will need to code weak κ-models M and M-ultrafilters U on κ as subsets of V_κ in some simple way, and since we are only really interested in the case when κ is an inaccessible cardinal, we may assume this to be the case whenever necessary. Our definition will be tailored so that any transitive weak κ-model that can be coded will have to be a superset of V_κ, with elements x of V_κ being coded as ordered pairs of the form $\langle 0, x\rangle$, and we code κ by 0.

Definition 13.1. We say that $\mathcal{M} \subseteq V_\kappa$ is a *code for a transitive weak κ-model* if $\mathcal{M} \subseteq V_\kappa$ with the following properties:

- $\mathcal{M}$ is a binary relation on V_κ, such that $\mathrm{dom}(\mathcal{M}) = V_\kappa$,
- for all $x, y \in V_\kappa$, $\langle 0, x\rangle \mathcal{M} \langle 0, y\rangle$ if and only if $x \in y$,
- for all x, $x \,\mathcal{M}\, 0$ if and only if $x \in \kappa$,
- $\mathcal{M}$ is well-founded and extensional, and
- $\langle V_\kappa, \mathcal{M}\rangle \models \mathrm{ZFC}^-$.

Note that the weak κ-model that is coded here is the model M such that $\langle M, \in\rangle$ is the transitive collapse of $\langle V_\kappa, \mathcal{M}\rangle$. On the other hand, any transitive weak κ-model $M \supseteq V_\kappa$ has a code as described above, using a suitable bijection between M and V_κ. Let $\pi_{\mathcal{M}}$ denote the transitive collapsing map of $\langle V_\kappa, \mathcal{M}\rangle$. If $X = \pi_{\mathcal{M}}(x)$, we say that *x is the code of X* (within $\mathcal{M}$).

Lemma 13.2. *The property that $\mathcal{M}$ is a code for a transitive weak κ-model is a Δ^1_1-property over $\langle V_\kappa, \mathcal{M}\rangle$.*

Proof. All but the final item in the above list can easily be phrased as first order properties within $\langle V_\kappa, \mathcal{M}\rangle$. The final item can be seen to be a Δ^1_1 property, for we need to say that either there is a satisfaction relation (coded as a subset of V_κ in some obvious way) for $\langle V_\kappa, \mathcal{M}\rangle$ that contains all axioms of ZFC$^-$, or that this is the case for all satisfaction relations, and being a satisfaction relation for $\langle V_\kappa, \mathcal{M}\rangle$ is a first order property, which is seen as usual: We require a satisfaction relation S to code finite tuples of the form $\langle \varphi, \vec{a}\rangle$, where of course we identify first order formulas φ with their Gödel codes. We require that if φ is an atomic formula, then its truth is correctly encoded by S. Recursively, we then require S to correctly encode the truth of all first order formulas, proceeding along the recursive construction of first order formulas. That is for example, we require S to consider $(\varphi \wedge \psi)(\vec{a})$ to be true just in case it considers both $\varphi(\vec{a})$ and $\psi(\vec{a})$ to be true, etc. [27] □

Note that we can easily shift between subsets X of V_κ in M and their codes within $\mathcal{M}$ – For $X \subseteq V_\kappa$ in M and $x \in V_\kappa$, the property $\pi^{-1}_{\mathcal{M}}(X) = x$ is a first order property over $\langle V_\kappa, \in, \mathcal{M}, X\rangle$: $\pi^{-1}_{\mathcal{M}}(X) = x$ in case $\forall y\ \ [\langle 0, y\rangle \mathcal{M} x \iff y \in X]$.

Next, we want to define what it means to code an M-ultrafilter on κ, which will easily be seen to be a first order property.

Definition 13.3. Given a code $\mathcal{M}$ for a transitive weak κ-model, we say that $\mathcal{U} \subseteq V_\kappa$ is a *code for an M-ultrafilter on κ* if $\langle V_\kappa, \mathcal{M}, \mathcal{U}\rangle$ thinks that $\mathcal{U}$ is an ultrafilter on 0 (note that our setup is so that 0 codes κ).

For our desired applications, we will need our operators to satisfy some properties of simple definability.

Definition 13.4. Let $\langle \Psi, \Omega\rangle$ be a pair of formulas, and let $\mathcal{O}$ be an ideal operator.

- $\langle \Psi, \Omega\rangle$ is *simple* in case ZFC proves the following:
  - (a) whenever M is a transitive weak κ-model, and U is an M-ultrafilter on κ, then $\Psi(M, U)$ translates to a Δ^1_1-property of any pair of codes $\langle \mathcal{M}, \mathcal{U}\rangle$ for $\langle M, U\rangle$ over V_κ, and
  - (b) whenever the property $X \in I^+$ is definable over V_κ by a Π^1_β-formula $\varphi(X)$ for some $0 < \beta < \kappa$, then $\Omega(U, I)$ translates to a Π^1_β-property of any code $\mathcal{U}$ of U over V_κ.
- $\langle \Psi, \Omega\rangle$ is *always simple* in case ZFC additionally proves that if in (b), the property $X \in I^+$ is first order definable over V_κ, then $\Omega(U, I)$ translates to a Δ^1_1-property of any code $\mathcal{U}$ of U over V_κ.
- $\mathcal{O}$ is *simple* or *always simple* in case ZFC proves that $\mathcal{O}$ can be characterized in the form $\mathcal{O} = \mathfrak{O}\Psi\Omega$ for some pair of formulas $\langle \Psi, \Omega\rangle$ that is simple or always simple respectively.

Definition 13.4(a) is immediate if Ψ can be expressed as a first order property of the structure $\langle M, \in, U\rangle$, for example if $\Psi(M, U)$ denotes the statement that U is κ-amenable for M.

If $\Psi(M, U)$ denotes the property that U is countably complete, this translates to the first order statement that for any countable sequence $\langle u_i \mid i < \omega\rangle$ of elements of $\mathcal{U}$,[28] there is x such that $x\mathcal{M}u_i$ for every $i < \omega$.

[27] In order to avoid any possible confusion here, let us emphasize that we are not claiming to be able to define a satisfaction predicate by a first order formula, but that given any particular predicate, we have a first order formula which uses this predicate as a second order parameter and which decides whether it actually is a satisfaction predicate.

[28] If κ is inaccessible (regular and uncountable suffices), then these countable sequences are elements of V_κ.

Consider the statement that M is closed under $<\kappa$-sequences. This translates to the following first order statement about any code $\mathcal{M}$ of M over V_κ:

$$\forall p\, \exists t\, \forall x\ \ (x\, \mathcal{M}\, t \iff x \in p)\,.$$ [29]

Finally, let $\Psi(M,U)$ denote the statement that the ultrapower of M by U is well-founded. Let us say that $\langle N, R\rangle$ *represents an ultrapower* of M by U if $N \subseteq V_\kappa$ consists of codes for functions f with domain κ such that

- for every g for which $\pi_{\mathcal{M}}^{-1}(g)$ is a function with domain κ, there is $f \in N$ such that $\{\alpha < \kappa \mid \pi_{\mathcal{M}}^{-1}(f)(\alpha) = \pi_{\mathcal{M}}^{-1}(g)(\alpha)\}$) is coded by a set in $\mathcal{U}$,
- for all $f, g \in N$, $\{\alpha < \kappa \mid \pi_{\mathcal{M}}^{-1}(f)(\alpha) = \pi_{\mathcal{M}}^{-1}(g)(\alpha)\}$ is not coded by a set in $\mathcal{U}$, and
- if $f, g \in N$, $f\, R\, g$ iff $\{\alpha < \kappa \mid \pi_{\mathcal{M}}^{-1}(f)(\alpha) \in \pi_{\mathcal{M}}^{-1}(g)(\alpha)\}$ is coded by a set in $\mathcal{U}$.

That is, essentially, a class $[f]_U$ in a usual ultrapower of M by U is taken to be represented by one of its elements. But now, asking that $\langle N, R\rangle$ is well-founded is clearly equivalent to asking the ultrapower of M by U to be well-founded, and moreover, as for the case of countable completeness above, this translates to a first order statement over V_κ, using that κ is regular and uncountable: it requires asking that no countable sequence of elements of N is decreasing with respect to R.

Let us look at some examples regarding Definition 13.4(b).

- If $\Omega(U, I)$ denotes the statement that $U \subseteq I^+$, then this translates to the statement that $\forall x\, \mathcal{M}\, \mathcal{U}\, \forall X\ [\pi_{\mathcal{M}}^{-1}(X) = x \to \varphi(X)]$, where φ is a Π^1_β-formula defining I^+ over V_κ.
- If $\Omega(U, I)$ denotes the property that countable intersections from U are in I^+, then this translates to the statement that for any countable sequence $\langle u_\beta \mid \beta < \omega\rangle$ of elements of $\mathcal{U}$,
$$\varphi(\{\alpha < \kappa \mid \forall \beta < \omega\ \langle 0, \alpha\rangle\, \mathcal{M}\, u_\beta\}).$$
- If $\Omega(U, I)$ denotes the property that $\Delta U \in I^+$, then this translates to the statement that for any κ-enumeration $\langle u_\beta \mid \beta < \kappa\rangle$ of the elements of $\mathcal{U}$,
$$\varphi(\{\alpha < \kappa \mid \forall \beta < \alpha\ \langle 0, \alpha\rangle\, \mathcal{M}\, u_\beta\}).$$

If the property $X \in I^+$ is first order definable, observe that we obtain a Δ^1_1-statement in the first two cases above, for we can equivalently rephrase the above to use existential rather than universal second order quantifiers. However this does not work in the third case (see the remarks made in Footnote 2). In particular, this means that the operators $\mathbf{T}$, $\mathbf{wf}$, $\mathcal{R}$ and $\mathbf{T}_{\mathrm{cl}}$ are always simple, while $\mathcal{I}_{mod}$ is simple.

The following lemma extracts what we will actually need in the next section.

Lemma 13.5. *If $\mathcal{O}$ is simple, then the following hold.*

- *If $y \subseteq \kappa$ and $\beta < \kappa$, then the statement that $X \in \mathcal{O}^y(\Pi^1_\beta(\kappa))$ can be expressed as a $\Pi^1_{\beta+2}$-property of X and y over V_κ.*

[29] Given any regular $\alpha < \kappa$, it is also easy to express closure of M under $<\alpha$-sequences by a first order statement about $\mathcal{M}$ over V_κ. We will however not make use of such properties in our paper.

- *If $\beta < \kappa$, then the statement that $X \in \mathcal{O}(\Pi^1_\beta(\kappa))^+$ can be expressed as a $\Pi^1_{\beta+3}$-property of X over V_κ.*

If $\langle\Omega, \Psi\rangle$ is always simple, then the above also hold in case $\beta = -1$.

Proof. Immediate from Lemma 3.3 by a simple counting of quantifiers in Definition 10.1, making use of the fact that $\mathcal{O}$ is simple. □

14. A test application: Generalized Pre-Operators

In this section, we want to provide a sample result, showing that simple ineffable operators are structurally well-behaved, by providing a basic theorem about their relationship to their corresponding pre-operators. This result generalizes the case when $\alpha = 1$ of [6, Theorem 6.1], and shows that our pre-operators have the same key role with respect to their corresponding operators as does the subtle operator with respect to the ineffability operator, and the pre-Ramsey operator with respect to the Ramsey operator by classical results of Baumgartner from his [3] and [4][30] – these results are instances of our general result below, which in particular also provides valid new instances for the operators $\mathbf{T}$, $\mathbf{wf}$ and $\mathbf{T}_{\mathrm{cl}}$. The proof follows the proof in [6] for the most part, but there are some subtleties involved in how to make use of the machinery that we developed in Section 13 above, and therefore we would like to provide the complete argument. If $\mathcal{A}$ is a collection of subsets of a cardinal κ, we write $\overline{\mathcal{A}}$ to denote the ideal on κ that is generated by $\mathcal{A}$: This is the collection of all subsets of κ that are contained in some finite union of elements of $\mathcal{A}$.

Theorem 14.1. *If $\mathcal{O}$ is ineffable and simple, and $\beta < \kappa$, then*

$$\mathcal{O}(\Pi^1_\beta(\kappa)) = \overline{\mathcal{O}_0(\Pi^1_\beta(\kappa)) \cup \Pi^1_{\beta+2}(\kappa)}.$$

If $\mathcal{O}$ is ineffable and always simple, then the above also holds in case $\beta = -1$.

Proof. Fix some β, and let $J = \overline{\mathcal{O}_0(\Pi^1_\beta(\kappa)) \cup \Pi^1_{\beta+2}(\kappa)}$. We show that $X \in J^+$ if and only if $X \in \mathcal{O}(\Pi^1_\beta(\kappa))^+$.

Suppose for a contradiction that $X \in J^+$, however $X \in \mathcal{O}(\Pi^1_\beta(\kappa))$. Let $y \subseteq \kappa$ be such that $X \in \mathcal{O}^y(\Pi^1_\beta(\kappa))$, that is whenever M is a transitive weak κ-model with $y \in M$ and U is an M-ultrafilter on κ with $X \in U$ and with $\Psi(M, U)$, then $\Omega(U, \Pi^1_\beta(\kappa))$ fails. By Lemma 13.5, this can be expressed by a $\Pi^1_{\beta+2}$-sentence φ over V_κ, and thus

$$C = \{\xi < \kappa \mid V_\xi \models \varphi(X \cap \xi, y \cap \xi)\} \in \Pi^1_{\beta+2}(\kappa)^*.$$

Since $X \notin J$, X is not the union of a set in $\mathcal{O}_0(\Pi^1_\beta(\kappa))$ and a set in $\Pi^1_{\beta+2}(\kappa)$, and since $X = (X \cap C) \cup (X \setminus C)$, it follows that $X \cap C \notin \mathcal{O}_0(\Pi^1_\beta(\kappa))$. Thus we may find $\xi \in (X \cap C) \setminus (\beta + 1)$, a weak ξ-model $\bar{M}$ with $y \cap \xi \in \bar{M}$ and an $\bar{M}$-ultrafilter $\bar{U}$ on ξ with $X \cap C \cap \xi \in \bar{U}$ such that $\Psi(\bar{M}, \bar{U})$ and $\Omega(\bar{U}, \Pi^1_\beta(\xi))$ hold, contradicting the above.

[30] Baumgartner has also verified a version of Theorem 14.1 for the weakly ineffable ideal in [3, Section 7]. Namely, he has shown that the weakly ineffable ideal on a cardinal κ is generated by the subtle ideal together with the Π^1_1-indescribable ideal. We do not know whether this result could also be obtained via Theorem 14.1, by using the operator $\mathcal{I}_{mod}$ or perhaps some slight variant. Some strongly related issues will be discussed in Section 15.

Now suppose $X \in \mathcal{O}(\Pi^1_\beta(\kappa))^+$. By [6, Remark 2.1], it suffices to show that $X \in \mathcal{O}_0(\Pi^1_\beta(\kappa))^+$ and $X \in \Pi^1_{\beta+2}(\kappa)^+$, where the latter is immediate from Corollary 10.5. We are thus left to show that $X \in \mathcal{O}_0(\Pi^1_\beta(\kappa))^+$. Fix $y \subseteq \kappa$ and a club $C \subseteq \kappa$. By the third item in Observation 10.4, it follows that $X \cap C \in \mathcal{O}(\Pi^1_\beta(\kappa))^+$. Thus, there are $\mathcal{M}$ and $\mathcal{U}$ such that the following $\Pi^1_{\beta+1}$-sentence φ holds over the structure $\langle V_\kappa, \in, y, X \cap C, M, U\rangle$: $\mathcal{M}$ is (a code for) a transitive weak κ-model M with $y \in M$ and $\mathcal{U}$ is (a code for) an M-ultrafilter U on κ with $X \cap C \in U$ such that $\Psi(M, U)$ and $\Omega(U, \Pi^1_\beta(\kappa))$ hold. Since $X \cap C \in \Pi^1_{\beta+2}(\kappa)^+$, there is $\xi \in (X \cap C) \setminus (\beta + 1)$ such that

$$\langle V_\xi, \in, y \cap \xi, X \cap C \cap \xi, M \cap \xi, U \cap \xi\rangle \models \varphi,$$

and hence $X \cap C \cap \xi \in \mathcal{O}^y(\Pi^1_\beta(\xi))^+$, as witnessed by the code $\mathcal{M} \cap \xi$ for a weak ξ-model and the code $\mathcal{U} \cap \xi$ for an M-ultrafilter on ξ, yielding that $X \in \mathcal{O}_0(\Pi^1_\beta(\kappa))^+$. □

By similar means as in Theorem 14.1, many of the results from [6] for the Ramsey operator can be extended to our generalized operators in a fairly straightforward way. Amongst other things, these further generalizations are planned to be included in a follow-up paper [7]. Let us present one final easy sample result here, namely that applications of our operators to different indescribability ideals give rise to a proper hierarchy of large cardinal notions (this is a weak generalized analogue of results from [6]).

Proposition 14.2. *If $\mathcal{O}$ is ineffable and simple, $\beta < \kappa$, and $\kappa \in \mathcal{O}(\Pi^1_{\beta+1}(\kappa))^+$, then κ is a stationary limit of cardinals α for which $\alpha \in \mathcal{O}(\Pi^1_\beta(\alpha))^+$. If $\mathcal{O}$ is ineffable and always simple, then the above also holds in case $\beta = -1$.*

Proof. Assume that $\kappa \in \mathcal{O}(\Pi^1_{\beta+1}(\kappa))^+$. By the monotonicity of $\mathcal{O}$, we thus also have $\kappa \in \mathcal{O}(\Pi^1_\beta(\kappa))^+$. But by Corollary 10.5, our assumption implies that $\kappa \in \Pi^1_{\beta+3}(\kappa)^+$, and by Lemma 13.5, $\kappa \in \mathcal{O}(\Pi^1_\beta(\kappa))^+$ can be expressed as a $\Pi^1_{\beta+3}$-property over V_κ. Hence, the set of cardinals α for which $\alpha \in \mathcal{O}(\Pi^1_\beta(\alpha))^+$ is contained in the $\Pi^1_{\beta+3}$-indescribable filter on κ, and hence in particular this set is stationary in κ. □

Let us close this section by providing some additional information about the relationship between the operators $\mathcal{I}$ and $\mathcal{R}$.

Observation 14.3. *Assume that $\beta \in \{-1\} \cup \mathrm{Ord}$, and that κ is least such that $\kappa \in \mathcal{R}(\Pi^1_\beta(\kappa))^+$. Then $\kappa \notin \mathcal{I}(\Pi^1_{\beta+1}(\kappa))^+$.*

Proof. Assume for a contradiction that $\kappa \in \mathcal{I}(\Pi^1_{\beta+1}(\kappa))^+$. Then, by Theorem 14.1, $\kappa \in \Pi^1_{\beta+3}(\kappa)^+$. By Lemma 13.5, $\kappa \in \mathcal{R}(\Pi^1_\beta(\kappa))^+$ can be expressed as a $\Pi^1_{\beta+3}$-property over V_κ. Combining these, we find some $\alpha < \kappa$ such that $\alpha \in \mathcal{R}(\Pi^1_\beta(\alpha))^+$, contradicting the leastness of κ. □

15. Some remarks on weak ineffability

In this final section, we want to treat the seemingly problematic case of applying operators of the form $\mathfrak{O}\Psi\Omega$ with $\Omega(U, I)$ being the statement that $\Delta U \in I^+$ to ideals of the form $[\kappa]^{<\kappa}$. Note that by the convention from Section 2, $\Delta U \in I^+$ abbreviates the statement that every diagonal intersection of (all the elements of) U is in I^+. If we strengthen this to require that every diagonal intersection of any κ-many (and not necessarily all) elements of U is in I^+, we can show the following, which is based on the observation from [12] that the notions of *genuinity* and *normality* from [17] coincide for weak κ-models.

Observation 15.1. *If $\Omega(U, I)$ denotes the property that every diagonal intersection of elements of U is in I^+, and $\mathcal{O} = \mathfrak{D}\Psi\Omega$ for some first order formula Ψ, then for any ideal I on κ and $y \subseteq \kappa$, $\mathcal{O}^y(I) = \mathcal{O}^y(\overline{I \cup \mathrm{NS}_\kappa})$. In particular, this implies that $\mathcal{O}(I) = \mathcal{O}(\overline{I \cup \mathrm{NS}_\kappa})$, and hence that $\mathcal{O}([\kappa]^{<\kappa}) = \mathcal{O}(\mathrm{NS}_\kappa)$. Moreover, given a sequence $\vec{I}$ of ideals, this also implies that $\mathcal{O}_0(\vec{I}) = \mathcal{O}_0(\overline{\langle I_\alpha \cup \mathrm{NS}_\alpha \mid \alpha \leq \kappa \rangle})$, and hence that $\mathcal{O}_0([\kappa]^{<\kappa}) = \mathcal{O}_0(\mathrm{NS}_\kappa)$.*

Proof. It is immediate that $\mathcal{O}(I) \subseteq \mathcal{O}(\overline{I \cup \mathrm{NS}_\kappa})$. Assume thus that $y \subseteq \kappa$, and that $A \in \mathcal{O}^y(I)^+$. That is, there is a transitive weak κ-model M with $y \in M$ and an M-ultrafilter U on κ with $A \in U$ such that $\Psi(M, U)$ holds and every diagonal intersection of elements of U is in I^+, and thus in particular an unbounded subset of κ. In the notation of [17], this means that U is a *genuine* M-ultrafilter. But by [12, Proposition 17.2], this implies that U is in fact a *normal* M-ultrafilter, meaning that ΔU is a stationary subset of κ. Making use of [6, Remark 2.1], this shows that $A \in \mathcal{O}^y(\overline{I \cup \mathrm{NS}_\kappa})^+$, as desired. The remaining statements follow by the very definitions of the operators involved. □

The above tells us for example that we cannot characterize the weakly ineffable ideal $\mathcal{I}([\kappa]^{<\kappa})$ by using the operator $\mathcal{O} = \mathfrak{D}\Psi\Omega$, where $\Psi(M, U)$ is trivial and $\Omega(U, I)$ is the statement that any diagonal intersection of elements of U is in I^+, for $\mathcal{O}([\kappa]^{<\kappa})$ already yields the ineffable ideal on κ whenever κ is ineffable (or all of $\mathcal{P}(\kappa)$ whenever it is not). As was already observed in [12, Section 17], such a characterization (of weak ineffability only) was wrongly claimed in [17, Theorem 3.2 (ii)]. Concerning our original operator $\mathcal{I}_{mod}$, we do not know as to whether $\mathcal{I}_{mod}([\kappa]^{<\kappa})$ is the weakly ineffable ideal on κ, however it seems unlikely to us. Let us introduce yet another variant of the operator $\mathcal{I}_{mod}$.

Definition 15.2. We define the operator $\mathcal{I}^*_{mod}$ as follows. First, for any $y \subseteq \kappa$, we introduce its local instance at y, letting

- $x \in \mathcal{I}^{*y}_{mod}(I)^+$ if there is a transitive weak κ-model M with $y \in M$ such that for any κ-enumeration $\vec{x}$ of $\mathcal{P}(\kappa) \cap M$, there is an M-ultrafilter U on κ that flips $\vec{x}$ such that $x \in U$ and $\Delta U \in I^+$, and we let
- $\mathcal{I}^*_{mod}(I)^+ = \bigcap_{y \subseteq \kappa} \mathcal{I}^{*y}_{mod}(I)^+$.

Assume that κ is weakly ineffable. Then, by the very definitions of the operators involved, $I^*_{mod}([\kappa]^{<\kappa}) \subseteq I_{coll}([\kappa]^{<\kappa})$, and we have shown the latter to be equal to the weakly ineffable ideal on κ in Proposition 2.2. However, the reverse inclusion seems to be potentially problematic, and thus we ask the following, which we conjecture to have a negative answer.

Question 15.3. Is $\mathcal{I}^*_{mod}([\kappa]^{<\kappa})$ the weakly ineffable ideal on κ?

A positive answer to Question 15.3 would show that a cardinal κ is weakly ineffable if and only if $\kappa \in I^*_{mod}([\kappa]^{<\kappa})^+$, and this was in fact claimed in [12, Paragraph after Proposition 17.2] without proof, however we do not know how to verify this claim, and would like to pose it as an open question.[31] It seems

[31]The problem that occurs if trying to verify the claim made in [12] is with collections of subsets of κ which are not of the form $\mathcal{P}(\kappa) \cap M$ for a weak κ-model M. While this may seem like a minor technical difficulty at first, it appears to be a serious obstacle on second sight.

likely that a positive answer to Question 15.4 would also yield a positive answer for Question 15.3.

Question 15.4. Does $\kappa \in I^*_{mod}([\kappa]^{<\kappa})^+$ imply that κ is weakly ineffable?

References

[1] Fred G. Abramson, Leo A. Harrington, Eugene M. Kleinberg, and William S. Zwicker. Flipping properties: a unifying thread in the theory of large cardinals. *Ann. Math. Logic*, 12(1):25–58, 1977.

[2] Joan Bagaria. Derived topologies on ordinals and stationary reflection. *Trans. Amer. Math. Soc.*, 371:1981–2002, 2019.

[3] James E. Baumgartner. Ineffability properties of cardinals. I. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. I*, pages 109–130. Colloq. Math. Soc. János Bolyai, Vol. 10. North-Holland, Amsterdam, 1975.

[4] James E. Baumgartner. Ineffability properties of cardinals. II. In *Logic, foundations of mathematics and computability theory (Proc. Fifth Internat. Congr. Logic, Methodology and Philos. of Sci., Univ. Western Ontario, London, Ont., 1975), Part I*, pages 87–106. Univ. Western Ontario Ser. Philos. Sci., Vol. 9. Reidel, Dordrecht, 1977.

[5] Brent Cody. Large cardinal ideals. To appear in *Research Trends in Contemporary Logic*.

[6] Brent Cody. A refinement of the Ramsey hierarchy via indescribability. *J. Symbolic Logic*, 85(2):773–808, 2020.

[7] Brent Cody and Peter Holy. Large cardinal operators and higher indescribability. Submitted, 2021.

[8] Qi Feng. A hierarchy of Ramsey cardinals. *Ann. Pure Appl. Logic*, 49(3):257–277, 1990.

[9] Victoria Gitman. *Applications of the proper forcing axiom to models of Peano arithmetic*. ProQuest LLC, Ann Arbor, MI, 2007. Thesis (Ph.D.)–City University of New York.

[10] Victoria Gitman. Ramsey-like cardinals. *J. Symbolic Logic*, 76(2):519–540, 2011.

[11] Victoria Gitman and Philip D. Welch. Ramsey-like cardinals II. *J. Symbolic Logic*, 76(2):541–560, 2011.

[12] Peter Holy and Philipp Lücke. Small models, large cardinals, and induced ideals. *Ann. Pure Appl. Logic*, 172(2), 2021.

[13] Peter Holy, Philipp Lücke, and Ana Njegomir. Small embedding characterizations for large cardinals. *Ann. of Pure Appl. Logic*, 170(2):251–271, 2019.

[14] Peter Holy and Philipp Schlicht. A hierarchy of Ramsey-like cardinals. *Fund. Math.*, 242(1):49–74, 2018.

[15] Ronald B. Jensen and Kenneth Kunen. Some combinatorial properties of L and V. Handwritten notes, 1969.

[16] William Mitchell. Ramsey cardinals and constructibility. *J. Symbolic Logic*, 44(2):260–266, 1979.

[17] Dan Saattrup Nielsen and Philip D. Welch. Games and Ramsey-like cardinals. *J. Symbolic Logic*, 84(1):408–437, 2019.

[18] Ian Sharpe and Philip D. Welch. Greatly Erdos cardinals with some generalizations to the Chang and Ramsey properties. *Ann. Pure Appl. Logic*, 162(11):863–902, 2011.

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